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Solving Bratu's problem by double exponential Sinc method

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Abstract. In this study, improved Sinc-Galerkin and Sinc-collocation methods are developed based on double exponential transformation to solve a one-dimensional Bratu-type equation. The properties of these methods are used to reduce the solution of the nonlinear problem to the solution of nonlinear algebraic equations. For simplicity in solving the nonlinear system, a matrix vector form of the nonlinear system is found. The upper bound of the error for the Sinc-Galerkin is determined. Also the numerical approximations are compared with the best results reported in the literature. The results confirm that both the Sinc-Galerkin and the Sinc-collocation methods have the same accuracy, but they are significantly more accurate than the other existing methods.

Keywords: Sinc-Galerkin, Sinc-collocation, Bratu's problem, double exponential transformation, boundary value problems.
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1 Introduction

In this study, a classical Liouville-Bratu-Gelfand problem is addressed as follows:

$$\begin{cases} u''(x) + \lambda \exp(u(x)) = 0, \qquad 0 < x < 1 \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

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where λ is a physical parameter. This problem has an analytical solution as below:

$$u(x) = -2\log\left(\frac{\cosh(0.5(x-0.5)\theta)}{\cosh(\theta/4)}\right) \tag{2}$$

in which θ is the solution of $\theta = \sqrt{2\lambda}\cosh(\theta/4)$ [3, 4, 6–8]. This nonlinear two-point boundary value problem appears in several engineering and physical problems, e.g., the theory of combustion, the Chandrasekhar model of expansion of the universe, questions of geometry and relativity in Chandrasekhar model [7, 19], the process in a rigid material, steady-state of heat diffusion and transfer condition, chemical reaction theory, radiative heat transfer, and non-technology [12]. Recently, this equation appeared in the fuel ignition model as considered by Raja [21]. He applied a procedure based on the neural network approach to solve it. An extended summary of the equation's history can be found in references [2, 4, 6, 11].

Several researchers have proposed methods to solve Bratu's equation numerically and analytically. Abbasbandy et al. [1] used the Lie-group shooting method. Caglar et al. [8] applied the B-Spline method, Abukhaled et al. [3] suggested a spline based method. Jalilian et al. [9] proposed a class of new method base on a septic non-polynomial spline function and discussed convergence analysis of the method. Khuri [15] recommended the Laplace method. Zarebnia et al. [29] utilized a parametric spline method. Rashidinia et al. [22, 23] developed a Sinc method based on the single exponential transformation. Following Zarebnia et al. [30] investigated the convergence of the Sinc-Galerkin method based on this transformation. Temimi et al. [5, 27] considered an iterative finite difference (IFD) scheme. They introduced a transformation to convert Bratu's problem into a simpler one and then applied the classical finite difference. The application of the Legendre spectral element method coupled with the qusi-linearization method was studied by Lotfi et al. [17]. Igbal et al. [13] studied the higher dimensional Gelfand Bratu model by a nonlinear multigrid method. Vazquez-Lead et al. [28] applied the novel Lead-polynomial to the approximation of various nonlinear differential equations. They considered Bratu's problem in their study. Singh et al. [24] used the Legendre spectral collocation method to find solutions for the fractional Bratu's problem. Hajipour et al. [10] developed a fourth-order nonstandard compact finite difference to solve 1-D, 2-D and 3-D Bratu's problem. Kazemi Nasab et al. [14] investigated a Chebyshev wavelet analysis method to solve Troesch's and Bratu's problems.

This problem has an analytic closed-form solution for several parameters, introduced in Eq. (1), which is a transcendental equation. It is, therefore, a suitable criterion for validating the accuracy and effectiveness of the obtained solution. On the other hand, this problem has more applications in science and engineering as already stated. Therefore, we were motivated to apply the Sinc-Galerkin and Sinc-collocation methods based on the double exponential (DE) transformation. The points to consider when applying of these methods are the order of accuracy $\mathcal{O}\Big(\exp\Big(-n/\log n\Big)\Big)$ [26] and the ability to handle singular problems. Interested readers can refer to the references of [18, 20, 25] for more information on the Sinc function.

This paper develops, Sinc-collocation and Sinc-Galerkin methods based on the DE transformation with a new strategy to solve Bratu's problem. The properties of these methods are useful to reduce the solution of a nonlinear problem to the solution of nonlinear algebraic equations. To simplify the solution by programming language, the matrix-vector forms of these nonlinear algebraic equations are achieved, with full details. Since the convergence of the Sinc-Galerkin method based on a single exponential transformation has been investigated by Zarebnia et al. [30] just the upper bound of error is obtained here. The results are compared with an exact solution and the numerical solution of other existing methods. As presented in the next sections, the results demonstrated that the DE Sinc methods are easy to implement, rapidly converge, and provide an effective mathematical tool to solve such nonlinear problems.

The rest of this paper is organized as follows. Section 2 outlines the theorems and notations and also some of the main properties of the Sinc function based on the DE transformation, which are needed by our method. In Section 3, the Sinc-Galerkin and Sinc-collocation approaches based on a double exponential transformation are developed to solve problem (1), and the corresponding discrete systems of algebraic equations are achieved. Section 4, compares the results of our method with each other and with some existing numerical results reported in the literature. Finally, the paper is concluded in Section 5

2 A survey of some properties of the Sinc method

This section provides, a brief overview of some properties of the Sinc function, theorems, and notations that are needed in other sections. References [18, 25, 26], discuss the Sinc method and applications thoroughly.

The original Sinc approximation is expressed on a whole real line as

$$f(x) \cong \sum_{j=-N}^{N} f(jh)S(j,h)(x) \tag{3}$$

where $h > 0$, and $S(j,h)(x)$ is the translated j^{th} Sinc function given by

$$S(j,h)(x) = Sinc\Big(\frac{x-jh}{h}\Big) = \begin{cases} \frac{\sin((\pi/h)(x-jh))}{(\pi/h)(x-jh)}, & x \neq jh, \\ 1, & x = jh. \end{cases} \quad (4)$$

Lemma 1. *[18] Let $S(k,h)(x)$ be the k^{th} Sinc functions with step size h, then*

$$\delta^{(0)}_{jk} = S(j,h)(kh) = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases}$$

$$\delta^{(1)}_{jk} = h\frac{d}{dx}[S(j,h)(x)](kh) = \begin{cases} 0, & j = k, \\ \frac{(-1)^{k-j}}{k-j}, & j \neq k, \end{cases}$$

$$\delta^{(2)}_{jk} = h^2\frac{d^2}{dx^2}[S(j,h)(x)](kh) = \begin{cases} \frac{-\pi^2}{3}, & j = k, \\ \frac{-2(-1)^{k-j}}{(k-j)^2}, & j \neq k. \end{cases}$$

For the assembly of the discrete system, it is convenient to define the following matrices:

$$I^{(l)} = [\delta^{(l)}_{jk}], \qquad l = 0, 1, 2, \quad (5)$$

where $I^{(l)}$'s are Toeplitz matrix. The following notation will be needed to write down the system of algebraic equations. Let $D(g)$ be an $m \times m$ diagonal matrix as follows:

$$D\Big(g(x)\Big) = diag\Big(g(-Nh),\ g((-N+1)h), \ldots,\ g(Nh)\Big). \quad (6)$$

Definition 1. *[18]* Let D_d denote the infinite strip region with $2d(d > 0)$ in a complex plane: $D_d \equiv \Big\{z \in \mathrm{C}\Big| |Imz| < d\Big\}$, and for $0 < \varepsilon < 1$, let $D_d(\varepsilon)$ be defined by $D_d(\varepsilon) \equiv \Big\{z \in \mathrm{C}\Big| |Rez| < 1/(\varepsilon), |Imz| < d(1-\varepsilon)\Big\}$, then $H^1(D_d)$ be the Hardy space over the region D_d, i.e., the set of functions f analytic in D_d such that

$$\lim_{\varepsilon \to 0} \int_{\partial D_d(\varepsilon)} |f(z)||dz| < \infty.$$

Theorem 1. *[26] Assume that a function f satisfies the following conditions*
1) $f \in H^1(D_d)$,
2) $\forall x \in \boldsymbol{R}: \quad | f(x) | \leq A\exp(-B\exp(\gamma \mid x \mid))$,

for positive constants A, B, γ and d where $\gamma d \le \frac{\pi}{2}$. Then, there exists a constant C independent of N, such that:

$$\sup_{-\infty<x<\infty}\Big|\ f(x)-\sum_{k=-N}^{N} f(kh)S(k,h)(x)\Big| \le C\exp\Big(-\frac{\pi d\gamma N}{log(\pi d\gamma N/B)}\Big),$$

where

$$h=\frac{\log(\pi d\gamma N/B)}{\gamma N}.$$

Theorem 2. *[26] For $d>0$, let f be a holomorphic function on D_d satisfying the following conditions*

1) $f\in H^1(D_d)$,

2) $\forall x\in \boldsymbol{R}:\quad |\ f(x)\ |\le A\exp(-B\exp(\gamma\ |\ x\ |))$,

for constants A, $B>0$ and $\gamma>0$ with $\gamma d\le\frac{\pi}{2}$. Then, there exists a constant C independent of N, such that:

$$|\int_{-\infty}^{\infty} f(x)dx-h\sum_{k=-N}^{N} f(kh)\ |\le C\exp\Big(-\frac{2\pi d\gamma N}{log(2\pi d\gamma N/B)}\Big), \tag{7}$$

where

$$h=\frac{\log(2\pi d\gamma N/B)}{\gamma N}. \tag{8}$$

3 Methodology

3.1 Using DE transformation for converting Beratu-equation to $(-\infty,\infty)$

Consider the Bratu's equation of the form

$$\begin{cases} L(u(x))=u''(x)+\lambda\exp(u(x))=0, & 0<x<1 \\ u(0)=u(1)=0, \end{cases} \tag{9}$$

From Theorems 1 and 2, it is clear that the original domain of the Sinc method is a whole real line. Therefore in problems with a different domain one can change the variable into a new variable in which the, problem has a whole real line as a domain. For this purpose, the following conformal map, which is known as the double exponential (DE) transformation is used

$$x = \psi_{DE}(t) = \frac{1}{2}\tanh\Big(\frac{\pi}{2}\sinh(t) + \frac{1}{2}\Big), \tag{10}$$

$$t = \phi_{DE}(x) = \psi_{DE}^{-1}(x) = \log\Big[\frac{1}{\pi}\log(\frac{x}{1-x}) + \sqrt{1 + \{\frac{1}{\pi}\log(\frac{x}{1-x})\}^2}\ \Big]. \tag{11}$$

By using this transformation, problem (9) is converted to:

$$\begin{cases} L\Big(u(\psi(t))\Big) = u''\Big(\psi(t)\Big) + \lambda \exp\Big(u(\psi(t))\Big) = 0, \\ \lim\limits_{t\to\pm\infty} u\Big(\psi(t)\Big) = 0. \end{cases} \tag{12}$$

Considering $v(t) = u(\psi(t))$, by using the chain role of differentiation, then

$$u'(\psi(t)) = \frac{d}{dx}u(\psi(t)) = \frac{d}{dt}u(\psi(t))\frac{dt}{dx} = \frac{1}{\psi'(t)}v'(t), \tag{13}$$

$$u''(\psi(t)) = \frac{d^2}{dx^2}u(\psi(t)) = \frac{d}{dx}u'(\psi(t)) = \Big(\frac{1}{\psi'(t)}\Big)^2 v''(t) - \frac{\psi''(t)}{(\psi'(t))^3}v'(t) \tag{14}$$

Substituting equations (13), (14) into problem (12), and multiplying by $\psi'(t)$, the following equation is obtained

$$\begin{cases} L\Big(v(t)\Big) = \Big(\frac{1}{\psi'(t)}\Big)v''(t) - \Big(\frac{\psi''(t)}{(\psi'(t))^2}\Big)v'(t) + \lambda\psi'(t)\exp\Big(v(t)\Big) = 0, \\ \lim_{t\to\pm\infty} v(t) = 0. \end{cases} \tag{15}$$

Now problem (15) is defined on a whole real line, so one can use the Sinc method to solve it. To approximate the solution of problem (15), the Sinc approximation is considered in the following form:

$$v_m(t) = \sum_{j=-N}^{N} C_j S_j(t), \qquad m = 2N+1, \tag{16}$$

where $\{C_j\}_{j=-N}^{N}$ is unknown and $S_j(t) = S(j,h)(t)$ are bases of the Sinc functions defined in Eq. (4). Note that $\lim_{t\longrightarrow\pm\infty} S_j(t) = 0$, so $v_m(t)$ satisfies the boundary conditions of problem (15). To specify the unknown coefficients in equation (16) two methods as Galerkin and collocation are applied.

3.2 The Sinc-Galerkin method

In this section, an approximation solution is obtained for problem (15) by the Galerkin method. In other words, $v_n(t)$ is calculated by orthogonalizing the residual $L(v(t))$ with respect to m-basis functions S_j, for $j = -N, ..., N$. At first, consider the following form of inner product for an arbitrary function f, g

$$\prec f, g \succ = \int_{-\infty}^{\infty} f(t)g(t)w(t)dt, \tag{17}$$

in which $w(t)$ is a weight function. This weight function may be chosen for different reasons. Lund [18] used a kind of w for symmetrization of the resulting system in a self-adjoint linear problem. But the selection here is due to the requirement of vanishing the boundary terms when using the integration by part in inner products. This is why weight function is chosen as $w(t) = \psi'(t)$.

With this interpretation, the orthogonality of $L(v(t))$ and S_j lead to

$$\prec Lv - \psi' f, S_j \succ = 0, \qquad j = -N, -N+1, \dots, N. \tag{18}$$

So we have:

$$\prec \Big(\frac{1}{\psi'(t)}\Big)v''(t) - \Big(\frac{\psi''(t)}{(\psi'(t))^2}\Big)v'(t) + \lambda\psi'(t)\exp\Big(v(t)\Big), S_j(t) \succ = 0,$$
$$j = -N, -N+1, \dots, N. \tag{19}$$

Then

$$\prec \Big(\frac{1}{\psi'(t)}\Big)v''(t), S_j(t) \succ - \prec \Big(\frac{\psi''(t)}{(\psi'(t))^2}\Big)v'(t), S_j(t) \succ$$
$$+ \prec \lambda\psi'(t)\exp\Big(v(t)\Big), S_j(t) \succ = 0, \qquad j = -N, -N+1, \dots, N. \tag{20}$$

By using the inner product (17), and the integration by part formulas, one can find

$$\begin{aligned} \prec \Big(\frac{1}{\psi'(t)}\Big)v''(t), S_j(t) \succ &= \int_{-\infty}^{\infty} v''(t)S_j(t)dt \\ &= B_1 + \int_{-\infty}^{\infty} v(t)S_j''(t)dt, \end{aligned} \tag{21}$$

$$\begin{aligned} \prec \Big(\frac{\psi''(t)}{(\psi'(t))^2}\Big)v'(t), S_j(t) \succ &= \int_{-\infty}^{\infty} \Big(\frac{\psi''(t)}{(\psi'(t))}\Big)v'(t)S_j(t)dt \\ &= B_2 - \int_{-\infty}^{\infty} v(t)\Big(\Big(\frac{\psi''(t)}{(\psi'(t))}\Big)S_j(t)\Big)'dt, \end{aligned} \tag{22}$$

Finally,

$$\prec \psi'(t)\exp\Big(v(t)\Big), S_j(t) \succ = \int_{-\infty}^{\infty}\Big(\psi'(t)\Big)^2 \exp\Big(v(t)\Big)S_j(t)dt, \tag{23}$$

where

$$B_1 = v'(t)S_j(t) - v(t)S_j'(t)\Big|_{-\infty}^{\infty}, \qquad B_2 = v(t)\Big(\frac{\psi''(t)}{(\psi'(t))}\Big)S_j(t)\Big|_{-\infty}^{\infty}.$$

Because of $w(t) = \psi'(t)$, then B_1 and B_2 are vanished.

Theorem 3. *Let $v(t)S_j''(t)$, $v(t)\Big(\Big(\frac{\psi''(t)}{(\psi'(t))}\Big)S_j(t)\Big)'$, $\Big(\psi'(t)\Big)^2\exp\Big(v(t)\Big)S_j(t)$ satisfy the conditions of Theorem 2, then*

$$\begin{aligned}\Big| \prec L(v(t)), S_j(t) \succ -h\sum_{k=-N}^{N}\Big[v(kh)\Big\{S_j''(kh) - \Big(\frac{\psi''}{\psi'}\Big)(kh)S_j'(kh)\\ + \Big(\frac{\psi''}{\psi'}\Big)'(kh)S_j(kh)\Big\} - \lambda\Big(\psi'(kh)\Big)^2\exp\Big(v(kh)\Big)S_j(kh)\Big]\Big|\\ \le C\exp\Big(\frac{-K'N}{\log(K'N/B)}\Big),\end{aligned} \tag{24}$$

where $h = \frac{\log(2\pi d\gamma N/B)}{\gamma N}$ and $K' = 2\pi d\gamma$.

Proof. Using triangular inequality

$$\begin{aligned}&\Big| \prec L(v(t)), S_j(t) \succ -h\sum_{k=-N}^{N}v(kh)\Big\{S_j''(kh) + \Big(\frac{\psi''}{\psi'}\Big)(kh)S_j'(kh)\\ &+ \Big(\frac{\psi''}{\psi'}\Big)'(kh)S_j(kh) - \lambda\Big(\psi'(kh)\Big)^2\exp\Big(v(kh)\Big)S_j(kh)\Big\}\Big|\\ &\le \Big| \prec \Big(\frac{1}{\psi'(t)}\Big)v''(t), S_j(t) \succ -h\sum_{k=-N}^{N}v(kh)S_j''(kh)\Big|\\ &+ \Big| \prec \Big(\frac{\psi''(t)}{(\psi'(t))^2}\Big)v'(t), S_j(t) \succ -h\sum_{k=-N}^{N}v(kh)\Big\{\Big(\frac{\psi''}{\psi'}\Big)(kh)S_j'(kh)\\ &+ \Big(\frac{\psi''}{\psi'}\Big)'(kh)S_j(kh)\Big\}\Big| + \Big| \prec \psi'(t)\exp\Big(v(t)\Big), S_j(t) \succ\\ &- h\sum_{k=-N}^{N}\lambda\Big(\psi'(kh)\Big)^2\exp\Big(v(kh)\Big)S_j(kh)\Big|,\end{aligned} \tag{25}$$

and regarding equations (21), (22) and (23), it is concluded that

$$\leq \Big| \int_{-\infty}^{\infty} v(t) S_j''(t)dt - h \sum_{k=-N}^{N} v(kh) S_j''(kh) \Big| + \Big| \int_{-\infty}^{\infty} v(t) \Big(\Big(\frac{\psi''(t)}{(\psi'(t))} \Big) S_j(t) \Big)' dt$$
$$- h \sum_{k=-N}^{N} v(kh) \Big\{ \Big(\frac{\psi''}{\psi'} \Big)(kh) S_j'(kh) + \Big(\frac{\psi''}{\psi'} \Big)'(kh) S_j(kh) \Big\} \Big|$$
$$+ \Big| \int_{-\infty}^{\infty} \Big(\psi'(t) \Big)^2 \exp\Big(v(t) \Big) S_j(t)dt - h \sum_{k=-N}^{N} \lambda \Big(\psi'(kh) \Big)^2 \exp\Big(v(kh) \Big) S_j(kh) \Big| \tag{26}$$

Using Theorem 1, the following upper bound for Eq. (26) is obtained

$$C_1 \exp\Big(\frac{-k'N}{\log(k'N/B)} \Big) + C_2 \exp\Big(\frac{-k'N}{\log(k'N/B)} \Big) + C_3 \exp\Big(\frac{-k'N}{\log(k'N/B)} \Big)$$
$$\leq C \exp\Big(\frac{-k'N}{\log(k'N/B)} \Big), \tag{27}$$

which completes the proof. □

Deleting the error term $\mathcal{O}\Big(\exp\Big(\frac{-k'N}{\log(k'N/B)} \Big) \Big)$, replacing $v(kh)$ by C_k and dividing by h, together with $\delta_{kj}^{(l)}$ defined in Lemma 1 imply following nonlinear system

$$\sum_{j=-N}^{N} C_j \Big\{ \frac{1}{h^2} \delta_{kj}^{(2)} + \Big(\frac{\psi''}{\psi'} \Big)(jh) \frac{1}{h} \delta_{kj}^{(1)} + \Big(\frac{\psi''}{(\psi')} \Big)'(kh) \delta_{kj}^{(0)} \Big\}$$
$$+ \lambda \Big(\psi' \Big)^2 (kh) \exp\Big(C_k \Big) = 0, \qquad k = -N, -N+1, \ldots, N. \tag{28}$$

Conveniently, by recalling the notations introduced in (5) and (6), it is possible to obtain a matrix-vector form nonlinear system (28) as follows:

$$AC + B\exp(C) = 0, \tag{29}$$

where

$$A = I^{(2)} + I^{(1)} D\Big(\frac{\psi''}{\psi'} \Big) + I^{(0)} D\Big(\Big(\frac{\psi''}{(\psi')} \Big)' \Big),$$
$$B = D\Big(\lambda \psi'^2 \Big),$$
$$C = \Big(C_{-N}, C_{-N+1}, \ldots, C_N \Big)^T,$$
$$\exp(C) = \Big(\exp(C_{-N}), \exp(C_{-N+1}), \ldots, \exp(C_N) \Big)^T.$$

To solve this nonlinear system, Newton's method is applied. In this regard, starting with an initial guess Ω_0, Newton's iteration is used as follows:

$$\Omega_{k+1} = \Omega_k - J^{-1}(\Omega_k)\Big\{F(\Omega_k)\Big\}, \tag{30}$$

where

$$F(\Omega_k) = A\Omega_k + B\exp(\Omega_k), \tag{31}$$

$$J(\Omega_k) = A + BD\Big(\exp(\Omega_k)\Big). \tag{32}$$

By solving this system and obtaining $C = \Big(C_{-N}, C_{-N+1}, \ldots, C_N\Big)^T$, a numerical solution of problem (1) is calculated by (16).

3.3 The Sinc-collocation method

In this section, v_m is calculated as the numerical solution of problem (1) by the Sinc-collocation method. A collocation scheme is defined by substituting $v_m(t) = \sum_{j=-N}^{N} C_j S_j(t)$ into (1) and calculating the result at $x_k = kh$, $k = -N, \ldots, N$, so

$$v'_m(t) = \sum_{j=-N}^{N} C_j \frac{d}{dt} S_j(t), \tag{33}$$

$$v''_m(t) = \sum_{j=-N}^{N} C_j \frac{d^2}{dt^2} S_j(t). \tag{34}$$

Therefore based on (15)

$$\frac{1}{\psi'(t)}\Big(\sum_{j=-N}^{N} C_j \frac{d^2}{dt^2} S_j(t)\Big) - \frac{\psi''}{\psi'^2}(t)\Big(\sum_{j=-N}^{N} C_j \frac{d}{dt} S_j(t)\Big)$$
$$+\lambda\psi'(t)\exp\Big(\sum_{j=-N}^{N} C_j S_j(t)\Big) = 0, \quad t = -Nh, -(N+1)h, \ldots, Nh, \tag{35}$$

and multiplying (35) by ψ', and some calculations, the nonlinear system (35) is rewritten as follows:

$$\sum_{j=-N}^{N} C_j \Big\{\frac{d^2}{dt^2} S_j(t) - \frac{\psi''}{\psi'^2}(t)\frac{d}{dt} S_j(t)\Big\} + \lambda\Big(\psi'(t)\Big)^2 \exp\Big(\sum_{j=-N}^{N} C_j S_j(t)\Big) = 0,$$
$$t = -Nh, -(N+1)h, \ldots, Nh. \tag{36}$$

By using lemma (1) and knowing that $\delta_{jk}^{(0)} = \delta_{kj}^{(0)}$, $\delta_{jk}^{(1)} = -\delta_{kj}^{(1)}$, and $\delta_{jk}^{(2)} = \delta_{kj}^{(2)}$, the above system can be rewritten as follows:

$$\sum_{j=-N}^{N} C_j \Big\{ \frac{1}{h^2}\delta_{jk}^{(2)} + \frac{1}{h}\delta_{jk}^{(1)} \frac{\psi''}{\psi'^2}(kh) \Big\} + \lambda \Big(\psi'(kh)\Big)^2 \exp\Big(C_k\Big) = 0,$$
$$k = -N, -N+1, \ldots, N. \tag{37}$$

By applying the diagonal matrix, which was introduced in (6), and the notations of $I^{(l)}$ in lemma (1), we have:

$$AC + B\exp(C) = 0, \tag{38}$$

where

$$\begin{aligned} A &= I^{(2)} + D\Big(\frac{\psi''}{\psi'}\Big) I^{(1)}, \\ B &= D\Big(\lambda \psi'^2\Big), \\ C &= \Big(C_{-N}, C_{-N+1}, \ldots, C_N\Big)^T, \\ \exp(C) &= \Big(\exp(C_{-N}), \exp(C_{-N+1}), \ldots, \exp(C_N)\Big)^T. \end{aligned}$$

This nonlinear system can be solved by Newton's method, which was discussed in Subsection 3.2

4 Numerical Results

In order to demonstrate the accuracy and efficiency of the numerical solution obtained by the DE Sinc method, Bratu's problem is solved by a variety of parameters reported in the literature. Furthermore, the solution is compared with the exact solution at certain points, and also with the best numerical results reported by existing methods in the literature.

For several parameters, the maximum absolute error in the solution over 999 equally spaced gride points Λ is calculated in which

$$\begin{aligned} \Lambda &= \{x_1, \ldots, x_{999}\}, \\ x_k &= \frac{k}{1000}, \quad k = 1, 2, \ldots, 999, \end{aligned}$$

Table 1: Absolute error in the solution for $\lambda = 1$.

x	DE Sinc method Galerkin	DE Sinc method collocation	B-Spline [8]	Laplace [15]	LSHM [1]	OSM [3]	PSM [29]	SESG(N=60) [30]	IFDM [5]
0.1	1.52 E-16	2.08 E-16	2.97 E-06	1.97 E-06	7.50 E-07	4.63 E-08	5.85 E-10	2.01 E-10	5.60 E-11
0.2	0.0000	1.52 E-16	5.46 E-06	3.93 E-06	1.01 E-06	1.02 E-07	2.58 E-10	1.69 E-10	3.62 E-11
0.3	2.77 E-17	3.88 E-16	7.33 E-06	5.85 E-06	5.85 E-06	1.44 E-07	2.59 E-11	1.82 E-11	2.80 E-11
0.4	1.38 E-16	3.88 E-16	8.49 E-06	7.70 E-06	7.70 E-06	1.71 E-07	8.77 E-11	1.14 E-11	1.00 E-12
0.5	1.94 E-16	4.16 E-16	8.89 E-06	9.46 E-06	9.46 E-06	1.81 E-07	1.38 E-10	1.18 E-11	1.30 E-11
0.6	1.66 E-16	5.27 E-16	8.49 E-06	1.11 E-05	1.11 E-05	1.71 E-07	8.77 E-11	1.14 E-11	3.50 E-11
0.7	1.24 E-16	5.27 E-16	7.33 E-06	1.25 E-05	1.25 E-05	1.44 E-07	5.59 E-11	1.82 E-11	1.20 E-11
0.8	2.22 E-16	2.91 E-16	5.46 E-06	1.34 E-05	1.34 E-05	1.02 E-07	2.58 E-10	1.69 E-10	2.19 E-11
0.9	3.46 E-17	2.91 E-16	2.97 E-06	1.19 E-05	1.19 E-05	4.63 E-08	5.87 E-10	2.01 E-10	6.70 E-12
MaxError	3.46 E-17	5.27 E-16	8.89 E-06	1.34 E-05	1.34 E-05	1.81 E-07	5.87 E-10	2.01 E-10	6.70 E-11

Table 2: Maximum absolute error and CPU time over Λ for $\lambda = 1$.

N	DE Sinc-collocation MAE	DE Sinc-collocation CPU-time	DE Sinc-collocation CondJ	DE Sinc-Galerkin MAE	DE Sinc-Galerkin CPU-time	DE Sinc-Galerkin CondJ	SE-SC [22]	SE-SG [23]
10	$1.8647E-07$	$1.1486s$	67.37	$3.0073E-07$	$1.1671s$	91.57	$3.02E-04$	———
16	$8.9636E-10$	$1.1870s$	124.2	$4.4568E-10$	$1.1767s$	166.57	———	$5.4E-05$
25	$1.0605E-12$	$1.2544s$	239.9	$1.3833E-13$	$1.2337s$	311.99	$6.46E-06$	———
32	$8.2434E-15$	$1.2601s$	353.9	$2.6784E-10$	$1.2477s$	450.34	———	$1.6E-06$
50	$4.7878E-15$	$1.6496s$	743.4	$5.8287E-16$	$1.2950s$	900.37	$7.30E-08$	———
64	$5.3291E-15$	$1.3784s$	1139	$4.7184E-15$	$1.3769s$	1339	———	$9.5E-09$

and the maximum absolute error (MAE) in the equally spaced points defined by:

$$MAESG = \max_{1\leq k\leq 999} |u_{exat-solution}(x_k) - u_{m,Sinc-Galerkin}(x_k)|,$$
$$MAESC = \max_{1\leq k\leq 999} |u_{exat-solution}(x_k) - u_{m,Sinc-collocation}(x_k)|.$$

All the computations developed on (PC) by MATLAB and the initial estimate for Newton's iteration are chosen the zero vector. Also to show the CPU execution time, the "tic" and "toc" functions are used, and results are reported in the tables.

Table 1 presents the maximum absolute error (MAE) over Λ for our method with $N = 50$, B-spline method [8], Laplace method [15], decomposition method [16], Lie group shooting method [1], optimal spline method [3], parametric spline method [29], SE Sinc-Galerkin method for $N = 60$ [30], and iterative finite difference method (IFDM) [5], for $\lambda = 1$. In the last row of the table the maximum error calculated at points 0.1, 0.2, ..., 0.9 is listed. Clearly, the DE Sinc method is more accurate than of other methods reported in the literature. Table 2 shows MAE for the Sinc-Galerkin [23]

Table 3: Absolute error in the solution for $\lambda = 2$.

	DE Sinc method								
x	Galerkin	collocation	B-Spline [8]	Laplace [15]	DCM [16]	LSHM [1]	PSM [29]	SESG(N=60) [30]	IFDM [5]
0.1	4.99 E-16	6.93 E-17	1.72 E-05	2.13 E-03	2.68 E-03	1.52 E-02	1.25 E-08	1.48 E-09	1.42 E-10
0.2	3.88 E-16	8.60 E-16	3.23 E-05	4.12 E-03	2.02 E-03	1.46 E-02	1.95 E-08	1.34 E-09	1.01 E-10
0.3	3.33 E-16	1.22 E-15	4.49 E-05	6.19 E-03	1.52 E-04	5.58 E-03	2.73 E-08	1.68 E-10	1.11 E-10
0.4	6.10 E-16	1.27 E-15	5.28 E-05	8.00 E-03	2.20 E-03	3.24 E-03	3.31 E-08	2.35 E-10	1.60 E-10
0.5	7.21 E-16	1.44 E-15	5.56 E-05	9.60 E-03	3.01 E-03	6.98 E-03	3.53 E-08	7.39 E-11	1.27 E-10
0.6	6.66 E-16	1.49 E-15	5.28 E-05	1.09 E-03	2.20 E-03	3.24 E-03	3.31 E-08	2.35 E-10	1.39 E-10
0.7	4.44 E-16	1.22 E-15	4.49 E-05	1.19 E-03	1.52 E-04	5.88 E-03	2.73 E-08	1.68 E-10	1.46 E-10
0.8	4.44 E-16	7.77 E-16	3.28 E-05	1.24 E-02	2.02 E-03	1.46 E-02	1.95 E-08	1.34 E-09	1.65 E-10
0.9	6.93 E-16	0	1.72 E-05	1.09 E-02	2.68 E-03	1.52 E-02	1.25 E-08	1.48 E-09	9.10 E-11
MaxError	7.21 E-16	1.49 E-15	5.56 E-05	1.09 E-02	3.01 E-03	1.52 E-02	3.53 E-08	1.48 E-09	1.65 E-10

Table 4: Maximum absolute error and CPU time over Λ for $\lambda = 2$.

	DE Sinc-collocation			DE Sinc-Galerkin				
N	MAE	CPU-time	CondJ	MAE	CPU-time	CondJ	SE-SC [22]	SE-SG [23]
10	$5.5338E-07$	1.1264	75	$1.0902E-06$	1.1637	101.92	$8.60E-04$	———
16	$1.6036E-08$	1.1436	138.12	$6.0332E-09$	1.1774	185.31	———	$1.5E-04$
25	$4.2510E-11$	1.1719	266.8	$1.3783E-11$	1.2250	347.01	$1.84E-05$	——-
32	$5.4937E-13$	1.2333	393.65	$1.6445E-13$	1.2631	500.85	———	$4.04E-06$
50	$1.7764E-15$	1.2703	826.74	$1.4433E-15$	1.3124	1001.30	$2.07E-07$	———
64	$1.5987E-15$	1.3532	1267	$1.6320E-14$	1.4132	1489	——-	$2.7E-08$

Table 5: Absolute error in the solution for $\lambda = 3.513830719$.

	DE Sinc method				
x	Sinc-Galerkin	Sinc-collocation	B-Spline [8]	LSHM [1]	IFDM [5]
0.1	$3.3683E-06$	$5.2052E-06$	$3.8417E-02$	$4.45E-05$	$4.74E-08$
0.2	$6.5536E-06$	$1.0127E-05$	$7.4813E-02$	$7.12E-05$	$8.97E-08$
0.3	$9.2633E-06$	$1.4315E-05$	$1.0582E-01$	$7.30E-05$	$1.25E-07$
0.4	$1.1121E-05$	$1.7186E-05$	$1.2711E-01$	$4.46E-05$	$1.49E-07$
0.5	$1.1787E-05$	$1.8215E-05$	$1.3475E-01$	$6.75E-07$	$1.57E-07$
0.6	$1.1121E-05$	$1.7186E-05$	$1.2711E-01$	$4.56E-05$	$1.48E-07$
0.7	$9.2633E-05$	$1.4315E-05$	$1.0582E-01$	$7.20E-05$	$1.23E-07$
0.8	$6.5536E-06$	$1.0112E-05$	$7.4813E-02$	$7.05E-05$	$8.78E-08$
0.9	$3.3683E-06$	$5.2052E-06$	$3.8417E-02$	$4.41E-05$	$4.52E-08$
MaxError	$9.2633E-05$	$1.8215E-05$	$1.3475E-01$	$7.30E-05$	$1.57E-07$

and Sinc-collocation [22] methods based on single exponential (SE) transformation, CPU time and condition number of Jacobian matrix introduced in Eq. (32) for the Sinc-collocation method and the Sinc-Galerkin method based on double exponential (DE) transformation, over Λ, for $\lambda = 1$ and different values of N. Based on the results, the two methods are much less time-consuming in terms of CUP time.

Table 3 enumerates MAE over Λ for our method with $N = 50$, B-spline method [8], Laplace method [15], decomposition method [16], Lie group shooting method (LSHM) [1], parametric spline method [29], SE Sinc-Galerkin method for $N = 60$ [30], and IDFM [5], for $\lambda = 2$. The last row of the table is devoted to the maximum error calculated at points $0.1, 0.2, \ldots, 0.9$. Table 4 presents MAE for the Sinc-Galerkin [23] and Sinc-collocation [22] methods based on single exponential (SE) transformation, CPU time and condition number of Jacobian matrix introduced in Eq. (32) for the Sinc-collocation method and the Sinc-Galerkin method based on double exponential (DE) transformation over Λ for $\lambda = 2$ and different values of N.

Table 5 includes MAE over Λ for our method with $N = 50$, B-spline method [8], LSHM [1], and IFDM [5], for $\lambda = 3.513830719$. In this special case IFDM outperforms the other methods. Tables 6 and 7 present MAE for the Sinc-collocation method [22] based on (SE) transformation, CPU time and condition number of Jacobian matrix introduced in Eq. (32) for the Sinc-collocation method and the Sinc-Galerkin method based on (DE) transformation over Λ for different values of N for $\lambda = 3.513830719$ and $\lambda = -\pi^2$ respectively.

Table 8 shows the MAE over Λ obtained by our method for different values of N and λ.

In Table 9, the absolute error in the points obtained by the DE Sinc methods and IFDM [5] for $\lambda = 0.1, 0.01$ tabulated. As the results show, the Sinc methods have the so smaller error.

Figures 1 and 2 depict the approximation solution obtained by the DE Sinc-Galerkin and the DE Sinc-collocation methods for different values of λ.

5 Conclusions

This study developed efficient Sinc-Galerkin and Sinc-collocation methods based on a double exponential transformation by a new strategy to solve one-dimensional Bratu-type problem. Bratu's problem was discretized to a nonlinear system of algebraic equations by using the Sinc method. Also, the upper bound of the error for Sinc-Galerkin was found. The nonlinear system was rearranged in matrix-vector forms and solved by Newton's method. The results obtained from the DE Sinc-Galerkin and the DE Sinc-collocation methods were compared with each other and with the best

Table 6: Maximum absolute error and CPU time over Λ for $\lambda = 3.513830719$.

	DE Sinc-collocation			DE Sinc-Galerkin			
N	MAE	CPU-time	CondJ	MAE	CPU-time	CondJ	SE-SC [22]
10	$9.9276E-03$	$1.1475s$	112.3	$1.1049E-02$	$1.1620s$	144.37	$3.42E-02$
25	$1.0915E-04$	$1.1885s$	356.93	$2.8514E-05$	$1.2136s$	451.53	$9.83E-04$
50	$9.9392E-06$	$1.2795s$	1015.4	$9.9490E-06$	$1.3169s$	1241.50	$1.12E-05$

Table 7: Maximum absolute error and CPU time over Λ for $\lambda = -\pi^2$.

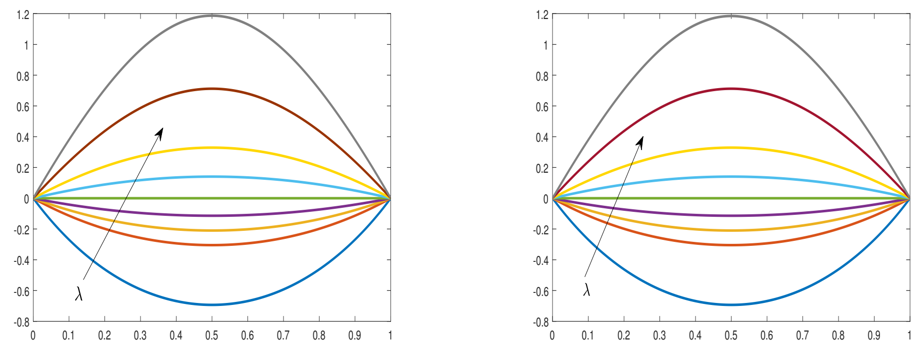
	DE Sinc-collocation			DE Sinc-Galerkin				
N	MAE	CPU-time	CondJ	MAE	CPU-time	CondJ	SE-SC [22]	SE-SG [23]
10	$1.1239E-06$	$1.1536s$	36.31	$1.3599E-06$	$1.1497s$	49.49	$1.52E-03$	———
16	$2.9987E-09$	$1.1578s$	68.00	$1.2821E-09$	$1.1929s$	90.86	———	$4.3E-04$
25	$2.6386E-12$	$1.2072s$	131.51	$7.1355E-13$	$1.2316s$	170.88	$3.21E-05$	———
32	$2.1719E-14$	$1.2458s$	194.14	$5.9536E-15$	$1.2641s$	246.96	———	$1.6E-05$
50	$1.3501E-15$	$1.2839s$	408.19	$1.4594E-15$	$1.3186s$	494.37	$3.61E-07$	———
64	$1.6100E-15$	$1.3412s$	626.01	$1.6320E-15$	$1.3708s$	735.73	———	$6.0E-08$

Table 8: Absolute error in the points for different values of λ.

	DE Sinc-Galerkin				DE Sinc-collocation			
N	$\lambda = -\pi^2$	$\lambda = 1$	$\lambda = 2$	$\lambda = 3.513830719$	$\lambda = -\pi^2$	$\lambda = 1$	$\lambda = 2$	$\lambda = 3.513830719$
2	0.2643	0.0556	0.1442	1.0761	0.2624	$5.50E-02$	$1.42E-01$	1.0758
5	1.70 E-03	$3.24E-04$	$9.21E-04$	$1.88E-01$	$1.61E-03$	$3.19E-04$	$9.18E-04$	$1.87E-01$
10	1.35 E-06	$3.00E-07$	$1.09E-06$	$1.10E-02$	$1.12E-06$	$1.86E-07$	$5.53E-07$	$9.92E-03$
15	3.43 E-09	$7.63E-10$	$1.13E-08$	$6.54E-04$	$3.13E-09$	$2.42E-09$	$3.28E-08$	$1.43E-03$
20	2.61 E-11	$1.58E-11$	$3.74E-10$	$1.01E-04$	$2.96E-11$	$4.34E-11$	$1.08E-09$	$7.49E-05$
30	2.99 E-14	$9.41E-15$	$5.80E-13$	$1.17E-05$	$8.38E-14$	$2.97E-14$	$1.85E-12$	$1.82E-05$
40	1.89 E-15	$4.16E-16$	$2.74E-15$	$9.96E-06$	$2.45E-15$	$5.27E-16$	$5.41E-16$	$9.21E-06$

Table 9: Absolute error in the points for different values of λ.

	$\lambda = 0.1$			$\lambda = 0.01$		
x	DESC	DESG	IFDM [5]	DESC	DESG	IFDM [5]
0.1	$1.38E-16$	$1.20E-16$	$7.63E-12$	$2.20E-16$	$2.22E-16$	$4.73E-13$
0.2	$2.70E-16$	$2.51E-16$	$7.70E-12$	$1.50E-17$	$1.23E-17$	$1.31E-12$
0.3	$1.16E-16$	$1.43E-16$	$8.70E-12$	$1.70E-16$	$1.73E-16$	$3.38E-12$
0.4	$1.82E-16$	$1.68E-16$	$7.60E-12$	$1.02E-16$	$1.05E-16$	$4.95E-12$
0.5	$1.38E-17$	$2.77E-17$	$3.50E-12$	$2.78E-16$	$2.81E-16$	$5.22E-12$
0.6	$2.06E-16$	$1.85E-16$	$1.00E-12$	$1.02E-16$	$1.04E-16$	$6.33E-12$
0.7	$9.02E-17$	$1.11E-16$	$3.30E-12$	$1.71E-16$	$1.71E-16$	$4.63E-12$
0.8	$2.34E-16$	$2.68E-16$	$7.74E-12$	$1.39E-17$	$1.36E-17$	$2.85E-12$
0.9	$2.44E-16$	$1.31E-16$	$8.02E-12$	$2.20E-16$	$2.22E-16$	$7.13E-13$



(a) (b)

Figure 1: Solution of Bratu's problem with $\lambda = -\pi^2, -\pi, -2, -1, 0, 1, 2, \pi, 3.513830719$ for (a) the DE Sinc-Galerkin method, (b) the DE Sinc-collocation method and (c) Exact solution.

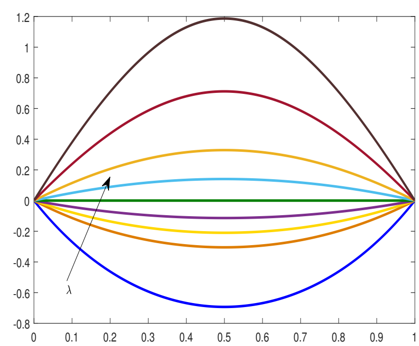


Figure 2: Exact solution of Bratu's problem with $\lambda = -\pi^2, -\pi, -2, -1, 0, 1, 2, \pi, 3.513830719$.

numerical results of the other existing methods reported in the literature. The absolute error and CPU execution time, as shown in tables, clearly indicate that the DE Sinc methods are much more accurate than existing methods in almost all of the parameters. The results show that both the DE Sinc-Galerkin and the DE Sinc-collocation methods are equally accurate. The DE Sinc-collocation method is, however, easier to use than the DE Sinc-Galerkin method for solving such problems.

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