Ulam stabilities for nonlinear fractional integro–differential equations with constant coefficient via Pachpatte’s inequality

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Abstract. In this article, we study some existence, uniqueness and Ulam type stability results for a class of boundary value problem for nonlinear fractional integro–differential equations with positive constant coefficient involving the Caputo fractional derivative. The main tools used in our analysis is based on Banach contraction principle, Schaefer’s fixed point theorem and Pachpatte’s integral inequality. Finally, results are illustrated with suitable example.

Keywords: Boundary value conditions, Caputo’s fractional derivative, fixed point, integral inequality, Stability.

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1 Introduction

The fractional calculus is an old branch of mathematics, but even then it is still new. This is an old branch because this branch was born in the time when Newton and Leibniz had introduced the concept of differential calculus. On September 30, 1695, Leibniz and L’Hospital discussed the
derivative of one half order. This was believed to be the moment of starting of fractional calculus. Since then many mathematicians have contributed to the basic concept of fractional calculus, see \cite{1, 4, 15, 17, 19, 31, 34, 35, 42} and the references therein.

Fractional differential equations have been proved to be valuable tools in the modeling of many phenomena in various fields such as control theory, signal processing, rheology, fractals, chaotic dynamics, modelling, bioengineering and biomedical applications and so on, for example, see \cite{12, 24} and the references therein. Recently, many researchers studied the fractional differential and integro-differential equations and obtained many interesting existence and uniqueness results, for detail see \cite{3, 6, 10, 11, 25–29, 39} and the references therein.

Recently, many mathematicians have been attracted by the research field of the stability problems of fractional differential equations and fractional integro-differential equations. This research field started from the speech given at Wisconsin University by Ulam in 1940. In this speech, Ulam \cite{32, 33} asked the question about the stability of the functional equation. Hyers \cite{7} was the first who gave the answer of this question in Banach space. Rassias \cite{20} studied the Ulam-Hyers stability of linear and nonlinear mapping. Jung \cite{8, 9} established Ulam-Hyers stability for more general mapping on restricted domain. In 1993, Obloza \cite{16} made the first study of Ulam-Hyers stability for linear differential equations. Later many researchers studied the Ulam type stability, for detail see \cite{2, 5, 21–23, 36–38, 40, 41}.

In \cite{30}, Tate et al. studied the existence, uniqueness and various types of Ulam stability of the following nonlinear Caputo fractional integro–differential equations of order $\alpha$ ($0 < \alpha \leq 1$):

\[ ^cD^\alpha y(t) = \lambda y(t) + f\left(t, y(t), \int_0^t h(t, s)y(s)ds\right), \quad t \in J := [0, T], \quad T > 0, \]

\[ y(0) + g(y) = y_0 \in \mathbb{R}, \]

where $f : J \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $g : C(J, \mathbb{R}) \rightarrow \mathbb{R}$ are continuous functions.

The above results motivate us and therefore, in this paper, we obtain the existence, uniqueness and various types of Ulam stability for the following boundary value problems (BVP for short) for nonlinear fractional integro–differential equations with constant coefficient $\lambda > 0$ of the type:

\[ ^cD^\alpha y(t) = \lambda y(t) + f\left(t, y(t), \int_0^t h(t, s)y(s)ds\right), \quad t \in J := [0, T], \quad T > 0, \quad (1) \]

\[ ay(0) + by(T) = c, \quad (2) \]
where \( cD^\alpha(0 < \alpha \leq 1) \) denotes the Caputo fractional derivative, \( f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is a given continuous function, and \( a, b, c \) are real constants with \( a + b \neq 0 \).

The rest of the paper is organized as follows. In Section 2, some definitions, notations and basic results are given. Section 3 is devoted to study the existence, uniqueness and stability of the problem (1)-(2). Illustrative example is given in the last section.

2 Preliminaries

In this section, we introduce some definitions, notations and results which are useful for further discussion. For \( T > 0 \) and \( J = [0, T] \), \( C(J, \mathbb{R}) \) denotes the Banach space of all continuous functions from \( J \) into \( \mathbb{R} \) with the norm

\[
||y||_\infty = \sup \{||y(t)|| : t \in J\}.
\]

Suppose \( L^1(J) \) denotes the space of Lebesgue-integrable functions \( y : J \to \mathbb{R} \) with the norm

\[
||y||_{L^1} = \int_0^T |y(t)| \, dt.
\]

**Definition 1** ([19]). The Riemann–Liouville fractional integral of a function \( h \in L^1([0,T],\mathbb{R}_+) \) of order \( \alpha \in \mathbb{R}_+ \) is defined by

\[
I^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) \, ds,
\]

where \( \Gamma(\cdot) \) is the Euler gamma function.

**Definition 2** ([12]). The Caputo fractional derivative of order \( \alpha > 0 \) of a function \( h \in L^1([0,T],\mathbb{R}_+) \) is defined as

\[
cD^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} h^{(n)}(s) \, ds, \quad n-1 < \alpha < n,
\]

where \( n = [\alpha] + 1 \) and \( [\alpha] \) denotes the integer part of the real number \( \alpha \).

**Lemma 1** ([12]). Let \( \alpha > 0 \) and \( n = [\alpha] + 1 \), then

\[
I^\alpha (cD^\alpha f(t)) = f(t) - \sum_{i=0}^{n-1} \frac{f^{(i)}(0)}{i!} t^i,
\]

where \( f^k(t) \) is the usual derivative of \( f(t) \) of order \( k \).
Lemma 2 ([19]). Let $\alpha > 0$. Then the fractional differential equation
\[ cD^\alpha h(t) = 0, \]
has a solution $h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$, where $c_i, i=0,1,2,\ldots,n-1$ are constant and $n = [\alpha] + 1$.

The following Pachpatte’s inequality plays an important role in obtaining our main results.

Theorem 1 ([18], p. 39). Let $u(t)$, $f(t)$ and $q(t)$ be nonnegative continuous functions defined on $\mathbb{R}_+$, and $n(t)$ be a positive and nondecreasing continuous function defined on $\mathbb{R}_+$ for which the inequality
\[ u(t) \leq n(t) + \int_0^t f(s) \left[ u(s) + \int_0^s q(\tau) u(\tau) d\tau \right] d\tau, \]
holds for $t \in \mathbb{R}_+$. Then
\[ u(t) \leq n(t) \left[ 1 + \int_0^t f(s) \exp \left( \int_0^s [f(\tau) + q(\tau)] d\tau \right) d\tau \right], \]
for $t \in \mathbb{R}_+$.

The following definitions are useful in the study of stability results.

Definition 3 ([5, 23]). The equation (1) is Ulam-Hyers stable if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C^1(J, \mathbb{R})$ satisfying the inequality
\[ \left\| cD^\alpha z(t) - \lambda z(t) - f \left( t, z(t), \int_0^t h(t, s) z(s) ds \right) \right\| \leq \epsilon, \ t \in J, \]
there exists a solution $y \in C^1(J, \mathbb{R})$ of equation (1) with
\[ \| z(t) - y(t) \| \leq c_f \epsilon, \ t \in J. \]

Definition 4 ([5, 23]). The equation (1) is generalized Ulam-Hyers stable if there exists $\psi_f \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\psi_f(0) = 0$, such that for each solution $z \in C^1(J, \mathbb{R})$ satisfying the inequality
\[ \left\| cD^\alpha z(t) - \lambda z(t) - f \left( t, z(t), \int_0^t h(t, s) z(s) ds \right) \right\| \leq \epsilon, \ t \in J, \]
there exists a solution $y \in C^1(J, \mathbb{R})$ of equation (1) with
\[ \| z(t) - y(t) \| \leq \psi_f(\epsilon), \ t \in J. \]
Definition 5 ([5,23]). The equation (1) is Ulam-Hyers-Rassias stable with respect to $\varphi \in C(J, \mathbb{R}_+)$ if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C^1(J, \mathbb{R})$ satisfying the inequality

$$\left\| c D^\alpha z(t) - \lambda z(t) - f \left( t, z(t), \int_0^t h(t, s)z(s)ds \right) \right\| \leq \epsilon \varphi(t), \; t \in J,$$

there exists a solution $y \in C^1(J, \mathbb{R})$ of equation (1) with

$$\|z(t) - y(t)\| \leq c_f \epsilon \varphi(t), \; t \in J.$$

Definition 6 ([5,23]). The equation (1) is generalized Ulam-Hyers-Rassias stable with respect to $\varphi \in C(J, \mathbb{R}_+)$ if there exists a real number $c_{f,\varphi} > 0$ such that for each solution $z \in C^1(J, \mathbb{R})$ satisfying the inequality

$$\left\| c D^\alpha z(t) - \lambda z(t) - f \left( t, z(t), \int_0^t h(t, s)z(s)ds \right) \right\| \leq \varphi(t), \; t \in J,$$

there exists a solution $y \in C^1(J, \mathbb{R})$ of equation (1) with

$$\|z(t) - y(t)\| \leq c_{f,\varphi} \varphi(t), \; t \in J.$$

Remark 1 ([5,23]). A function $z \in C^1(J, \mathbb{R})$ satisfies the inequality

$$\left\| c D^\alpha z(t) - \lambda z(t) - f \left( t, z(t), \int_0^t h(t, s)z(s)ds \right) \right\| \leq \epsilon, \; t \in J,$$

if and only if there exists a function $g \in C(J, \mathbb{R})$ (which depends on solution $y$) such that

(i) $\|g(t)\| \leq \epsilon, \; \forall t \in J$;

(ii) $c D^\alpha z(t) = \lambda z(t) + f \left( t, z(t), \int_0^t h(t, s)z(s)ds \right) + g(t), \; t \in J$.

Remark 2. Clearly,

(i) Definition 3 implies Definition 4.

(ii) Definition 5 implies Definition 6.

Remark 3. A solution satisfying the inequality

$$\left\| c D^\alpha z(t) - \lambda z(t) - f \left( t, z(t), \int_0^t h(t, s)z(s)ds \right) \right\| \leq \epsilon, \; t \in J,$$

is called an fractional $\epsilon$-solution of the nonlinear fractional integro–differential equation (1).
3 Existence and Ulam-Hyers stability of the boundary value problem

In this section we obtain existence, uniqueness and stability results for the problem (1)-(2). Now we introduce the following set of conditions:

\((H_1)\) There exists a constant \(L > 0\) such that
\[
||f(t, x, y) - f(t, \bar{x}, \bar{y})|| \leq L(||x - \bar{x}|| + ||y - \bar{y}||), \text{ for each } t \in J \text{ and } x, y, \bar{x}, \bar{y} \in \mathbb{R}.
\]

\((H_2)\) The function \(f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) is continuous.

\((H_3)\) There exists a constant \(a_f > 0\) such that \(||f(t, x, y)|| \leq a_f(1 + ||x|| + ||y||), \text{ for each } t \in J \text{ and } x, y \in \mathbb{R}.\)

**Lemma 3** ([13]). Let \(0 < \alpha < 1\) and let \(h : J \to \mathbb{R}\) be continuous. A function \(y\) is a solution of the fractional integral equation
\[
y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} h(s) \, ds,
\]
if and only if \(y\) is a solution of the initial value problem for the fractional differential equation
\[
cD^{\alpha}y(t) = h(t), \quad t \in J = [0, T], \quad T > 0,
\]
\[
y(0) = y_0.
\]

**Lemma 4** ([3]). Let \(0 < \alpha < 1\) and let \(h : J \to \mathbb{R}\) be continuous. A function \(y\) is a solution of the fractional integral equation
\[
y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} h(s) \, ds - \frac{b}{a + b} \left[ \frac{b}{\Gamma(\alpha)} \int_0^T (T - s)^{\alpha - 1} h(s) \, ds - c \right],
\]
if and only if \(y\) is a solution of the fractional BVP
\[
cD^{\alpha}y(t) = h(t), \quad t \in J = [0, T], \quad T > 0,
\]
\[
ay(0) + by(T) = c.
\]

As a consequence of Lemma 3 and Lemma 4 and [14], we have the following result which is useful in our main results.
Lemma 5. If \( f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is a continuous function, then the problem (1)-(2) is equivalent to the following integral equation

\[
y(t) = \tilde{A} + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) \, ds \\
+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, y(s), \int_0^s h(t, \tau) y(\tau) \, d\tau \right) \, ds,
\]

for \( t \in J \), and

\[
\tilde{A} = \frac{1}{a+b} \left[ c - \frac{b\lambda}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} y(s) \, ds \\
- \frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f\left(s, y(s), \int_0^s h(t, \tau) y(\tau) \, d\tau \right) \, ds \right].
\]

Theorem 2. Assume that \((H_1)\) holds. If

\[
\left[ \frac{(\lambda+L)T^\alpha + LhT^{\alpha+1}}{\Gamma(\alpha+1)} \right] \left(1 + \frac{|b|}{|a+b|} \right) < 1,
\]

where \( h_T = \sup\{|h(t,s)| | 0 \leq s \leq t \leq T \} \), then the BVP (1)-(2) has a unique solution on \( J \).

Proof. Transform the problem (1)-(2) into a fixed point problem. Consider the operator \( F : C(J, \mathbb{R}) \to C(J, \mathbb{R}) \) defined by

\[
F(y)(t) = \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) \, ds \\
+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, y(s), \int_0^s h(t, \tau) y(\tau) \, d\tau \right) \, ds \\
- \frac{1}{a+b} \left[ \frac{b\lambda}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} y(s) \, ds \\
+ \frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f\left(s, y(s), \int_0^s h(t, \tau) y(\tau) \, d\tau \right) \, ds \right].
\]

Let \( x, y \in C(J, \mathbb{R}) \). Then for each \( t \in J \), we have

\[
||F(x)(t) - F(y)(t)|| \leq \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ||x(s) - y(s)|| \, ds
\]
\[
\begin{align*}
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\| f \left( s, x(s), \int_0^s h(t, \tau)x(\tau)d\tau \right) \right\| ds \\
&- f \left( s, y(s), \int_0^s h(t, \tau)y(\tau)d\tau \right) \left\| ds \\
&+ \frac{|b|}{\Gamma(\alpha)} \left( \lambda + L \right) \int_0^T (T-s)^{\alpha-1} \left\| x(s) - y(s) \right\| ds \\
&+ \frac{|b|}{\Gamma(\alpha)} \left( \lambda + L \right) \int_0^T (T-s)^{\alpha-1} \left\| f \left( s, x(s), \int_0^s h(t, \tau)x(\tau)d\tau \right) \right\| ds \\
&+ Lh_T T^{\alpha-1} \left( \int_0^s \| x(\tau) - y(\tau) \| d\tau \right) ds \\
&\leq \left[ \left( \lambda + L \right) T^{\alpha} + Lh_T T^{\alpha+1} \right] \left( 1 + \frac{|b|}{|a+b|} \right) \| x - y \|_\infty.
\end{align*}
\]

Thus
\[
\| F(x) - F(y) \|_\infty \leq \left[ \left( \lambda + L \right) T^{\alpha} + Lh_T T^{\alpha+1} \right] \left( 1 + \frac{|b|}{|a+b|} \right) \| x - y \|_\infty.
\]

Thus, \( F \) is a contraction due to the inequality (4).

As a consequence of Banach contraction principle, it is deduced that \( F \) has a unique fixed point which is just the unique solution of the problem (1)-(2). \( \square \)

The second result is based on Schaefer’s fixed point theorem.

**Theorem 3.** Assume that \( (H_2) \) and \( (H_3) \) hold. Then the BVP (1)-(2) has at least one solution on \( J \).

**Proof.** Let the operator \( F \) be defined as in (5). We complete the proof in the following four steps.

Step 1: \( F \) is continuous.
Let \( \{y_n\} \) be a sequence such that \( y_n \to y \) in \( C(J, \mathbb{R}) \). Then for each \( t \in J \), we have

\[
\|F(y_n)(t) - F(y)(t)\|
\leq \frac{\lambda T^\alpha}{\Gamma(\alpha + 1)} \|y_n(s) - y(s)\|_{\infty}
+ \frac{T^\alpha}{\Gamma(\alpha + 1)} \left\| f\left(s, y_n(s), \int_0^s h(t, \tau)y_n(\tau)d\tau\right) - f\left(s, y(s), \int_0^s h(t, \tau)y(\tau)d\tau\right) \right\|_{\infty}
+ \frac{|b| \lambda T^\alpha}{\Gamma(\alpha + 1) |a + b|} \|y_n(s) - y(s)\|_{\infty}
+ \frac{|b| T^\alpha}{\Gamma(\alpha + 1) |a + b|} \left\| f\left(s, y_n(s), \int_0^s h(t, \tau)y_n(\tau)d\tau\right) - f\left(s, y(s), \int_0^s h(t, \tau)y(\tau)d\tau\right) \right\|_{\infty}
\leq \frac{\lambda}{\Gamma(\alpha + 1)} \left(1 + \frac{|b|}{|a + b|}\right) T^\alpha \|y_n(s) - y(s)\|_{\infty}
+ \frac{1}{\Gamma(\alpha + 1)} \left(1 + \frac{|b|}{|a + b|}\right) T^\alpha \left\| f\left(s, y_n(s), \int_0^s h(t, \tau)y_n(\tau)d\tau\right) - f\left(s, y(s), \int_0^s h(t, \tau)y(\tau)d\tau\right) \right\|_{\infty}
\]

Since \( f \) is a continuous function and \( y_n \to y \), we have

\[
\|F(y_n)(t) - F(y)(t)\|_{\infty} \to 0,
\]
as \( n \to \infty \). Consequently, \( F \) is continuous.

Step 2: \( F \) maps bounded sets into bounded sets in \( C(J, \mathbb{R}) \).

We need to show that for any \( \mu^* > 0 \), there exists a positive constant \( l \) such that for each \( y \in B_{\mu^*} = \{y \in C(J, \mathbb{R}) : \|y\|_{\infty} \leq \mu^*\} \), we have \( \|F(y)\|_{\infty} \leq l \).

By condition \((H_3)\), we have for each \( t \in [0, T] \),

\[
\|F(y)\| \leq \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} \|y(s)\| 
\]
\[
+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| f \left( s, y(s), \int_0^s h(t, \tau)y(\tau)d\tau \right) \right| ds \\
+ \frac{|b|}{|a+b|} \frac{\lambda}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} ||y(s)|| ds \\
+ \frac{|b|}{|a+b|} \frac{\lambda}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \left| f \left( s, y(s), \int_0^s h(t, \tau)y(\tau)d\tau \right) \right| ds \\
+ \frac{|c|}{|a+b|} \\
\leq \frac{\lambda \mu^* T^{\alpha}}{\Gamma(\alpha + 1) \Gamma(\alpha + 1)} \left[ 1 + \frac{|b|}{|a+b|} + \frac{a f(1 + \mu^* + \mu^* h T) T^{\alpha}}{\Gamma(\alpha + 1)} \right] \\
+ \frac{|c|}{|a+b|}. 
\]

Thus

\[
||F(y)||_{\infty} \leq \left[ \frac{\lambda \mu^* T^{\alpha}}{\Gamma(\alpha + 1)} \frac{a f(1 + \mu^* + \mu^* h T) T^{\alpha}}{\Gamma(\alpha + 1)} \right] \left[ 1 + \frac{|b|}{|a+b|} \right] \\
+ \frac{|c|}{|a+b|} := l.
\]

Step 3: \( F \) maps bounded sets into equicontinuous sets of \( C(J, \mathbb{R}) \).

Let \( t_1, t_2 \in (0, T), t_1 < t_2, B_{\mu^*} \) be a bounded set in \( C(J, \mathbb{R}) \) as in step 2, and let \( y \in B_{\mu^*} \). Then

\[
||F(y)(t_1) - F(y)(t_2)|| \leq \frac{\lambda}{\Gamma(\alpha)} \int_0^{t_1} \left\{ (t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1} \right\} ||y(s)|| ds \\
+ \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left\{ (t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1} \right\} ||f \left( s, y(s), \int_0^s h(t, \tau)y(\tau)d\tau \right)|| ds 
\]
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\[- \frac{\lambda}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ||y(s)|| \, ds \]

\[- \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ||f(s, y(s), \int_0^s h(t, \tau)y(\tau) d\tau)|| \, ds \]

\[\leq \frac{\lambda ||y(s)||}{\Gamma(\alpha)} \int_0^{t_1} \{(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}\} \, ds \]

\[+ \frac{a_f(1 + ||y(s)|| + \int_0^s |h(t, \tau)||y(\tau)||d\tau)}{\Gamma(\alpha)} \int_0^{t_1} \{(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}\} \, ds \]

\[+ \frac{\lambda ||y(s)||}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \, ds \]

\[+ \frac{a_f(1 + ||y(s)|| + \int_0^s |h(t, \tau)||y(\tau)||d\tau)}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \, ds \]

\[\leq \frac{(\lambda \mu^* + a_f(1 + \mu^* + \mu^* h_T T))}{\Gamma(\alpha + 1)} \{2(t_2 - t_1)^{\alpha} + (t_1^\alpha - t_2^\alpha)\}. \tag{6} \]

As \( t_1 \to t_2 \), the right-hand side of the above inequality tends to zero. As a consequence of steps 1 to 3 together with the Arzelà-Ascoli theorem, we can conclude that \( F : C(J, \mathbb{R}) \to C(J, \mathbb{R}) \) is continuous and completely continuous.

Step 4: A priori bounds. Now it remains to show that the set

\[ \mathcal{E} = \{y \in C(J, \mathbb{R}) : y = \beta F(y), \text{ for some } \beta \in (0, 1)\}, \]

is bounded.

Let \( y \in \mathcal{E} \), then \( y = \beta F(y) \), for some \( \beta \in (0, 1) \). Thus, for each \( t \in J \), we have

\[ y(t) = \beta \left\{ \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} y(s) \, ds \right. \]

\[ + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f \left( s, y(s), \int_0^s h(t, \tau)y(\tau) \, d\tau \right) \, ds \]

\[ - \frac{1}{a + b} \left[ \frac{b\lambda}{\Gamma(\alpha)} \int_0^T (T - s)^{\alpha-1} y(s) \, ds \right. \]

\[ + \frac{b}{\Gamma(\alpha)} \int_0^T (T - s)^{\alpha-1} f \left( s, y(s), \int_0^s h(t, \tau)y(\tau) \, d\tau \right) \, ds - c \left. \right\}. \tag{6} \]
From condition \((H_3)\), for each \(t \in J\), we have

\[
\|F(y)(t)\| \leq \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|y(s)\| \, ds
\]

\[
+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\| f\left(s, y(s), \int_0^s h(t, \tau) y(\tau) \, d\tau\right) \right\| \, ds
\]

\[
+ \frac{|b|}{|a + b| \Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \|y(s)\| \, ds
\]

\[
+ \frac{|b|}{a + b \Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \left\| f\left(s, y(s), \int_0^s h(t, \tau) y(\tau) \, d\tau\right) \right\| \, ds
\]

\[
+ \frac{|c|}{|a + b|}
\]

\[
\leq \frac{\lambda u^*}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \, ds + \frac{a f(1 + \mu^* + \mu^* T^\alpha)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \, ds
\]

\[
+ \frac{|b|}{|a + b| \Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \, ds + \frac{|c|}{|a + b|}
\]

\[
\leq \frac{\lambda u^* T^\alpha}{\Gamma(\alpha + 1)} \left[ 1 + \frac{|b|}{|a + b|} \right] + \frac{a f(1 + \mu^* + \mu^* T^\alpha) T^\alpha}{\Gamma(\alpha + 1)} \left[ 1 + \frac{|b|}{|a + b|} \right]
\]

\[
+ \frac{|c|}{|a + b|}.
\]

Thus for every \(t \in J\), we have

\[
\|F(y)\|_\infty \leq \left[ \frac{\lambda u^* T^\alpha}{\Gamma(\alpha + 1)} + \frac{a f(1 + \mu^* + \mu^* T^\alpha) T^\alpha}{\Gamma(\alpha + 1)} \right] \left[ 1 + \frac{|b|}{|a + b|} \right]
\]

\[
+ \frac{|c|}{|a + b|} := R.
\]

This shows that the set \(E\) is bounded. Now applying Schaefer’s fixed point theorem, we deduce that \(F\) has a fixed point which is a solution of the problem (1)-(2).

\[\square\]

**Theorem 4.** Assume that \((H_1)\) and inequality (4) hold. Then the BVP (1)-(2) is Ulam-Hyers stable.

**Proof.** Let \(\epsilon > 0\) and let \(z \in C^1(J, \mathbb{R})\) be a function which satisfies the
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inequality:

\[
\left| cD^\alpha z(t) - \lambda z(t) - f\left(t, z(t), \int_0^s h(t, \tau)z(\tau)d\tau\right) \right| \leq \epsilon, \text{ for any } t \in J \quad (7)
\]

and let \( y \in C(J, \mathbb{R}) \) be the unique solution of the following Cauchy problem

\[
cD^\alpha y(t) = \lambda y(t) + f\left(t, y(t), \int_0^s h(t, \tau)y(\tau)d\tau\right), \quad t \in J; \quad 0 < \alpha \leq 1
\]

\[
y(0) = z(0), \quad y(T) = z(T).
\]

Using Lemma 5, we obtain

\[
y(t) = \tilde{A}y + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}y(s)ds
\]

\[
+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}f\left(s, y(s), \int_0^s h(t, \tau)y(\tau)d\tau\right)ds,
\]

and

\[
\tilde{A}y = \frac{1}{a+b} \left[ c - \frac{b\lambda}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}y(s)ds
\right.
\]

\[
- \frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}f\left(s, y(s), \int_0^s h(t, \tau)y(\tau)d\tau\right)ds \right].
\]

If \( y(T) = z(T) \) and \( y(0) = z(0) \) then we find

\[
\left| \tilde{A}y - \tilde{A}z \right| \leq \frac{|b| \lambda}{\Gamma(\alpha) |a+b|} \int_0^T (T-s)^{\alpha-1} \|y(s) - z(s)\| ds
\]

\[
+ \frac{|b|}{\Gamma(\alpha) |a+b|} \int_0^T (T-s)^{\alpha-1}L(\|y(s) - z(s)\|) ds
\]

\[
+ \int_0^T |h(t, \tau)| \|y(\tau) - z(\tau)\| d\tau ds
\]

\[
\leq \frac{|b| (\lambda + L)}{\Gamma(\alpha) |a+b|} \int_0^T (T-s)^{\alpha-1} \|y(s) - z(s)\| ds
\]

\[
+ \frac{|b| Lh_T}{\Gamma(\alpha) |a+b|} \int_0^T (T-s)^{\alpha-1}\left( \int_0^s \|y(\tau) - z(\tau)\| d\tau \right) ds
\]

\[
\leq \frac{|b| (\lambda + L)}{\Gamma(\alpha) |a+b|} \int_0^T (T-s)^{\alpha-1} \|y(s) - z(s)\| ds
\]
Thus
\[ \tilde{A}_y = \tilde{A}_z. \]

Then we have
\[ y(t) = \tilde{A}_z + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) \, ds \]
\[ + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s), \int_0^s h(t, \tau) y(\tau) \, d\tau) \, ds, \]
on integration of the inequality (7), we obtain
\[ \left\| z(t) - \tilde{A}_z - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) \, ds \right\| \]
\[ - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, z(s), \int_0^s h(t, \tau) z(\tau) \, d\tau) \, ds \leq \frac{\epsilon t^\alpha}{\Gamma(\alpha + 1)}. \]

For any \( t \in J \) we have
\[ \|z(t) - y(t)\| \leq \left\| z(t) - \tilde{A}_z - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) \, ds \right. \]
\[ - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, z(s), \int_0^s h(t, \tau) z(\tau) \, d\tau) \, ds \]
\[ + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|z(s) - y(s)\| \, ds \]
\[ + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\| f(s, z(s), \int_0^s h(t, \tau) z(\tau) \, d\tau) \right\| \, ds. \]

Using inequality (8) and condition \((H_1)\), we obtain
\[ \|z(t) - y(t)\| \leq \frac{\epsilon t^\alpha}{\Gamma(\alpha + 1)} + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|z(s) - y(s)\| \, ds \]
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\[ + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} L \left( \|z(s) - y(s)\| \right) ds \]
\[ + \int_0^s |h(t, \tau)| \|z(\tau) - y(\tau)\| d\tau \right) ds \]
\[ \leq \frac{e^{t^\alpha}}{\Gamma(\alpha + 1)} + \frac{\lambda + L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|z(s) - y(s)\| ds \]
\[ + \frac{Lh_T}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left( \int_0^s \|z(\tau) - y(\tau)\| d\tau \right) ds, \]
\[ \leq \frac{e^{t^\alpha}}{\Gamma(\alpha + 1)} + \int_0^t \frac{(\lambda + L)}{\Gamma(\alpha)} (T-s)^{\alpha-1} \left( \|z(s) - y(s)\| \right) \]
\[ + \int_0^s \frac{Lh_T}{\lambda + L} \|z(\tau) - y(\tau)\| d\tau \right) ds. \] (9)

Applying Pachpatte’s inequality given in the Theorem 1 to the inequality (9) with \( u(t) = \|z(t) - y(t)\| \),

\[ n(t) = \frac{e^{t^\alpha}}{\Gamma(\alpha + 1)}, \quad f(s) = \frac{(\lambda + L)}{\Gamma(\alpha)} (T-s)^{\alpha-1}, \quad q(\tau) = \frac{Lh_T}{\lambda + L}, \]

we obtain

\[ \|z(t) - y(t)\| \leq \frac{e^{t^\alpha}}{\Gamma(\alpha + 1)} \left[ 1 + \int_0^t \frac{(\lambda + L)}{\Gamma(\alpha)} (T-s)^{\alpha-1} \right. \]
\[ \times \exp \left( \int_0^s \left\{ \frac{(\lambda + L)}{\Gamma(\alpha)} (T-\tau)^{\alpha-1} + \frac{Lh_T}{\lambda + L} \right\} d\tau \right) ds \]
\[ \leq C\epsilon, \]

for all \( t \in J \), where

\[ C = \frac{T^{\alpha}}{\Gamma(\alpha + 1)} \left[ 1 + \int_0^T \frac{(\lambda + L)}{\Gamma(\alpha)} (T-s)^{\alpha-1} \right. \]
\[ \times \exp \left( \int_0^s \left\{ \frac{(\lambda + L)}{\Gamma(\alpha)} (T-\tau)^{\alpha-1} + \frac{Lh_T}{\lambda + L} \right\} d\tau \right) ds \].

We conclude the problem (1)-(2) is Ulam-Hyers stable.

[\Box]

**Corollary 1.** If \( f \) in the problem (1)-(2) satisfies the condition \((H_1)\) and the inequality (4) holds, then the problem (1)-(2) is generalized Ulam-Hyers stable.

**Theorem 5.** Assume that \((H_1)\) and inequality (4) hold. Further suppose there exists an increasing function \( \varphi \in C(J, \mathbb{R}_+) \) and there exists \( \kappa_\varphi > 0 \) such that for any \( t \in J \)

\[ \Gamma^\alpha \varphi(t) \leq \kappa_\varphi \varphi(t) \]
are satisfied. Then the BVP (1)-(2) is Ulam-Hyers-Rassias stable.

Proof. Let \( z \in C^1(J, \mathbb{R}) \) be satisfies the following inequality:

\[
\left\| cD^\alpha z(t) - \lambda z(t) - f\left(t, z(t), \int_0^s h(t, \tau)z(\tau)d\tau\right) \right\| \leq \epsilon \varphi(t),
\]

for any \( t \in J, \epsilon > 0 \). Let \( y \in C(J, \mathbb{R}) \) be the unique solution of the following Cauchy problem

\[
cD^\alpha y(t) = \lambda y(t) + f\left(t, y(t), \int_0^s h(t, \tau)y(\tau)d\tau\right), \quad t \in J; \quad 0 < \alpha \leq 1,
\]

\[
y(0) = z(0), \quad y(T) = z(T).
\]

By Lemma 5, we have

\[
y(t) = \tilde{A}z + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}y(s)ds
\]

\[
+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}f\left(s, y(s), \int_0^s h(t, \tau)y(\tau)d\tau\right)ds,
\]

and

\[
\tilde{A}z = \frac{1}{a+b} \left[ c - \frac{b\lambda}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}z(s)ds
\right.
\]

\[
- \frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}f\left(s, z(s), \int_0^s h(t, \tau)z(\tau)d\tau\right)ds \right].
\]

Integrating both sides of inequality (10), we obtain

\[
\left\| z(t) - \tilde{A}z - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}z(s)ds
\right.
\]

\[
- \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}f\left(s, z(s), \int_0^s h(t, \tau)z(\tau)d\tau\right)ds \right\|
\]

\[
\leq \frac{\epsilon}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}\varphi(t)ds = \epsilon I^\alpha \varphi(t) \leq \epsilon \kappa \varphi(t).
\]  

(11)

On the other hand, we have

\[
\|z(t) - y(t)\| \leq \left\| z(t) - \tilde{A}z - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}z(s)ds
\right.
\]
Using inequality (11) and condition (H1), we obtain

\[ ||z(t) - y(t)|| \leq \epsilon \kappa \varphi(t) + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ||z(s) - y(s)|| \, ds \]

\[ + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left( \int_0^s |h(t,\tau)||z(\tau) - y(\tau)|| \, d\tau \right) \, ds \]

\[ \leq \epsilon \kappa \varphi(t) + \left( \frac{\lambda + L}{\Gamma(\alpha)} \right) \int_0^t (t-s)^{\alpha-1} ||z(s) - y(s)|| \, ds \]

\[ + \frac{Lh_T}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left( \int_0^s |z(\tau) - y(\tau)|| \, d\tau \right) \, ds \]

\[ \leq \epsilon \kappa \varphi(t) + \left( \frac{(\lambda + L)}{\Gamma(\alpha)} \right) \left( T - s \right)^{\alpha-1} \left[ ||z(s) - y(s)|| \right] \]

\[ + \int_0^s \left( \frac{Lh_T}{\lambda + L} \right) ||z(\tau) - y(\tau)|| \, d\tau \] \, ds.

By applying Pachpatte’s inequality given in the Theorem 1 with \( u(t) = ||z(t) - y(t)|| \),

\[ n(t) = \epsilon \kappa \varphi(t), \quad f(s) = \frac{(\lambda + L)}{\Gamma(\alpha)} (T - s)^{\alpha-1}, \quad q(\tau) = \frac{Lh_T}{\lambda + L}, \]

we obtain

\[ ||z(t) - y(t)|| \leq \epsilon \kappa \varphi(t) \left[ 1 + \int_0^t \left( \frac{(\lambda + L)}{\Gamma(\alpha)} (T - s)^{\alpha-1} \right. \right. \]

\[ \times \exp \left( \int_0^s \left\{ \frac{(\lambda + L)}{\Gamma(\alpha)} (T - \tau)^{\alpha-1} + \frac{Lh_T}{\lambda + L} \right\} d\tau \right) ds \] \]

\[ \leq C \epsilon \varphi(t), \]
for all \( t \in J \), where
\[
C = \kappa \varphi \left[ 1 + \int_0^T \frac{(\lambda + L)}{\Gamma(\alpha)}(T - s)^{\alpha - 1} \right.
\times \exp \left( \int_0^s \left\{ \frac{(\lambda + L)}{\Gamma(\alpha)}(T - \tau)^{\alpha - 1} + \frac{Lh_T}{(\lambda + L)} \right\} d\tau \right) ds \right].
\]

The proof is complete. \( \square \)

**Corollary 2.** Under the assumptions of Theorem 5, the problem (1)-(2) is generalized Ulam-Hyers-Rassias stable.

## 4 Examples

In this section, we illustrate our main results with the help of following example.

**Example 1.** Consider
\[
\begin{align*}
^cD^{\frac{1}{2}} x(t) &= \frac{1}{10} x(t) + \frac{x(t)}{t^2 + 9} + \frac{1}{9} \int_0^t \frac{x(s)}{(2 + t)^2} ds, \quad t \in [0,1] \quad (12) \\
x(0) + x(1) &= 0. \quad (13)
\end{align*}
\]

Define
\[
f(t, x(t), Hx(t)) = \frac{x(t)}{t^2 + 9} + \frac{1}{9} Hx(t), \quad t \in [0,1],
\]

\( \alpha = \frac{1}{2}, \lambda = \frac{1}{10}, \) where
\[
Hx(t) = \int_0^t \frac{1}{(2 + t)^2} x(s) ds.
\]

Clearly, the function \( f \) is continuous. For any \( x_1, x_2 \in \mathbb{R} \) and \( t \in [0,1] \)
\[
||f(t, x_1, Hx_1) - f(t, x_2, Hx_2)|| \leq \frac{1}{9} \left[ ||x_1 - x_2|| + ||Hx_1 - Hx_2|| \right]. \quad (14)
\]

Hence condition \((H_1)\) is satisfied with \( L = \frac{1}{9}. \)

As \( h_T = \frac{1}{4}, a = b = T = 1, c = 0 \) and \( \alpha = \frac{1}{2} \) we have
\[
\left[ \frac{(\lambda + L)T^{\alpha} + Lh_T T^{\alpha + 1}}{\Gamma(\alpha + 1)} \right] \left( 1 + \frac{|b|}{|a|} \right) = \left[ \frac{(\frac{1}{10} + \frac{1}{4}) + \frac{1}{36}}{\Gamma(\frac{3}{2} + 1)} \right] \left( 1 + \frac{1}{2} \right)
\]
\[
= \frac{43}{60\sqrt{\pi}} < 1.
\]

So all conditions of Theorem 2 hold. Thus Theorem 2 implies that the problem (12)-(13) has a unique solution on \([0,1]\). Moreover, Theorem 4 implies that the problem (12)-(13) is Ulam-Hyers stable.
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