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Inner and outer estimations of the generalized solution sets and an application in economic

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Abstract. Generalized intervals (intervals whose bounds are not constrained to be increasingly ordered) extend classical intervals and present algebraic completion of conventional interval arithmetic, allowing efficient solution for interval linear systems. In this paper, we use the Cholesky decomposition of a symmetric generalized interval matrix $\mathbf{A}$ introduced by Zhao et al. (A generalized Cholesky decomposition for interval matrix, Adv. Mat. Res. 479 (2012) 825–828), to construct the algebraic solution of the triangular interval linear system of equations. Also we utilize this decomposition to find inner and outer estimations of the generalized solution set of the symmetric interval linear systems. Finally some numerical experiments and an application in economic are given to show the efficiency of the presented technique.

Keywords: Interval arithmetic, Kaucher arithmetic, Cholesky decomposition.
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1 Introduction

In our paper, the main object under study is the interval linear system

$$\mathbf{A}x = \mathbf{b}, \tag{1}$$

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with symmetric $n \times n$ interval matrix $\mathbf{A}$ and n-dimensional interval vector $\mathbf{b}$. The interval system of equations (1) frequently appears in the cases when the components of the input data are accompanied by error.

The most important solution sets of the interval system (1) which have been the subject of more active research in modern interval analysis are united solution set, tolerable solution set and controllable solution set which we introduce them later. Obtaining united, tolerable and controllable solution sets is an NP-hard problem [34].

Here, we utilize a generalized interval arithmetic, namely Kaucher arithmetic due to its interesting algebraic properties to simply obtain good estimations to the solution set of the interval system (1). Generalized intervals are intervals whose bounds are not constrained to be increasingly ordered. The set of generalized intervals is a group for addition and for multiplication of zero free generalized intervals. For an introduction of the generalized interval arithmetic (also called Kaucher arithmetic), we refer the interested reader to [18, 19, 25]. In order to emphasize that a generalized interval can be considered as a pair of a proper interval and a direction, sometimes Kaucher arithmetic is called "directed interval arithmetic" [22]. A variant of Kaucher arithmetic adopted to semantic problems has been proposed and developed by Gardeñes et al., namely "modal interval arithmetic" [1, 14]. Another generalizations of the interval arithmetic and their applications can be seen in [2, 16, 36, 37].

The algebraic properties of Kaucher arithmetic make it a suitable environment for solving interval algebraic problems [27]. Zhao et al. [38] introduced the Cholesky decomposition of a generalized interval matrix $\mathbf{A}$ such that $\mathbf{A} = \mathbf{L}\mathbf{L}^T$ instead of the weaker relation $\mathbf{A} \subseteq \mathbf{L}\mathbf{L}^T$ in classical interval computations. Using this decomposition, we obtain the algebraic solution of the triangular interval linear system $\mathbf{A}x = \mathbf{b}$ in such a way that it solves the equation exactly, i.e., the obtained solution $\mathbf{x}$ exactly satisfies $\mathbf{A}\mathbf{x} = \mathbf{b}$ instead of containing the solution set or including in the solution set that occur commonly in classical interval computations. We also introduce the concept of generalized solution set to the interval system of equations (1). Then using the proposed technique for obtaining the algebraic solution of the triangular interval systems, we propose some approaches to obtain inner and outer estimations of the interval generalized solution sets. The new approaches help us to estimate the important united, tolerable and controllable solution sets with good quality in the context of classical interval computations.

Some approaches for solving interval linear systems can be seen in [5–9, 17, 20, 21, 23, 26, 30]. In [32] one can see some applications of Kaucher interval arithmetic in identification, tolerance and control problems for the interval linear equations. Another application of generalized interval arithmetic for solutions of fuzzy equations can be seen in [31].

The rest of the paper is organized as follows. In Section 2 an overview of the generalized intervals is given. Section 3 presents the algebraic solution of the triangular interval linear systems. Some approached for inner and outer estimations of the generalized solution sets are given in Section 4. In Section 5, we present an application of our approach in input-output models of economic. Finally we complete the paper with some concluding remarks in Section 6.

2 Generalized intervals

The set of proper intervals is denoted by $\mathbb{IR} := \{\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}] : \underline{\mathbf{x}} \leq \overline{\mathbf{x}},\ \underline{\mathbf{x}}, \overline{\mathbf{x}} \in \mathbb{R}\}$. This set is extended by the set of improper intervals with bounds ordered decreasingly which is denoted by $\overline{\mathbb{IR}} := \{\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}] : \underline{\mathbf{x}} \geq \overline{\mathbf{x}},\ \underline{\mathbf{x}}, \overline{\mathbf{x}} \in \mathbb{R}\}$. Totally these two sets construct the set of generalized intervals which is denoted by $\mathbb{KR} := \{\mathbf{a} = [\underline{\mathbf{a}}, \overline{\mathbf{a}}] : \underline{\mathbf{a}}, \overline{\mathbf{a}} \in \mathbb{R}\}$. For example $[-1, 1]$ and $[1, -1]$ are generalized intervals. The "dual" is an important monadic operator that reverses the endpoints of the intervals. For $\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}]$, its dual is defined by

$$\text{dual}(\mathbf{x}) := [\overline{\mathbf{x}}, \underline{\mathbf{x}}].$$

Also, for theoretical aspects, we introduce proper projection of a generalized interval by $\text{pro}([\underline{\mathbf{x}}, \overline{\mathbf{x}}]) := [\min\{\underline{\mathbf{x}}, \overline{\mathbf{x}}\}, \max\{\underline{\mathbf{x}}, \overline{\mathbf{x}}\}]$. The generalized intervals are partially ordered by the inclusion order relation $[\underline{\mathbf{x}}, \overline{\mathbf{x}}] \subseteq [\underline{\mathbf{y}}, \overline{\mathbf{y}}] \Leftrightarrow (\underline{\mathbf{y}} \leq \underline{\mathbf{x}})$ and $(\overline{\mathbf{x}} \leq \overline{\mathbf{y}})$, which extends the inclusion relation of the classical intervals. More details about Kaucher interval arithmetic can be found in [19, 34].

Generalized interval arithmetic has better algebraic properties than the classical interval arithmetic. For example, the addition in $\mathbb{KR}$ is a group and the opposite of an interval $\mathbf{x}$ is $-\text{dual}(\mathbf{x})$, i.e.,

$$\mathbf{x} + (-\text{dual}(\mathbf{x})) = \mathbf{x} - \text{dual}(\mathbf{x}) = [0, 0].$$

On the other hand, the multiplication in $\mathbb{KR}$ restricted to zero free intervals, is also a group and the inverse of such interval $\mathbf{x}$ is $\frac{1}{\text{dual}(\mathbf{x})}$, i.e.,

$$\mathbf{x} \cdot \frac{1}{\text{dual}(\mathbf{x})} = \frac{\mathbf{x}}{\text{dual}(\mathbf{x})} = [1, 1].$$

In Kaucher arithmetic, the monotonicity with respect to inclusion is maintained, i.e., if $* \in \{+, -, \cdot, /\}$ and $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{KR}$ then we have

$$\mathbf{a} \subseteq \mathbf{b} \quad \text{and} \quad \mathbf{c} \subseteq \mathbf{d} \Rightarrow \mathbf{a} * \mathbf{c} \subseteq \mathbf{b} * \mathbf{d}.$$

Note that if $\mathbf{x}, \mathbf{y} \in \mathbb{KR}$, then $\text{dual}(\mathbf{xy}) = \text{dual}(\mathbf{x})\text{dual}(\mathbf{y})$. The set of m-by-n Kaucher interval matrices is denoted by $\mathbb{KR}^{m\times n}$. Similar to the scalar case, for two Kaucher interval matrices $\mathbf{A} \in \mathbb{KR}^{m\times n}$ and $\mathbf{B} \in \mathbb{KR}^{n\times p}$, we have $\text{dual}(\mathbf{AB}) = \text{dual}(\mathbf{A})\text{dual}(\mathbf{B})$. For interval number $\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}] \in \mathbb{KR}$, define

$$\begin{aligned} \Omega &:= \{\mathbf{x} \in \mathbb{KR} : \underline{\mathbf{x}} > 0 \wedge \overline{\mathbf{x}} > 0\}, & \overline{\Omega} &:= \{\mathbf{x} \in \mathbb{KR} : \underline{\mathbf{x}} \geq 0 \ \wedge \ \overline{\mathbf{x}} \geq 0\}, \\ -\Omega &:= \{-\mathbf{x} : \mathbf{x} \in \Omega\}, & -\overline{\Omega} &:= \{-\mathbf{x} : \mathbf{x} \in \overline{\Omega}\}. \end{aligned}$$

By the above notations for $\mathbf{x} \in \mathbb{KR}$, we say $\mathbf{x} > 0$ ($x \geq 0$), if $\mathbf{x} \in \Omega$ ($\mathbf{x} \in \overline{\Omega}$), and $\mathbf{x} < 0$ ($\mathbf{x} \leq 0$) if $\mathbf{x} \in -\Omega$ ($-\overline{\Omega}$).

Notation 1. *If $\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}] \in \mathbb{KR}$ is positive, we define $\sqrt{\mathbf{x}}$ as*

$$\sqrt{\mathbf{x}} := [\sqrt{\underline{\mathbf{x}}}, \sqrt{\overline{\mathbf{x}}}].$$

It is obvious that by this notation, we have $\sqrt{\mathbf{x}}\sqrt{\mathbf{x}} = \mathbf{x}$.

3 The Cholesky decomposition

Suppose $\mathbf{A} = (\mathbf{A}_{ij}) \in \mathbb{KR}^{n\times n}$ is symmetric, Zhao et al. [38] proposed the following decomposition, namely Cholesky decomposition, for $\mathbf{A}$

$$\mathbf{L}_{ii} := \sqrt{\mathbf{A}_{ii} - \sum_{k=1}^{i-1} \text{dual}(\mathbf{L}_{ik}\mathbf{L}_{ik})}, \qquad i = 1, ..., n, \tag{2}$$

$$\mathbf{L}_{ij} := \frac{\mathbf{A}_{ij} - \sum_{k=1}^{j-1} \text{dual}(\mathbf{L}_{ik}\mathbf{L}_{jk})}{\text{dual}(\mathbf{L}_{jj})}, \qquad i > j, \tag{3}$$

where $\mathbf{L}$ is a lower triangular interval matrix such that $\mathbf{A} = \mathbf{L}\mathbf{L}^T$.

Proposition 1. *[38] Let $\mathbf{A} \in \mathbb{KR}^{n\times n}$ be a symmetric interval matrix. Provided that the interval matrix $\mathbf{L}$ defined by Eqs. (2) and (3) can be constructed, it satisfies $\mathbf{A} = \mathbf{L}\mathbf{L}^T$.*

Example 1. Consider the symmetric interval matrix

$$\mathbf{A} = \begin{pmatrix} [9, 16] & [3, 8] & [-4, 4] \\ [3, 8] & [10, 20] & [2, 7] \\ [-4, 4] & [2, 7] & [1, 3] \end{pmatrix}.$$

For computing $\mathbf{LL}^T$ decomposition, using (2) and (3) we can write

$\mathbf{L}_{11} = \sqrt{\mathbf{A}_{11}} = [3, 4]$,
$\mathbf{L}_{21} = \frac{\mathbf{A}_{21}}{\mathrm{dual}(\mathbf{L}_{11})} = [1, 2]$,
$\mathbf{L}_{31} = \frac{\mathbf{A}_{31}}{\mathrm{dual}(\mathbf{L}_{11})} = [-1, 1]$,
$\mathbf{L}_{22} = \sqrt{\mathbf{A}_{22} - \mathrm{dual}(\mathbf{L}_{21}\mathbf{L}_{21})} = [3, 4]$,
$\mathbf{L}_{32} = \frac{\mathbf{A}_{32} - \mathrm{dual}(\mathbf{L}_{31}\mathbf{L}_{21})}{\mathrm{dual}(\mathbf{L}_{22})} = [\frac{4}{3}, \frac{5}{4}]$,
$\mathbf{L}_{33} = \sqrt{\mathbf{A}_{33} - \mathrm{dual}(\mathbf{L}_{31}\mathbf{L}_{31}) - \mathrm{dual}(\mathbf{L}_{32}\mathbf{L}_{32})} = [\frac{\sqrt{2}}{3}, \frac{\sqrt{7}}{4}]$.

Therefore, we obtain the following lower triangular interval matrix $\mathbf{L}$

$$\mathbf{L} = \begin{pmatrix} [3,4] & 0 & 0 \\ [1,2] & [3,4] & 0 \\ [-1,1] & [\frac{4}{3}, \frac{5}{4}] & [\frac{\sqrt{2}}{3}, \frac{\sqrt{7}}{4}] \end{pmatrix},$$

which satisfies $\mathbf{A} = \mathbf{LL}^T$.

3.1 Algebraic solution of a triangular interval linear system

Definition 1. *An interval vector* $\mathbf{x} \in \mathbb{KR}^n$ *is called an algebraic solution of the interval linear system* $\mathbf{A}x = \mathbf{b}$, *with* $\mathbf{A} \in \mathbb{KR}^{m \times n}$ *and* $\mathbf{b} \in \mathbb{KR}^m$, *if it satisfies* $\mathbf{Ax} = \mathbf{b}$.

As previously mentioned, in literature for an interval linear system $\mathbf{A}x = \mathbf{b}$ almost an interval vector $\mathbf{x}$ is computed which contains its solution set or includes in the solution set and does not exactly satisfy in the equation, i.e., almost we have $\mathbf{Ax} \neq \mathbf{b}$. Here, we want to utilize the interesting properties of the Kaucher interval arithmetic to construct the exact solution of the triangular interval linear systems. A triangular interval linear system is an interval linear system with triangular coefficient matrix. We want to find the algebraic solution of the lower triangular interval linear system $\mathbf{A}x = \mathbf{b}$ in which

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & 0 & \cdots & 0 \\ \vdots & \mathbf{A}_{22} & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ \mathbf{A}_{n1} & \cdots & \cdots & \mathbf{A}_{nn} \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_n \end{pmatrix}, \tag{4}$$

and $0 \notin \mathbf{A}_{ii}$. By forward substitution technique (similar to real arithmetic) and using opposite and inverse of a generalized interval, we propose the following answer

$$x_i := \frac{\mathbf{b}_i - \sum_{j=1}^{i-1} \mathrm{dual}(\mathbf{A}_{ij} x_j)}{\mathrm{dual}(\mathbf{A}_{ii})}, \qquad i = 1, ..., n. \tag{5}$$

For a triangular interval linear system with upper triangular coefficient matrix, the backward substitution technique can be used.

Lemma 1. *Let* $\mathbf{A}x = \mathbf{b}$ *be a triangular interval linear system with lower triangular coefficient matrix* $\mathbf{A} \in \mathbb{KR}^{n\times n}$ *and right-hand side* $\mathbf{b} \in \mathbb{KR}^n$ *denoted by (4), then* $\mathbf{x} = (\mathbf{x}_1, ..., \mathbf{x}_n)^T \in \mathbb{KR}^n$ *constructed by Eq. (5) satisfies* $\mathbf{Ax} = \mathbf{b}$*, i.e.,* $\mathbf{x}$ *is an algebraic solution to* $\mathbf{A}x = \mathbf{b}$*.*

Proof. It is sufficient to prove $(\mathbf{Ax})_i = \mathbf{b}_i$. Using Eq. (5), we have

$$\begin{aligned}
(\mathbf{Ax})_i &= \sum_{k=1}^{i} \mathbf{A}_{ik}\mathbf{x}_k = \sum_{k=1}^{i} \mathbf{A}_{ik} \frac{\mathbf{b}_k - \sum_{j=1}^{k-1} \mathrm{dual}(\mathbf{A}_{kj}\mathbf{x}_j)}{\mathrm{dual}(\mathbf{A}_{kk})} \\
&= \sum_{k=1}^{i-1} \mathbf{A}_{ik} \frac{\mathbf{b}_k - \sum_{j=1}^{k-1} \mathrm{dual}(\mathbf{A}_{kj}\mathbf{x}_j)}{\mathrm{dual}(\mathbf{A}_{kk})} + (\mathbf{b}_i - \sum_{j=1}^{i-1} \mathrm{dual}(\mathbf{A}_{ij}\mathbf{x}_j)) \\
&= \sum_{j=1}^{i-1} \mathbf{A}_{ij}\mathbf{x}_j + (\mathbf{b}_i - \mathrm{dual}(\sum_{j=1}^{i-1} \mathbf{A}_{ij}\mathbf{x}_j)) = \mathbf{b}_i.
\end{aligned}$$

□

Example 2. Consider the triangular interval linear system $\mathbf{A}x = \mathbf{b}$, with

$$\mathbf{A} = \begin{pmatrix} [1,2] & 0 & 0 \\ [2,5] & [-3,-1] & 0 \\ [0,1] & [1,3] & [2,4] \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} [1,2] \\ [2,3] \\ [3,4] \end{pmatrix}.$$

Using Eq. (5) we can write

$\mathbf{x}_1 = \frac{\mathbf{b}_1}{\mathrm{dual}(\mathbf{A}_{11})} = 1,$

$\mathbf{x}_2 = \frac{\mathbf{b}_2 - \mathrm{dual}(\mathbf{A}_{21}\mathbf{x}_1)}{\mathrm{dual}(\mathbf{A}_{22})} = [2,0],$

$\mathbf{x}_3 = \frac{\mathbf{b}_3 - \mathrm{dual}(\mathbf{A}_{31}\mathbf{x}_1) - \mathrm{dual}(\mathbf{A}_{32}\mathbf{x}_2)}{\mathrm{dual}(\mathbf{A}_{33})} = [\frac{1}{2}, \frac{3}{4}].$

Therefore we obtain the following generalized interval vector

$$\mathbf{x} = \begin{pmatrix} 1 \\ [2,0] \\ [\frac{1}{2}, \frac{3}{4}] \end{pmatrix},$$

which satisfies $\mathbf{Ax} = \mathbf{b}$.

4 Inner and outer estimations of the generalized solution sets

In this section, we consider the generalized solution set of the generalized symmetric interval linear system $\mathbf{A}x = \mathbf{b}$ in which $\mathbf{A} \in \mathbb{KR}^{n\times n}$ and

$\mathbf{b} \in \mathbb{KR}^n$, using the same convention in [15]. Define $\mathbf{A}^\dagger, \mathbf{A}^\ddagger \in \mathbb{IR}^{n\times n}$ and $\mathbf{b}^\dagger, \mathbf{b}^\ddagger \in \mathbb{IR}^n$ by

$$\mathbf{A}^\dagger_{ij} := \begin{cases} \mathbf{A}_{ij}, & \text{if } \ \mathbf{A}_{\rm ij} \in \mathbb{IR}, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathbf{A}^\ddagger_{ij} := \begin{cases} \mathrm{pro}(\mathbf{A}_{ij}), & \text{if } \ \mathbf{A}_{\rm ij} \in \overline{\mathbb{IR}}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\mathbf{b}^\dagger_{i} := \begin{cases} \mathbf{b}_{i}, & \text{if } \ \mathbf{b}_{\rm i} \in \mathbb{IR}, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathbf{b}^\ddagger_{i} := \begin{cases} \mathrm{pro}(\mathbf{b}_{i}), & \text{if } \ \mathbf{b}_{\rm i} \in \overline{\mathbb{IR}}, \\ 0, & \text{otherwise}. \end{cases}$$

It is obvious that $\mathbf{A} = \mathbf{A}^\dagger + \mathrm{dual}(\mathbf{A}^\ddagger)$ and $\mathbf{b} = \mathbf{b}^\dagger + \mathrm{dual}(\mathbf{b}^\ddagger)$. The generalized solution set $\Sigma(\mathbf{A}, \mathbf{b})$ is defined as

$$\sum(\mathbf{A}, \mathbf{b}) := \{x \in \mathbb{R}^n : (\forall A^\ddagger \in \mathbf{A}^\ddagger)(\forall b^\ddagger \in \mathbf{b}^\ddagger) \\ (\exists A^\dagger \in \mathbf{A}^\dagger)(\exists b^\dagger \in \mathbf{b}^\dagger)(A^\dagger + A^\ddagger)x = (b^\dagger + b^\ddagger)\}. \quad (6)$$

Note that the generalized solution set (6) is different from the introduced generalized solution set by Shary [34]. In the generalized solution set (6), a component of the input data $\mathbf{A}$ or $\mathbf{b}$ occurs with the existential quantifier "$\exists$" if it belongs to $\mathbb{IR}$ and occurs with the universal quantifier "$\forall$" if it belongs to $\overline{\mathbb{IR}}$. While in the generalized solution set introduced by Shary [34], these occurrences depend on the nature of the problem and are predetermined, i.e., a component in a problem may occur with the existential quantifier "$\exists$" while the same parameter occurs with the universal quantifier "$\forall$" in another problem.

Here we present an analytical result for the inner and outer estimations of the solution set (6). This analytic result can be used for the inner and outer estimations of the well-known united, tolerable, and controllable solution sets of the classical interval linear system $\mathbf{D}x = \mathbf{c}$ with $\mathbf{D} \in \mathbb{IR}^{n\times n}$ and $\mathbf{c} \in \mathbb{IR}^n$, as

- The united solution set

$$\sum_U(\mathbf{D}, \mathbf{c}) := \{x \in \mathbb{R}^n : (\exists D \in \mathbf{D})(\exists c \in \mathbf{c})(Dx = c)\},$$

conform with the solution set (6) to the generalized interval linear system $\mathbf{D}x = \mathbf{c}$.

- The tolerable solution set

$$\sum_T(\mathbf{D}, \mathbf{c}) := \{x \in \mathbb{R}^n : (\forall D \in \mathbf{D})(\exists c \in \mathbf{c})(Dx = c)\},$$

conform with the solution set (6) to the generalized interval linear system $\mathrm{dual}(\mathbf{D})x = \mathbf{c}$.

- The controllable solution set

$$\sum_C(\mathbf{D},\mathbf{c}) := \{x \in \mathbb{R}^n : (\forall c \in \mathbf{c})(\exists D \in \mathbf{D})(Dx = c)\},$$

conform with the solution set (6) to the generalized interval linear system $\mathbf{D}x = \mathrm{dual}(\mathbf{c})$.

We manipulate slightly Theorem 1 in [15], to build inner and outer estimations of the generalized solution set (6) of the symmetric interval linear systems.

Theorem 1. *Let $\mathbf{A} \in \mathbb{KR}^{n\times n}$ be symmetric and $\mathbf{b} \in \mathbb{KR}^n$. Suppose $\mathbf{L}$ constructed by Eqs. (2) and (3). If we define the generalized interval vectors $\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}' \in \mathbb{KR}^n$ by*

$$\mathbf{y}_i := \frac{\mathbf{b}_i - \sum_{j=1}^{i-1} \mathbf{L}_{ij}\mathrm{dual}(\mathbf{y}_j)}{\mathbf{L}_{ii}}, \qquad i = 1,\dots,n, \tag{7}$$

$$\mathbf{x}_i := \frac{\mathbf{y}_i - \sum_{j=i+1}^{n} \mathbf{L}_{ji}\mathrm{dual}(\mathbf{x}_j)}{\mathbf{L}_{ii}}, \qquad i = 1,\dots,n, \tag{8}$$

$$\mathbf{y}'_i := \frac{\mathbf{b}_i - \sum_{j=1}^{i-1} \mathbf{L}_{ij}\mathbf{y}'_j}{\mathbf{L}_{ii}}, \qquad i = 1,\dots,n, \tag{9}$$

$$\mathbf{x}'_i := \frac{\mathbf{y}'_i - \sum_{j=i+1}^{n} \mathbf{L}_{ji}\mathbf{x}'_j}{\mathbf{L}_{ii}}, \qquad i = 1,\dots,n. \tag{10}$$

then the following properties hold

1) If $\mathbf{L}$ is proper and $\mathbf{x}$ is proper then $\mathbf{x} \subseteq \Sigma(\mathbf{A},\mathbf{b})$.

2) Suppose $\mathbf{L}$ is improper. If $\mathbf{x}'$ is proper then $\Sigma(\mathbf{A},\mathbf{b}) \subseteq \mathbf{x}'$, otherwise $\Sigma(\mathbf{A},\mathbf{b}) = \emptyset$.

Example 3. *[24]*Consider the interval linear system $\mathbf{A}x = \mathbf{b}$ with

$$\mathbf{A} = \begin{pmatrix} [2,4] & [-1,1] \\ [-1,1] & [2,4] \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} [-3,3] \\ 0 \end{pmatrix},$$

The solution set $\sum(\mathbf{A},\mathbf{b})$ of this system is plotted in Fig. 1. Using Eqs. (2) and (3), the matrix $\mathbf{L}$ is obtained as

$$\mathbf{L} = \begin{pmatrix} [\sqrt{2},2] & 0 \\ [-\frac{1}{2},\frac{1}{2}] & [\frac{3}{2},\frac{\sqrt{15}}{2}] \end{pmatrix},$$

and this interval matrix satisfies $\mathbf{A} = \mathbf{L}\mathbf{L}^T$. Since $\mathbf{L}$ is proper, we compute vectors $\mathbf{y}$ and then $\mathbf{x}$, respectively, by Eqs. (7) and (8). We obtain

$\mathbf{x} = ([-\frac{3}{2}, \frac{3}{2}], 0)^T$, from Fig. 1, one can see $\mathbf{x} \subseteq \Sigma(\mathbf{A}, \mathbf{b})$ which confirms Theorem 1. It is to be noted that using the generalized interval LU decomposition introduced in [15], we obtain the same result but with more computational costs.

Example 4. *[20, 29]* Let us consider the symmetric interval linear system $\mathbf{A}x = \mathbf{b}$, where

$$\mathbf{A} = \begin{pmatrix} [2,4] & [-1,1] \\ [-1,1] & [2,4] \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} [0,2] \\ [0,2] \end{pmatrix},$$

Fig. 2 shows the solution set of this system. The coefficient matrix $\mathbf{A}$ of this system is similar to the one in previous example and so the triangular matrix in its Cholesky decomposition is

$$\mathbf{L} = \begin{pmatrix} [\sqrt{2}, 2] & 0 \\ [-\frac{1}{2}, \frac{1}{2}] & [\frac{3}{2}, \frac{\sqrt{15}}{2}] \end{pmatrix},$$

Now, since $\mathbf{L}$ is proper, we obtain the vectors $\mathbf{y}$ and $\mathbf{x}$ from Eqs. (7) and (8), respectively, which yield $\mathbf{x} = ([0,1], [0, 0.8889])^T$. Fig. 2 shows that $\mathbf{x} \subseteq \Sigma(\mathbf{A}, \mathbf{b})$ which confirms Theorem 1.

Example 5. Consider the interval linear system $\mathbf{A}x = \mathbf{b}$ with

$$\mathbf{A} = \begin{pmatrix} [100,150] & [3,5] & [1,7] \\ [3,5] & [100,140] & [3,5] \\ [1,7] & [3,5] & [60,80] \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} [0,14] \\ [0,9] \\ [0,3] \end{pmatrix}.$$

Since $\mathbf{A}$ is proper and nonnegative, we obtain an inner estimation of the united solution set using algorithm INonNeg introduced in [35]. By choosing the parameters $\lambda = \mu = 0.1$ (λ and μ are auxiliary scalar parameters in algorithm INonNeg which help to find inner estimation for the united solution set, see Algorithm INonNeg in [35]) and the initial point $\tilde{x} = (0,0,0)^T$ from the solution set $\Sigma(\mathbf{A}, \mathbf{b})$, this algorithm gives

$$\mathbf{x} = \begin{pmatrix} [0, 0.0141] \\ [0, 0.0090] \\ [0, 0.0494] \end{pmatrix}.$$

Now using Eqs. (2) and (3) for the Cholesky decomposition of the interval matrix $\mathbf{A}$, the following triangular interval matrix $\mathbf{L}$ is obtained

$$\mathbf{L} = \begin{pmatrix} [10.0000, 12.2475] & [0.0000, 0.0000] & [0.0000, 0.0000] \\ [0.3000, 0.4083] & [9.9954, 11.8252] & [0.0000, 0.0000] \\ [0.1000, 0.5716] & [0.2971, 0.4031] & [7.7396, 8.9169] \end{pmatrix}.$$

Since $\mathbf{L}$ is proper, using Eqs. (7) and (8) we obtain

$$\mathbf{x} = \begin{pmatrix} [0, 0.1371] \\ [0, 0.0846] \\ [0, 0.0435] \end{pmatrix}.$$

Since $\mathbf{L}$ and $\mathbf{x}$ are proper, according to Theorem 1, $\mathbf{x} \subseteq \Sigma(\mathbf{A}, \mathbf{b})$. As one can see, the result obtained by the new technique is wider than the one computed by algorithm INonNeg and so is more valuable. Also the proposed technique in [15] gives the same result but with more computational costs.

Example 6. Consider the tridiagonal interval linear system $\mathbf{A}x = \mathbf{b}$, where

$$\mathbf{A} = \begin{pmatrix} [3,1] & [2,1] & 0 \\ [2,1] & [5,4] & [2,1] \\ 0 & [2,1] & [3,2] \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} [1,2] \\ [2,3] \\ [1,4] \end{pmatrix}.$$

The generalized solution set $\sum(\mathbf{A}, \mathbf{b})$ of this system conform with the tolerable solution set $\sum_T(\text{dual}(\mathbf{A}), \mathbf{b})$. An outer approximation for the tolerable solution set $\sum_T(\text{dual}(\mathbf{A}), \mathbf{b})$ can be obtained by the generalized interval Gauss-Seidel (GIGS) method proposed in [33]. Starting from the initial box

$$([-1000, 1000], [-1000, 1000], [-1000, 1000])^T,$$

the GIGS method gives the following outer approximation

$$\mathbf{x} = \begin{pmatrix} [0.1249, 2.8001] \\ [-0.8001, 0.6251] \\ [0.1249, 2.4001] \end{pmatrix}.$$

Now, we utilize our new approach to obtain an outer estimation for the solution set. First using Eqs. (2) and (3), we obtain

$$\mathbf{L} = \begin{pmatrix} [1.7321, 1.0000] & 0 & 0 \\ [1.1547, 1.0000] & [1.9149, 1.7321] & 0 \\ 0 & [1.0445, 0.5774] & [1.3817, 1.2910] \end{pmatrix},$$

which satisfies $\mathbf{A} = \mathbf{L}\mathbf{L}^T$. Since $\mathbf{L}$ is improper so we apply Eqs. (9) and (10) which yields

$$\mathbf{x}' = \begin{pmatrix} [0.6419, 0.8378] \\ [-0.2964, 0.3581] \\ [0.2661, 1.8283] \end{pmatrix},$$

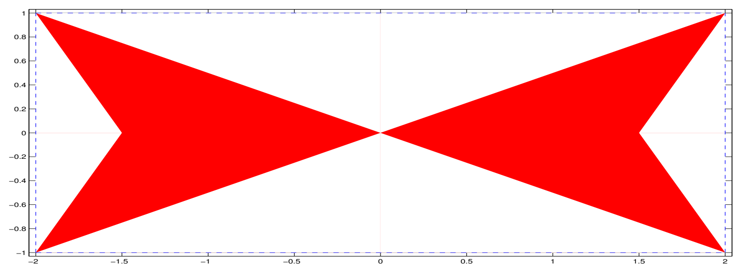

Figure 1: Solution set of Example 3

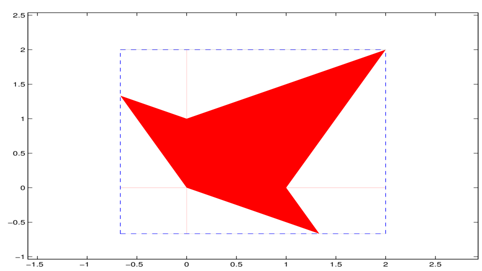

Figure 2: Solution set of Example 4

Since $\mathbf{L}$ is improper and $\mathbf{x}'$ is proper, our approach gives the following outer estimation for the solution set (6)

$$\mathbf{x} = \begin{pmatrix} [0.6419, 0.8378] \\ [-0.2964, 0.3581] \\ [0.2661, 1.8283] \end{pmatrix},$$

which is much sharper than the one computed by GIGS method. Also the generalized LU decomposition method introduced in [15] is not applicable here.

5 An application in economic

In economic, an input-output (I-O) model is a quantitative economic technique that represents the interdependencies between different branches of

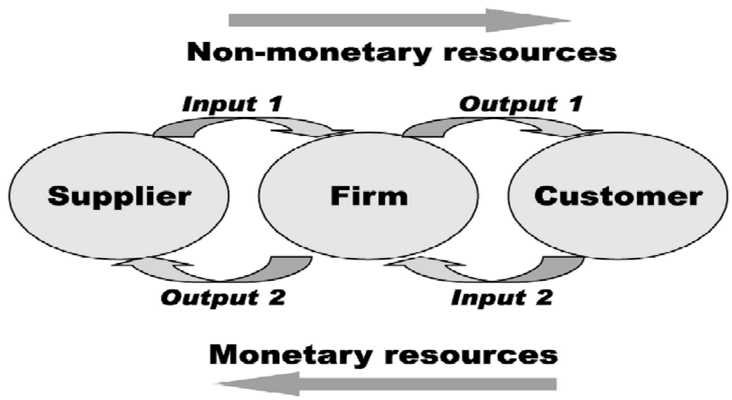

Figure 3: Two-way input-output model of the firm (see [28])

a national economy or different regional economies, Wassily Leontief developed this type of analysis [10]. In Fig. 3, one can see an image of two-way input-output model of the firm. As mentioned in [29], in compact form, an I-O model can be written as $x = Ax + b$, where x is a vector of sectorial outputs, A is the input-output technical coefficient matrix and b is the vector of final demands. The equation $x = Ax + b$ is equivalent to $Bx = b$ in which $B = I - A$ and B is called the Leontief matrix. If final demand is known, then the amount of the goods needed to satisfy this demand can be found by solving the linear system $Bx = b$. Now, we want to consider an I-O model of the 1987 Washington state direct purchase.

Table 1 presents the 1987 Washington State direct purchase coefficient table estimated from an aggregated model (see [4, 29]). As said in [29], to evaluate the total economic impact of a \$50 million increase in manufacturing exports, the author solved the system $Bx = b$, where B is the Leontief matrix and $b = (0, 50, 0, 0)^T$ MM\$. The vector x is obtained: $(3, 60, 27, 42)^T$ MM\$ (Base Case). That means, for example, that a \$50 million increase in manufacturing exports is expected to result in a \$3 million production increase in natural resource industries.

In most cases, due to some sources of uncertainties in I-O models, the entries of the Leontief matrix are not known exactly, but must be estimated and therefore are subject to some level of uncertainties. An approach for representing these uncertainties is utilizing the intervals. If we define interval matrix and interval vector $\mathbf{B}$ and $\mathbf{b}$, respectively, bounding Leontief matrix B and vector b, then we should find the united solution set of the

Table 1: 1987 Washington State direct purchase coefficient table

	Natural Resource	Manufacturing	Trade and Services	Personal Consumption
Natural Resource	0.10453	0.04279	0.00287	0.00305
Manufacturing	0.08263	0.10870	0.05835	0.03212
Trade and Services	0.08667	0.10188	0.20319	0.35550
Personal Consumption	0.62531	0.34483	0.61063	0.07981

interval linear system $\mathbf{B}x = \mathbf{b}$.

Now, suppose uncertainties have the effect shown in Table 2 for interval technology matrix. To evaluate the economic impact of a \$50 million increase in manufacturing exports taking into account the uncertainties on the technology coefficient matrix, we need to determine the united solution set of the symmetric interval linear system $\mathbf{B}x = b$, where $\mathbf{B}$ is the interval technology matrix shown in Table 2, and $b = (0, 50, 0, 0)^T$. Here we find an inner estimation of this solution set.

The Krawczyk method [23], interval Gauss-Seidel method [23], conjugate directions method [3], and verifylss function of Intlab, can not solve the interval system $\mathbf{B}x = b$ since the interval matrix $\mathbf{B}$ is not regular, i.e., there is a matrix $\tilde{B} \in \mathbf{B}$ which is singular. We want to obtain an inner box for its solution set by our new approach. Using Eqs. (2) and (3), the triangular matrix $\mathbf{L}$ is obtained as follows

$$\mathbf{L} = \begin{pmatrix} [0.3223, 1.0896] & 0 & 0 & 0 \\ [0.0393, 0.0836] & [0.3137, 1.4692] & 0 & 0 \\ [0.0065, 0.0799] & [-0.0002, 0.6762] & [0.4472, 1.1489] & 0 \\ [0.0085, 0.6314] & [0.0910, 0.6448] & [0.0224, 0.1164] & [0.2510, 0.2688] \end{pmatrix},$$

and it satisfies $\mathbf{B} = \mathbf{L}\mathbf{L}^T$. Since $\mathbf{L}$ is proper, using Eqs. (7) and (8) we obtain

$$\mathbf{x} = \begin{pmatrix} [-0.6110, 279.8203] \\ [43.6027, 920.6500] \\ [-98.4370, 52.2610] \\ [-329.7889, -200.8584] \end{pmatrix}.$$

Since $\mathbf{L}$ and $\mathbf{x}$ are proper, according to Theorem 1, $\mathbf{x} \subseteq \Sigma(\mathbf{A}, \mathbf{b})$. Note that the first three components of $\mathbf{x}$ contain the first three components of the solution $(3, 60, 27, 42)^T$ of the main system.

6 Concluding remarks

In this paper, using the Cholesky decomposition introduced by Zhao et al. [38], we proposed an algorithm to find the algebraic solution of the symmetric interval systems. We then utilized this decomposition to construct

Table 2: 1987 Washington State Input-Output study: Direct purchase coefficient table with uncertainty

	Natural Resource	Manufacturing	Trade and Services	Personal Consumption
Natural Resource	[0.1039, 1.1873]	[0.0126, 0.0911]	[0.0020, 0.0871]	[0.0027, 0.6879]
Manufacturing	[0.0126, 0.0911]	[0.1000, 2.1653]	[0.0000, 1.0000]	[0.0288, 1.0000]
Trade and Services	[0.0020, 0.0871]	[0.0000, 1.0000]	[0.2000, 1.7834]	[0.0100, 0.6201]
Personal Consumption	[0.0027, 0.6879]	[0.0288, 1.0000]	[0.0100, 0.6201]	[0.0718, 0.9002]

inner and outer estimations to the generalized solution set introduced in Section 4. Also we applied the new approach for solving an important problem in economic without computing the interval inverse matrix or need to have an initial guess. The numerical experiments showed the effectiveness of the proposed approaches.

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