Journal of Mathematical Modeling
Vol. 7, No. 4, 2019, pp. 469-496

JMM

Stabilized IMLS based element free Galerkin method for stochastic elliptic partial differential equations

Komeil Izadpanah[*] **and Ali Mesforush and Ali Nazemi**

Faculty of Mathematical Sciences, Shahrood University of Technology, Shahrood, Iran

Emails: k.izadpanah@shahroodut.ac.ir, ali.mesforush@shahroodut.ac.ir, nazemi20042003@gmail.com

Abstract. In this paper, we propose a numerical method to solve the elliptic stochastic partial differential equations (SPDEs) obtained by Gaussian noises using an element free Galerkin method based on stabilized interpolating moving least square shape functions. The error estimates of the method is presented. The method is tested via several problems. The numerical results show the usefulness and accuracy of the new method.

Keywords: Element free Galerkin method, stabilized interpolating moving least square, stochastic elliptic equation, Gussian noise, error estimates.
AMS Subject Classification: 65N30, 93E24, 60H40, 35J15, 65Gxx.

1 Introduction

Many natural phenomena and physical applications are modeled by stochastic partial differential equations (SPDEs) [1,3].

The numerical solution of SPDEs is becoming a fast-growing research area. Many authors try to solve SPDEs through various methods. The Finite element method is a technique for solving SPDEs [2,6], and the Finite difference method is another way, see [2,11,35,36]. The authors of [19] tried to solve SPDEs by the method of Wiener chaos expansions. The spectral Galerkin method [39], the stochastic spectral collocation method [22] and

[*]Corresponding author.
Received: 31 August 2019 / Revised: 21 September 2019 / Accepted: 30 September 2019.
DOI: 10.22124/jmm.2019.14278.1314

the Itô Taylor expansions method [20] are other numerical tools that were discussed for solving SPDEs.

Meshless methods are powerful numerical tools that have been applied for solving many problems in engineering and applied mathematics. These methods do not require a mesh to discretize the domain of the problem under consideration and the approximate solution is constructed entirely based on a set of scattered nodes.

Some authors have used mesh-free methods to solve SPDEs. Fasshauer et al. tried kernel-based collocation method [10, 17, 18], also Dehghan and Shirzadi used radial basis functions and dual reciprocity method [13–16].

Lancaster presented the moving least-squares (MLS) approximation to study surface fitting [23]. The advantage of the MLS approximation is to obtain the shape function in meshless methods with higher order continuity and consistency by employing the basis functions with lower order and choosing a suitable weight function. The shape function obtained with the MLS approximation, can achieve a very precise solution [34].

Based on the MLS approximation, Belytschko et al. proposed the element-free Galerkin (EFG) method [4]. Cheng carried out an error estimation and convergence analysis of the finite point method and the EFG method based on the MLS approximation [8, 9]. Mukherjee made an improvement on the MLS approximation in order to deal with boundary conditions conveniently in the EFG method [29]. Cheng proposed an improved moving least-squares approximation by orthogonalizing the basis functions in the MLS approximation, and based on it Cheng put forward a boundary element-free method [7, 26].

Since the shape function of the MLS approximation does not have the properties of the Kronecker Delta function, the meshless method based on it must use other methods, such as the penalty function method and Lagrange multiplier, to impose essential boundary conditions, which makes the weak form of the problem to be solved more complicated and the computational efficiency lower as a result. Therefore, it is important to study the interpolating moving least-squares method which has the properties of the Kronecker Delta function.

Based on the MLS approximation, Lancaster proposed the interpolating moving least-squares (IMLS) method [23]. Compare with the IMLS method, the shape function of the MLS approximation is much simpler. By using the IMLS method, Kaljević presented the improved EFG method in which the essential boundary condition can be applied directly [21]. Ren et al. proposed an improved IMLS method, and based on it the interpolating element-free Galerkin (IEFG) method and the improved boundary element-free method are presented [30–33]. In Refs. [27, 28], Mirzaei et al. developed a shifted and scaled polynomial basis function to stabilize the MLS approximation. In Ref. [28], the relationship between the condition numbers and the determinants of the moment matrix in the MLS and those in the stabilized MLS is presented and proved. Besides, in Ref. [27] it was proved that, using the new basis, the minimum eigenvalue of the corresponding moment matrix is bounded independent of the fill distance, which means that the stabilized MLS approximation is theoretically stable. Xiaolin Li developed a stabilized IMLS method. Theoretical analysis shows that both the determinant and the condition number of the moment matrix in the stabilized IMLS method are invariable with respect to the separation distance. Thus, the stabilized IMLS method prevents the instability occurrence [25].

In this paper we try to use the stabilized IMLS based element free Galerkin method to solve elliptic SPDEs.

2 Interpolating moving least-squares method

Let $\mathbf{X} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$ be a set of all nodes in the bounded domain $\mathscr{D} \subset \mathbb{R}^n$ where N is the number of nodes. The parameter ρ_I denotes the radius of the domain of influence of $\mathbf{x}_I$, and $\|\cdot\|$ denotes the Euclidean norm. The domain of influence of $\mathbf{x}_I$ is defined by $\mathscr{D}_I = \{\mathbf{x} : \|\mathbf{x} - \mathbf{x}_I\| \leq \rho_I, \mathbf{x} \in \mathscr{D}\}$. Let $\rho = \max_{\mathbf{x}_I \in \mathbf{X}} \{\rho_I\}$. For a given point $\mathbf{x} \in \mathscr{D}$, define the index set $\tau_{\mathbf{x}} = \{I | \|\mathbf{x} - \mathbf{x}_I| \leq \rho_I, \mathbf{x} \in \mathbf{X}\}$.

Let $u(\mathbf{x})$ be the function of the field variable defined in $\mathscr{D}$. The approximation function of $u(\mathbf{x})$ is denoted by $u^h(\mathbf{x})$. In order to let the approximation $u^h(\mathbf{x})$ in the IMLS method satisfy the interpolating property, we define a singular weight function, i.e.,

$$w(\mathbf{x}, \mathbf{x}_I) = w(\mathbf{x} - \mathbf{x}_I) = \begin{cases} \|\frac{\mathbf{x}-\mathbf{x}_I}{\rho_I}\|^{-\alpha}, & \|\mathbf{x} - \mathbf{x}_I\| \leq \rho_I, \\ 0, & \text{otherwise}, \end{cases} \quad (1)$$

where the parameter α is an even positive integer.

Define the inner product and the corresponding norm at $\mathbf{x}$

$$\langle f, g \rangle_{\mathbf{x}} = \sum_{I \in \tau_{\mathbf{x}}} w(\mathbf{x}, \mathbf{x}_I) f(\mathbf{x}_I) g(\mathbf{x}_I), \quad \forall f, g \in C^0(\mathscr{D}),$$

$$\|f\|_{\mathbf{x}} = \left[\sum_{I \in \tau_{\mathbf{x}}} w(\mathbf{x}, \mathbf{x}_I) f^2(\mathbf{x}_I) \right]^{\frac{1}{2}},$$

where the subscript $\mathbf{x}$ denotes a point in $\mathscr{D}$.

Let $p_0(\mathbf{x}) \equiv 1, p_1(\mathbf{x}), \ldots, p_{\bar{m}}(\mathbf{x})$ be given basis functions, where $\bar{m}+1$ denotes the number of the basis functions. Let us define

$$\tilde{p}_0(\mathbf{x};\bar{\mathbf{x}}) = \frac{1}{\left[\sum_{I\in\tau_{\mathbf{x}}} w(\mathbf{x},\mathbf{x}_I)\right]^{\frac{1}{2}}}, \tag{2}$$

where $\bar{\mathbf{x}}$ is the point in the domain of influence of $\mathbf{x}$.

Then we can generate new basis functions orthogonal to $\tilde{p}_0(\mathbf{x};\bar{\mathbf{x}})$ as

$$\tilde{p}_i(\mathbf{x},\bar{\mathbf{x}}) = p_i(\bar{\mathbf{x}}) - \mathcal{S}p_i(\mathbf{x}), \quad i = 1,2,\ldots,\bar{m}.$$

where $\mathcal{S}p_i$ is a linear operator defined as

$$\mathcal{S}p_i(\mathbf{x}) = \sum_{I\in\tau_{\mathbf{x}}} v(\mathbf{x},\mathbf{x}_I)p_i(\mathbf{x}_I),$$

and

$$v(\mathbf{x},\mathbf{x}_I) = \frac{w(\mathbf{x},\mathbf{x}_I)}{\sum_{J\in\tau_{\mathbf{x}}} w(\mathbf{x},\mathbf{x}_J)}.$$

The function $v(\mathbf{x},\mathbf{x}_I)$ has the following properties [23].

Lemma 1. *If the weight function of (1) is used, then* $v(\mathbf{x},\mathbf{x}_I) \in C^\infty(\bar{\mathscr{D}})$,

i) $v(\mathbf{x}_I,\mathbf{x}_J) = \delta_{IJ}, \quad \forall I,J \in \tau_{\mathbf{x}}$,

ii) $\sum_{I\in\tau_{\mathbf{x}}} v(\mathbf{x},\mathbf{x}_I) = 1, \quad \forall \mathbf{x} \in \tau_{\mathbf{x}}$,

iii) $0 \le v(\mathbf{x},\mathbf{x}_I) \le 1, \quad \forall \mathbf{x} \in \mathscr{D},\ v(\mathbf{x},\mathbf{x}_I) = 0 \Leftrightarrow (\mathbf{x} = \mathbf{x}_J\,, J \neq I)$,

iv) $\frac{\partial v(\mathbf{x}_I,\mathbf{x}_J)}{\partial \mathbf{x}} = 0, \quad \forall I,J \in \tau_{\mathbf{x}}$.

Lancaster and Salkauskas [23] defined a local approximation, i.e.,

$$u^h(\mathbf{x},\bar{\mathbf{x}}) = \tilde{p}_0(\mathbf{x};\bar{\mathbf{x}})a_0(\mathbf{x}) + \sum_{i=1}^{\bar{m}} \tilde{p}_i(\mathbf{x};\bar{\mathbf{x}})a_i(\mathbf{x}),$$

where $a_i(\mathbf{x})$, $i = 1,2,\ldots,\bar{m}$, are the unknown coefficients of basis functions. Then define the weighted discrete L_2 norm as

$$J(\mathbf{x}) = \sum_{I\in\tau_{\mathbf{x}}} w(\mathbf{x},\mathbf{x}_I)\left[u^h(\mathbf{x},\mathbf{x}_I) - u_I\right]^2, \tag{3}$$

where $w(\mathbf{x},\mathbf{x}_I)$, as shown in (1), is a weight function with compact support, $\mathbf{x}_I$ for $I \in \tau_{\mathbf{x}}$ are the nodes with domains of influence that cover the point $\mathbf{x}$, and $u_I = u(\mathbf{x}_I)$.

By minimizing the weighted discrete L_2 norm of (3) we have

$$\langle u(\cdot) - u^h(\mathbf{x}, \cdot), \tilde{p}_0 \rangle_{\mathbf{x}} = 0, \tag{4}$$

$$\langle u(\cdot) - u^h(\mathbf{x}, \cdot), \tilde{p}_i \rangle_{\mathbf{x}} = 0, \quad i = 1, 2, \ldots, \bar{m}. \tag{5}$$

In terms of the orthogonality, Eqs. (4) and (5) can be rewritten as

$$a_0(\mathbf{x}) = \langle u, \tilde{p}_0 \rangle_{\mathbf{x}},$$

$$a_0(\mathbf{x}) \langle \tilde{p}_0, \tilde{p}_i \rangle_{\mathbf{x}} + \sum_{i=1}^{\bar{m}} a_i(\mathbf{x}) \langle \tilde{p}_i, \tilde{p}_j \rangle_{\mathbf{x}} = \langle u, \tilde{p}_j \rangle_{\mathbf{x}}, \quad j = 1, 2, \ldots, \bar{m}. \tag{6}$$

According to (2) and the definition of inner product, we have

$$\tilde{p}_0(\mathbf{x}, \bar{\mathbf{x}}) a_0(\mathbf{x}) = \frac{1}{\left[\sum_{I \in \tau_{\mathbf{x}}} w(\mathbf{x}, \mathbf{x}_I)\right]^{\frac{1}{2}}} \langle u, \tilde{p}_0 \rangle_{\mathbf{x}} = \sum_{I \in \tau_{\mathbf{x}}} v(\mathbf{x}, \mathbf{x}_I) u_I = \mathcal{S}u.$$

Then (6) reduces to

$$\sum_{i=1}^{\bar{m}} a_i(\mathbf{x}) \langle \tilde{p}_i, \tilde{p}_j \rangle_{\mathbf{x}} = \langle u - \mathcal{S}u, \tilde{p}_j \rangle_{\mathbf{x}}, \quad j = 1, 2, \ldots, \bar{m}. \tag{7}$$

Lemma 2. *If the weight function of (1) is used, for all $\mathbf{x} \in \mathscr{D}$,*

$$\langle \mathcal{S}u, \tilde{p}_i \rangle_{\mathbf{x}} = 0, \quad i = 1, 2, \ldots, \bar{m}.$$

Proof. For the proof see [38]. □

According to lemma 2, Eq. (7) can be simplified as

$$\sum_{i=1}^{\bar{m}} a_i(\mathbf{x}) \langle \tilde{p}_i, \tilde{p}_j \rangle_{\mathbf{x}} = \langle u, \tilde{p}_j \rangle_{\mathbf{x}}, \quad j = 1, 2, \ldots, \bar{m}. \tag{8}$$

Eq. (8) is simpler than Eq. (7), and can be rewritten as

$$\mathbf{A}(\mathbf{x}) \mathbf{a}(\mathbf{x}) = \mathbf{F}_w(\mathbf{x}) \mathbf{u},$$

where

$$\mathbf{a}^T(\mathbf{x}) = (a_1(\mathbf{x}), a_2(\mathbf{x}) \ldots, a_{\bar{m}}(\mathbf{x})),$$

$$\mathbf{u}^T = (u_1, u_2, \ldots, u_N),$$

$$\mathbf{A}(\mathbf{x}) = \mathbf{F}_w(\mathbf{x}) \mathbf{F}^T(\mathbf{x}),$$

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} \tilde{p}_1(\mathbf{x};\mathbf{x}_1) & \tilde{p}_1(\mathbf{x};\mathbf{x}_2) & \dots & \tilde{p}_1(\mathbf{x};\mathbf{x}_N) \\ \tilde{p}_2(\mathbf{x};\mathbf{x}_1) & \tilde{p}_2(\mathbf{x};\mathbf{x}_2) & \dots & \tilde{p}_2(\mathbf{x};\mathbf{x}_N) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{p}_{\bar{m}}(\mathbf{x};\mathbf{x}_1) & \tilde{p}_{\bar{m}}(\mathbf{x};\mathbf{x}_2) & \dots & \tilde{p}_{\bar{m}}(\mathbf{x};\mathbf{x}_N) \end{bmatrix},$$

and $\mathbf{F}_w(\mathbf{x}) = \bar{\omega}_{kJ}(\mathbf{x})_{\bar{m}\times N}$ is a $\bar{m} \times N$ matrix, and

$$\bar{\omega}_{kJ}(\mathbf{x}) = \begin{cases} w(\mathbf{x},\mathbf{x}_J)\tilde{p}_k(\mathbf{x};\mathbf{x}_J), & \mathbf{x} \neq \mathbf{x}_J, \\ \\ \sum\limits_{I\in\tau_{\mathbf{x}}, I\neq J} w(\mathbf{x}_J,\mathbf{x}_I)\left[p_k(\mathbf{x}_J) - p_k(\mathbf{x}_I)\right], & \mathbf{x} = \mathbf{x}_J. \end{cases}$$

Then we can obtain

$$\mathbf{a}(\mathbf{x}) = \mathbf{A}^{-1}(\mathbf{x})\mathbf{F}_w(\mathbf{x})\mathbf{u}.$$

Then the local approximation function can be obtained as

$$u^h(\mathbf{x},\bar{\mathbf{x}}) = \mathcal{S}u + \sum_{i=1}^{\bar{m}} a_i(\mathbf{x})\tilde{p}_i(\mathbf{x};\bar{\mathbf{x}}).$$

Thus the global interpolating approximation function of $u(\mathbf{x})$ can be obtained as

$$u^h(\mathbf{x}) = \mathcal{S}u + \sum_{i=1}^{\bar{m}} a_i(\mathbf{x})g_i(\mathbf{x}) \equiv \Phi(\mathbf{x})\mathbf{u} = \sum_{I=1}^{N} \phi_I(\mathbf{x})u(\mathbf{x}_I), \tag{9}$$

where $\Phi(\mathbf{x})$ is a matrix of shape functions,

$$\Phi(\mathbf{x}) = (\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \dots, \phi_N(\mathbf{x})) = \mathbf{v}^T + \mathbf{p}^T(\mathbf{x})\mathbf{A}^{-1}(\mathbf{x})\mathbf{F}_w(\mathbf{x}), \tag{10}$$

where

$$\mathbf{v}^T = (v(\mathbf{x},\mathbf{x}_1), v(\mathbf{x},\mathbf{x}_2), \dots, v(\mathbf{x},\mathbf{x}_N)),$$
$$\mathbf{p}^T(\mathbf{x}) = (g_1(\mathbf{x}), g_2(\mathbf{x}), \dots, g_{\bar{m}}(\mathbf{x})),$$
$$g_i(\mathbf{x}) = p_i(\mathbf{x}) - \mathcal{S}p_i(\mathbf{x}).$$

Then the first partial derivatives of the shape functions of the IMLS method can be obtained as

$$\begin{aligned} \phi_{,i}(\mathbf{x}) = \mathbf{v}^T_{,i} &+ \mathbf{p}^T_{,i}(\mathbf{x})\mathbf{A}^{-1}(\mathbf{x})\mathbf{F}_w(\mathbf{x}) \\ &+ \mathbf{p}^T(\mathbf{x})\mathbf{A}^{-1}(\mathbf{x})\mathbf{F}_{w,i}(\mathbf{x}) + \mathbf{p}^T(\mathbf{x})\mathbf{A}^{-1}_{,i}(\mathbf{x})\mathbf{F}_w(\mathbf{x}), \end{aligned} \tag{11}$$

where

$$\mathbf{F}_w(\mathbf{x}) = \bar{\omega}_{kJ,i}(\mathbf{x})_{\bar{m}\times N},$$

$$\bar{\omega}_{kJ,i}(\mathbf{x}) = \begin{cases} w_{,i}(\mathbf{x},\mathbf{x}_J)\tilde{p}_k(\mathbf{x};\mathbf{x}_J) + w(\mathbf{x},\mathbf{x}_J)\tilde{p}_{k,i}(\mathbf{x};\mathbf{x}_J)(\mathbf{x}), & \mathbf{x} \neq \mathbf{x}_J, \\ \sum_{I\in\tau_{\mathbf{x}}, I\neq J} w_{,i}(\mathbf{x},\mathbf{x}_I)\left[p_k(\mathbf{x}_J) - p_k(\mathbf{x}_I)\right], & \mathbf{x} = \mathbf{x}_J, \end{cases}$$

$$\mathbf{A}_{,i}^{-1}(\mathbf{x}) = -\mathbf{A}^{-1}(\mathbf{x})\mathbf{A}_{,i}(\mathbf{x})\mathbf{A}^{-1}(\mathbf{x}),$$

and the notation "$_{,i}$" is the first derivative with respect to the i's element of $\mathbf{x}$.

Eq. (10) is the shape function of the IMLS method.

3 The stabilized interpolating moving least square method

A disadvantage of the IMLS method is bad-conditioning of the moment matrix $\mathbf{A}$ when the fill-distance is tend to zero. when the condition number of the moment matrix is tend to zero, it reduces the accuracy of the method. for overcoming this difficulty Xiaolin Li and Qingqing Wang proposed Stabilized IMLS method [25].

Let $\mathbf{X} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$ be a set of all nodes in the bounded domain $\mathscr{D} \subset \mathbb{R}^n$ where N is the number of nodes. The fill distance is defined as

$$h_{\mathbf{X},\mathscr{D}} = \sup_{\mathbf{x}\in\mathscr{D}} \min_{1\leq j\leq N} \|\mathbf{x} - \mathbf{x}_j\|_2,$$

and the separation distance is defined as

$$q_{\mathbf{X}} = \frac{1}{2}\min_{i\neq j}\|\mathbf{x} - \mathbf{x}_j\|_2.$$

$\mathbf{X}$ is said to be quasi-uniform with respect to a positive constant C if

$$q_{\mathbf{X}} \leq h_{\mathbf{X},\mathscr{D}} \leq C q_{\mathbf{X}}. \tag{12}$$

Assumption 1. *Assume that the data site X used in this paper satisfies the quasi-uniform condition (12) with the same quasi-uniform constant C.*

Assumption 2. *For any $\mathbf{x} \in \mathscr{D}$, assume that the matrix $\mathbf{A}$ is invertible.*

Theorem 1. *Suppose that Assumptions 1 and 2 hold. Then for the moment matrix $\mathbf{A}(x)$ in the IMLS method, there exists a number $C_d(\mathbf{x}, n, m)$ independent of $q_{\mathbf{X}}$ such that the determinant of $\mathbf{A}(x)$ is*

$$\det(\mathbf{A}(x)) = C_d(\mathbf{x}, n, m){q_{\mathbf{X}}}^{2\sum_{j=1}^{m} j}, \qquad \forall \mathbf{x} \in \mathscr{D},$$

In addition, there is a constant $h_0 > 0$, for any $q_{\mathbf{x}} \leq h_0$, we have a number $C_c(\mathbf{x}, n, m)$ independent of $q_{\mathbf{x}}$ such that the condition number of $\mathbf{A}(x)$ in the L^2 norm is

$$\operatorname{cond}(\mathbf{A}(x)) = C_c(\mathbf{x}, n, m) q_{\mathbf{x}}^{-2\hat{m}} \qquad \forall \mathbf{x} \in \mathscr{D}.$$

Proof. for the proof see [25]. □

Remark 1. *From Theorem 1 we can observe that*

$$\det(\mathbf{A}(x)) \mapsto 0 \text{ and } \operatorname{cond}(\mathbf{A}(x)) \mapsto \infty \text{ as } q_{\mathbf{x}} \mapsto 0.$$

These results indicate that, for $q_{\mathbf{x}}$ small enough, the moment matrix $\mathbf{A}(x)$ in the IMLS method becomes poorly conditioned, and the accuracy of computing the coefficient vector $\mathbf{a}(\mathbf{x})$ and thus the shape function $\phi_i(\mathbf{x})$ decreases. Since the condition number increases with the degree of ill-conditioning, we can finally conclude that the stability of the IMLS method decreases as the separation distance $q_{\mathbf{x}}$ decreases.

To improve the stability of the IMLS method, the following basis functions are used in the stabilized algorithm of the IMLS method,

$$\mathbf{p}^s(\mathbf{x}) \triangleq [\mathbf{p}_0^s(\mathbf{x}), \mathbf{p}_1^s(\mathbf{x}), \ldots, \mathbf{p}_m^s(\mathbf{x})]^T = \mathbf{p}\left(\frac{\mathbf{x} - \mathbf{x}^e}{q_{\mathbf{x}}}\right), \mathbf{x} \in \mathscr{D},$$

where $\mathbf{x}^e$ is fixed and depends on the evaluation point to be considered. If $\mathbf{x}$ is the evaluation point, the best result will be obtained if we finally set $\mathbf{x}^e = \mathbf{x}$. In all what follow we will set $\mathbf{x}^e = \mathbf{x}$ for the evaluation point $\mathbf{x}$. For example, in a one-dimensional domain ($x \in \mathbb{R}$)

$$\mathbf{p}^s(x) = \left[1, \frac{x - x^e}{q_x}, \frac{(x - x^e)^2}{q_x^2}, \ldots, \frac{(x - x^e)^{\hat{m}}}{q_x^{\hat{m}}}\right]^T,$$

whereas in a two-dimensional domain ($\mathbf{x} = (x_1, x_2)^T \in \mathbb{R}^2$),

$$\mathbf{p}^s(\mathbf{x}) = \begin{cases} \left[1, \frac{\mathbf{x}_1 - \mathbf{x}_1{}^e}{q_{\mathbf{x}}}, \frac{\mathbf{x}_2 - \mathbf{x}_2{}^e}{q_{\mathbf{x}}}\right]^T, & \hat{m} = 1, \\ \left[1, \frac{\mathbf{x}_1 - \mathbf{x}_1{}^e}{q_{\mathbf{x}}}, \frac{\mathbf{x}_2 - \mathbf{x}_2{}^e}{q_{\mathbf{x}}}, \frac{(\mathbf{x}_1 - \mathbf{x}_1{}^e)^2}{q_{\mathbf{x}}^2}, \frac{(\mathbf{x}_1 - \mathbf{x}_1{}^e)(\mathbf{x}_2 - \mathbf{x}_2{}^e)}{q_{\mathbf{x}}^2}, \frac{(\mathbf{x}_2 - \mathbf{x}_2{}^e)^2}{q_{\mathbf{x}}^2}\right]^T, & \hat{m} = 2. \end{cases}$$

If we use the new basis $\mathbf{p}^s(\mathbf{x})$ the moment matrix of the stabilized IMLS method is $\mathbf{A}^s$ and the stabilized IMLS shape function is

$$\Phi^s(\mathbf{x}) = (\phi_1^s(\mathbf{x}), \phi_2^s(\mathbf{x}), \ldots, \phi_N^s(\mathbf{x})) = (\mathbf{v}^s)^T + (\mathbf{P}^s)^T(\mathbf{x})(\mathbf{A}^s)^{-1}(\mathbf{x})\mathbf{F}^s{}_w(\mathbf{x}). \tag{13}$$

Theorem 2. *Suppose that assumptions 1 and 2 hold. Then for the moment matrix* $\mathbf{A}^s$ *in the stabilized IMLS method, there exist numbers* $C_d^s(\mathbf{x}, n, m)$ *and* $C_c^s(\mathbf{x}, n, m)$ *independent of* $q_{\mathbf{x}}$ *such that*

$$\det(\mathbf{A}^s(x)) = C_d^s(\mathbf{x}, n, m), \qquad \forall \mathbf{x} \in \mathscr{D},$$
$$\operatorname{cond}(\mathbf{A}^s(x)) = C_c^s(\mathbf{x}, n, m), \qquad \forall \mathbf{x} \in \mathscr{D}.$$

Proof. for the proof see [25]. □

Remark 2. *From theorem 2 we can observe that both the determinant and the condition number of the moment matrix* $\mathbf{A}^s(x)$ *in the stabilized algorithm are invariable with respect to the separation distance* $q_{\mathbf{x}}$. *Therefore, the stabilized IMLS (SIMLS) method has better stability and convergence than the original IMLS method.*

4 Spectral approximations and error estimates

From [5] we have the following error estimate for the spectral approximation of the elliptic SPDE

$$-\Delta u(x, \xi) = f(u(x, \xi)) + \dot{W}^Q(x, \xi) \quad x \in \mathscr{D}, \xi \in \Omega, \tag{14}$$
$$u(x, \xi) = \bar{u}(x, \xi), \qquad x \in \partial\mathscr{D}, \xi \in \Omega, \tag{15}$$

where $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space and $\mathscr{D} \subseteq \mathbb{R}^d$ be a bounded domain with regular boundary $\partial\mathscr{D}$, $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function, and $\dot{W}^Q$ is a class of centered Gaussian noises with covariance operator Q.

For $r \in \mathbb{N}$, we use $(\mathbb{H}^r, \|.\|_r)$ to denote the usual Sobolev space

$$\mathbb{H}^r := \left\{ v : \|v\|_{\mathbb{H}^r} := \left(\sum_{|k| \le r} \|D^k v\|^2 \right)^{\frac{1}{2}} < \infty \right\}.$$

When $r = 0$, $\mathbb{H}^0 := \mathbb{H}$ is the space of square integrable functions on $\mathscr{D}$, whose inner product and norm are denoted by $(.,.)$ and $\|.\|$, respectively. We also use $\mathbb{H}_0^1$ to denote the subspace of $\mathbb{H}^1$ whose elements vanish on $\partial\mathscr{D}$. For $s \in \mathbb{R}$, we use $(\dot{\mathbb{H}}^s, \|.\|_s)$ to denote the interpolation space

$$\dot{\mathbb{H}}^s := \left\{ v : \|v\|_s := \left(\sum_{k \in \mathbb{N}^+} \lambda_k^s (v, \varphi_k)^2 \right)^{\frac{1}{2}} < \infty \right\},$$

where $\{(\lambda_k, \varphi_k)\}_{k \in \mathbb{N}^+}$ is an eigensystem of the negative Dirichlet Laplacian.

Recall that an $\mathbb{H}$-valued random field u is said to be a solution of SPDE (14) if

$$u = A^{-1} f(u) + A^{-1} \dot{W}^Q, \tag{16}$$

where $A^{-1} = (-\Delta)^{-1}$ is the inverse of negative Dirichlet Laplacian.

The centered Gaussian noise $\dot{W}^Q$ is uniquely determined by its covariance operator Q. Assume that Q has $\{(\sigma_k, \psi_k)\}_{k=1}^{\infty}$ ass its eigensystem, i.e.,

$$Q\psi_m = \sigma_m \psi_m, \quad m \in \mathbb{N}^+,$$

where $\{\psi_k\}_{k=1}^{\infty}$ form a complete orthonormal basis in $\mathbb{H}$. Then we have the following expansion for the infinite dimensional noise $\dot{W}^Q$:

$$\dot{W}^Q(\omega) = \sum_{m=1}^{\infty} Q^{\frac{1}{2}} \psi_m \eta_m(\omega), \quad \omega \in \Omega, \tag{17}$$

where $\{\eta_m\}_{m=1}^{\infty}$ are independent and normal random variables [12].

Consider the following assumptions:

Assumption 3. *f is Lipschitz continuous, i.e.,*

$$\|f\|_{Lip} := \sup_{u,v \in \mathbb{R}, u \neq v} \frac{|f(u) - f(v)|}{|u - v|} < \infty.$$

We further assume that the Lipschitz constant $\|f\|_{Lip}$ is smaller than the positive constant γ in the Poincaré inequality:

$$\|\nabla v\|^2 \geq \gamma \|v\|^2, \quad \forall v \in \mathbb{H}_0^1.$$

We remark that the well-posedness of (14) is also valid for general assumptions on f possibly depending on the spatial variable which is proposed in [6], i.e., there exist two positive constants $L_1 < \gamma$ and L_2 such that for any $x \in \mathscr{D}$ and any $u, v \in \mathbb{R}$,

$$(f(x,u) - f(x,v), u - v) \geq -L_1 |u - v|^2$$

and

$$|f(x,u) - f(x,v)| \leq L_2 (1 + |u - v|).$$

In that case, all the convergence rates halve.

Assumption 4. *There exists a parameter $\beta \in [0, 2]$ such that*

$$\|A^{\frac{\beta-2}{2}}\|_{\mathcal{L}_2^0} < \infty,$$

where $\mathcal{L}_2^0 := HS(Q^{\frac{1}{2}}(\mathbb{H}), \mathbb{H})$ denotes the space of Hilbert-Schmidt operators from $Q^{\frac{1}{2}}(\mathbb{H})$ to $\mathbb{H}$ and $\|.\|_{\mathcal{L}_2^0}$ denotes the corresponding norm.

Let $\mathbb{P}_N : \mathbb{H} \to V_N = \text{span}\{\varphi_m\}_{m=1}^N$ be the projection operator from $\mathbb{H}$ to V_N such that $(\mathbb{P}_N u, v) = (u, v)$ for any $u \in \mathbb{H}, v \in V_N$, and let u_N be the solution of the SPDE with the noise term in (16) replaced by its spectral projection:

$$u_N = A^{-1} f(u_N) + A^{-1}\mathbb{P}_N \dot{W}^Q, \quad N \in \mathbb{N}^+. \tag{18}$$

Theorem 3. *Let $p \geq 1$ and Assumptions 3 and 4 hold. Then SPDE (14) possesses a unique solution $u \in L^p(\Omega; \mathbb{H}^\beta)$.*

Proof. for the proof see [5]. □

Theorem 4. *Let $P \geq 1$ and Assumptions 3 and 4 hold and u and $u_N, N \in \mathbb{N}^+$ be the solutions of (16) and (18), respectively. Then $u_N \in L^p(\Omega; \dot{\mathbb{H}}^2)$ and there exist a constant C independent of N such that*

$$\mathbb{E}\left[\|u_N\|_2^p\right] \leq C\lambda_N^{\frac{(2-\beta)p}{2}}(1 + \|A^{\frac{\beta-2}{2}}\|_{\mathcal{L}_2^0}^p). \tag{19}$$

where $\mathbb{E}[.]$ is the expected value. Assume furthermore that f has bounded derivatives up to order $r-1$ with $r \geq 2$, with its first derivative being bounded by γ, then $u_N \in L^p(\Omega; \dot{\mathbb{H}}^{r+1})$ and

$$\mathbb{E}\left[\|u_N\|_{r+1}^p\right] \leq C\lambda_N^{\frac{(r+1-\beta)p}{2}}(1 + \|A^{\frac{\beta-2}{2}}\|_{\mathcal{L}_2^0}^p). \tag{20}$$

Moreover,

$$(\mathbb{E}\left[\|u - u_N\|^p\right])^{\frac{1}{p}} \leq C\lambda_{N+1}^{-\frac{\beta}{2}}(1 + \|A^{\frac{\beta-2}{2}}\|_{\mathcal{L}_2^0}^p). \tag{21}$$

Proof. for the proof see [5]. □

5 Error estimates of element free Galerkin method for stochastic elliptic equations

Let

$$V_\rho = \{v | v \in \text{Span}\{\phi_I\}, v = 0 \quad \text{on} \quad \partial\mathscr{D}\},$$

where ϕ_I is the shape function of element free Galerkin method. The variational problem of (18) is to find a $u_N \in \mathbb{H}^1$ such that

$$a(u_N, v) = (f, v) + (\mathbb{P}_N \dot{W}^Q, v) \quad \forall v \in \mathbb{H}_0^1, \tag{22}$$

where

$$a(u,v)=\int_{\mathscr{D}}\nabla u\times\nabla v\mathrm{d}\mathscr{D}+\int_{\mathscr{D}}uv\mathrm{d}\mathscr{D},$$
$$(f,v)=\int_{\mathscr{D}}fv\mathrm{d}\mathscr{D},$$
$$(\mathbb{P}_N\dot{W}^Q,v)=\int_{\mathscr{D}}\mathbb{P}_N\dot{W}^Q v\mathrm{d}\mathscr{D}.$$

We can prove that the bilinear form $a(\cdot,\cdot)$ on Sobolev space $\mathbb{H}_0^1$ is bounded and coercive that is there exist constants $\bar{\alpha}>0,\bar{M}<\infty$ such that

$$|a(u,v)|\leq\bar{M}\|u\|_{\mathbb{H}^1}\|v\|_{\mathbb{H}^1}\quad\forall u,v\in\mathbb{H}_0^1,\tag{23}$$
$$a(u,v)\geq\bar{\alpha}\|v\|^2_{\mathbb{H}^1}\quad\forall v\in\mathbb{H}_0^1.\tag{24}$$

The IEFG method according to (14) is to find $u_N^\rho\in V_\rho$ such that

$$a(u_N^\rho,v)=(f,v)+(\mathbb{P}_N\dot{W}^Q,v)\quad\forall v\in V_\rho.\tag{25}$$

Theorem 5. *Suppose that u_N,u_N^ρ be the solutions of variational problem (22) and IEFG method (25) respectively, then there exist*

i) $a(u_N-u_N^\rho,v)=0\quad\forall v\in V_\rho,$

ii) $a(u_N-u_N^\rho,u_N-u_N^\rho)=\inf_{v\in V_\rho}a(u_N-v,u_N-v),$

iii) $\|u_N-u_N^\rho\|_{\mathbb{H}^1}\leq C\inf_{v\in V_\rho}\|u_N-v\|_{\mathbb{H}^1}.$

Proof. By subtracting (22) and (25) for all $v\in V_\rho$ we have

$$a(u_N-u_N^\rho,v)=a(u_N,v)-a(u_N^\rho,v)=0$$

Let us define the energy norm $\|u\|_E=\sqrt{a(u,u)}$. Then we have

$$\begin{aligned}\|u_N-u_N^\rho\|_E^2&=a(u_N-u_N^\rho,u_N-u_N^\rho)\\&=a(u_N-u_N^\rho,u_N-v)+a(u_N-u_N^\rho,v-u_N^\rho)\\&=a(u_N-u_N^\rho,u_N-v)\quad(v-u_N^\rho\in V_\rho)\\&\leq\|u_N-u_N^\rho\|_E\|u_N-v\|_E.\quad\text{(Schwarz' inequality)}\end{aligned}$$

If $\|u_N-u_N^\rho\|_E\neq0$, we can divide by it to obtain $\|u_N-u_N^\rho\|_E\leq\|u_N-v\|_E$, for any $v\in V_\rho$. If $\|u_N-u_N^\rho\|_E=0$, this inequality is trivial. Taking the infimum over $v\in V_\rho$ yields

$$\|u_N-u_N^\rho\|_E\leq\inf_{v\in V_\rho}\|u_N-v\|_E.$$

Since $v \in V_\rho$, we have

$$\inf_{v\in V_\rho} \|u_N - v\|_E \le \|u_N - u_N^\rho\|_E.$$

Therefore,

$$\|u_N - u_N^\rho\|_E = \inf_{v\in V_\rho} \|u_N - v\|_E,$$

i.e.,

$$a(u_N - u_N^\rho, u_N - u_N^\rho) = \inf_{v\in V_\rho} a(u_N - v, u_N - v).$$

For the third case we have

$$\begin{aligned}
\bar{\alpha}\|u_N - u_N^\rho\|_{\mathbb{H}^1}^2 &\le a(u_N - u_N^\rho, u_N - u_N^\rho) \quad \text{by (24)}\\
&= a(u_N - u_N^\rho, u_N - v) + a(u_N - u_N^\rho, v - u_N^\rho)\\
&= a(u_N - u_N^\rho, u_N - v) \quad (v - u_N^\rho \in V_\rho)\\
&\le \bar{M}\|u_N - u_N^\rho\|_{\mathbb{H}^1}\|u_N - v\|_{\mathbb{H}^1} \quad \text{by (23)}.
\end{aligned}$$

Therefore,

$$\|u_N - u_N^\rho\|_{\mathbb{H}^1} \le \frac{\bar{M}}{\bar{\alpha}}\|u_N - v\|_{\mathbb{H}^1}.$$

By taking the infimum over all $v \in V_\rho$ we have

$$\|u_N - u_N^\rho\|_{\mathbb{H}^1} \le \frac{\bar{M}}{\bar{\alpha}} \inf_{v\in V_\rho} \|u_N - v\|_{\mathbb{H}^1}.$$

This completes the proof. □

We define the interpolation operator of the IMLS method as

$$\mathcal{I}u = \mathcal{S}u + \sum_{i=1}^{m} a_i(x)g_i(x) = \Phi^T u.$$

Form [24,37] we have if $u \in \mathbb{H}^{m+1}$, then there exist bounded function $C'_k(x)$ and constant C_k such that

$$\begin{aligned}
&\frac{\partial^{|k|}}{\partial^{k_1}\partial^{k_2}\dots\partial^{k_n}}\Phi_I(x) = C'_k(x)\rho_x^{-|k|},\\
&\|\mathcal{I}u - u\|_{\mathbb{H}^k} \le C_k\rho^{m+1-|k|}\|u\|_{\mathbb{H}^{m+1}},
\end{aligned} \tag{26}$$

where $k = (k_1, k_2, \dots, k_n), 0 \le |k| \le m$.

Let $\|u_N - u_N^\rho\|_E^2 = a(u_N - u_N^\rho, u_N - u_N^\rho)$ then the following error estimates of the energy and $\mathbb{H}^1$ norms can be obtained.

Theorem 6. *Suppose that $u \in \mathbb{H}^{m+1}$ and u_N, u_N^ρ be the solutions of the problem (22) and (25) respectively, then there exist C_1 and C_2 which are independent with the parameter ρ such that*

$$\|u_N - u_N^\rho\|_E \le C_1 \rho^m \|u\|_{\mathbb{H}^{m+1}},$$
$$\|u_N - u_N^\rho\|_{\mathbb{H}^1} \le C_2 \rho^m \|u\|_{\mathbb{H}^{m+1}}.$$

Proof. From Theorem 5 and (26) we have

$$\begin{aligned}
\|u_N - u_N^\rho\|_E &= a(u_N - u_N^\rho, u_N - u_N^\rho) \\
&= \inf_{v \in V_\rho} a(u_N - v, u_N - v) \\
&\le a(u_N - \mathcal{I}u_N, u_N - \mathcal{I}u_N) \\
&\le \bar{M}\|u_N - \mathcal{I}u_N\|_{\mathbb{H}^1}^2 \le C_1 \rho^{2m} \|u_N\|_{\mathbb{H}^{m+1}}^2, \\
\|u_N - u_N^\rho\|_{\mathbb{H}^1} &\le C \inf_{v \in V_\rho} \|u_N - v\|_{\mathbb{H}^1} \\
&\le C\|u_N - \mathcal{I}u_N\|_{\mathbb{H}^1} \\
&\le C_2 \rho^m \|u_N\|_{\mathbb{H}^{m+1}}.
\end{aligned}$$

□

Theorem 7. *Suppose that $u \in \mathbb{H}^{m+1}$. Let u_N, u_N^ρ be the solutions of the problem (22) and (25), respectively, then there exist a constant C which is independent of the parameter ρ such that*

$$\|u_N - u_N^\rho\|_{L^2} \le C \rho^{m+1} \|u_N\|_{\mathbb{H}^{m+1}}.$$

Proof. For all $g \in L^2$, let $\varphi \in \mathbb{H}_0^1 \cap \mathbb{H}^2$ be the solutions of

$$a(\varphi, v) = (g, v) \quad \forall v \in \mathbb{H}_0^1,$$

then we have following estimates,

$$\|\varphi\|_{\mathbb{H}^2} \le \|g\|_{L^2}. \tag{27}$$

Let $v = u_N - u_N^\rho$ then we have

$$a(\varphi, u_N - u_N^\rho) = (g, u_N - u_N^\rho), \tag{28}$$

for arbitrary $v_\rho \in V_\rho$ from Theorem 5 we have

$$a(v_\rho, u_N - u_N^\rho) = 0. \tag{29}$$

It follows from (28) and (29) that

$$a(\varphi - v_\rho, u_N - u_N^\rho) = (g, u_N - u_N^\rho). \tag{30}$$

Let $v_\rho = \mathcal{I}\varphi$ and $g = u_N - u_N^\rho$ then there exist

$$\begin{aligned} \|u_N - u_N^\rho\|_{L^2}^2 &= (u_N - u_N^\rho, u_N - u_N^\rho) \\ &= a(\varphi - \mathcal{I}\varphi, u_N - u_N^\rho) \quad \text{by (30)} \\ &\le \bar{M}\|\varphi - \mathcal{I}\varphi\|_{\mathbb{H}^1}\|u_N - u_N^\rho\|_{\mathbb{H}^1} \quad \text{by (23)} \\ &\le \bar{M}C_1\rho\|\varphi\|_{\mathbb{H}^2}C_2\rho^m\|u_N\|_{\mathbb{H}^{m+1}}. \quad \text{by (26) and theorem 6} \end{aligned} \tag{31}$$

From (27) and (31), we have

$$\|u_N - u_N^\rho\|_{L^2} \le C\rho^{m+1}\|u_N\|_{\mathbb{H}^{m+1}}.$$

Then this theorem is proved. □

Theorem 8. *Let $p \ge 1$ and assumption 3 and 4 hold, and u_N, u_N^ρ be the solutions of (22) and (25) respectively. Then there exist a constant C independent of ρ and λ_N such that*

$$\left(\mathbb{E}\|u_N - u_N^\rho\|^p\right)^{\frac{1}{p}} \le C\rho^2\lambda_N^{\frac{2-\beta}{2}}\left(1 + \|A^{\frac{\beta-2}{2}}\|_{\mathcal{L}_2^0}\right). \tag{32}$$

Assume furthermore that f has bounded derivatives up to order $m-1$ for some $m \ge 2$, with its first derivative being bounded by γ then

$$\left(\mathbb{E}\|u_N - u_N^\rho\|^p\right)^{\frac{1}{p}} \le C\rho^{m+1}\lambda_N^{\frac{m+1-\beta}{2}}\left(1 + \|A^{\frac{\beta-2}{2}}\|_{\mathcal{L}_2^0}\right). \tag{33}$$

Proof. We have from Theorem 8 that

$$\|u_N - u_N^\rho\|_{L^2} \le C\rho^2\|u_N\|_{\mathbb{H}^2} \tag{34}$$

and

$$\|u_N - u_N^\rho\|_{L^2} \le C\rho^{m+1}\|u_N\|_{\mathbb{H}^{m+1}} \tag{35}$$

Then by (19) and (34) one can see that (32) is true. Also by (20) and (35) we can see that (33) is valid. □

6 The stabilized IMLS based EFG method for stochastic elliptic equations

In this section, the element free Galerkin method based on the SIMLS method will be presented for stochastic elliptic equations. The advantage of the IEFG method is that the essential boundary conditions can be applied directly and easily.

Consider the following stochastic elliptic equations

$$-\nabla^2 u(\mathbf{x}) + bu(\mathbf{x}) = g(\mathbf{x}) + \dot{W}^Q(\mathbf{x}) \quad \forall \mathbf{x} \in \mathscr{D}, \tag{36}$$

with boundary conditions of the Dirichlet type, i.e.,

$$u(\mathbf{x}) = \bar{u}(\mathbf{x}) \quad \forall \mathbf{x} \in \partial\mathscr{D}, \tag{37}$$

where $u(\mathbf{x})$ is an unknown function, $g \in \mathrm{L}^2(\mathscr{D})$ is a Lipschitz continuous function, b is a positive constant, $\bar{u}$ is known and $\dot{W}^Q$ is a class of Gaussian noises with mean zero and covariance operator Q.

The Galerkin weak form of (36) and (37) is

$$\int_{\mathscr{D}} \delta(\nabla u)^T \cdot \nabla u \mathrm{d}\mathscr{D} + b\int_{\mathscr{D}} \delta u^T \cdot u \mathrm{d}\mathscr{D} = \int_{\mathscr{D}} \delta u^T \cdot g(\mathbf{x}) \mathrm{d}\mathscr{D} + \int_{\mathscr{D}} \delta u^T \cdot \dot{W}^Q(\mathbf{x}) \mathrm{d}\mathscr{D}. \tag{38}$$

From the SIMLS method, the unknown function $u(\mathbf{x})$ at arbitrary point $\mathbf{x}$ can be expressed as

$$u(\mathbf{x}) \approx u^h(\mathbf{x}) = \Phi^s(\mathbf{x})\mathbf{u} = \sum_{I=1}^{N} \phi_I^s(\mathbf{x}) u_I, \tag{39}$$

where N is the number of nodes whose compact support domains cover the point $\mathbf{x}$.

Substituting (39) into (38) yields

$$\int_{\mathscr{D}} \delta\mathbf{u}^T(\mathbf{B}\mathbf{B}^T)\mathbf{u}\mathrm{d}\mathscr{D} + b\int_{\mathscr{D}} \delta\mathbf{u}^T(\Phi^s)^T(\mathbf{x}) \cdot \Phi^s(\mathbf{x})\mathbf{u}\mathrm{d}\mathscr{D} =$$
$$\int_{\mathscr{D}} \delta\mathbf{u}^T(\Phi^s)^T(\mathbf{x}) \cdot g(\mathbf{x})\mathrm{d}\mathscr{D} + \int_{\mathscr{D}} \delta\mathbf{u}^T(\Phi^s)^T(\mathbf{x}) \cdot \dot{W}^Q(\mathbf{x})\mathrm{d}\mathscr{D}.$$

where

$$B^T = \begin{bmatrix} \nabla\phi_1^s(x) & \nabla\phi_2^s(x) & \cdots & \nabla\phi_N^s(x) \end{bmatrix},$$

Because the nodal test function $\delta\mathbf{u}$ is arbitrary, the final discretized equation is obtained as

$$\mathbf{K}\mathbf{u} = \mathbf{F}, \tag{40}$$

where

$$\mathbf{K}_{IJ} = \int_{\mathscr{D}} \nabla\phi_I^s \cdot \nabla\phi_J^s \mathrm{d}\mathscr{D} + b\int_{\mathscr{D}} \phi_I^s \cdot \phi_J^s \mathrm{d}\mathscr{D}, \tag{41}$$

$$\mathbf{F}_I = \int_{\mathscr{D}} \Phi_I^s(\mathbf{x}) \cdot g(\mathbf{x})\mathrm{d}\mathscr{D} + \int_{\mathscr{D}} \Phi_I^s(\mathbf{x}) \cdot \dot{W}^Q(\mathbf{x})\mathrm{d}\mathscr{D}. \tag{42}$$

The matrix $\mathbf{K}$ is an invertible matrix since,

$$\mathbf{K} = \int_{\mathscr{D}} (\nabla\phi^s)^T \cdot \nabla\phi^s \mathrm{d}\mathscr{D} + b\int_{\mathscr{D}} (\phi^s)^T \cdot \phi^s \mathrm{d}\mathscr{D},$$

and for an arbitrary vector z we have,

$$\begin{aligned} z^T\mathbf{K}z &= z^T \left(\int_{\mathscr{D}} (\nabla\phi^s)^T \cdot \nabla\phi^s \mathrm{d}\mathscr{D} + b\int_{\mathscr{D}} (\phi^s)^T \cdot \phi^s \mathrm{d}\mathscr{D} \right) z \\ &= \int_{\mathscr{D}} z^T (\nabla\phi^s)^T \cdot \nabla\phi^s z \mathrm{d}\mathscr{D} + b\int_{\mathscr{D}} z^T (\phi^s)^T \cdot \phi^s z \mathrm{d}\mathscr{D} \\ &= \int_{\mathscr{D}} (\nabla\phi^s z)^T \cdot \nabla\phi^s z \mathrm{d}\mathscr{D} + b\int_{\mathscr{D}} (\phi^s z)^T \cdot \phi^s z \mathrm{d}\mathscr{D} \\ &= \int_{\mathscr{D}} \|\nabla\phi^s z\|_2^2 \mathrm{d}\mathscr{D} + b\int_{\mathscr{D}} \|\phi^s z\|_2^2 \mathrm{d}\mathscr{D}, \end{aligned}$$

then due to positiveness of constant b we have $z^T\mathbf{K}z > 0$, thus the matrix $\mathbf{K}$ is positive definite and also invertible. Therefore the solution of (40) exists and is unique.

Although the weight function is singular at nodes, the shape functions of the SIMLS method are not singular at any point. The integration in the meshless method based on the SIMLS method can be obtained by the general Gauss quadrature. Since the shape function of the SIMLS method satisfies the property of the Kronecker δ function, the essential boundary conditions can be applied directly into (40), and then we can obtain the numerical solutions at nodes.

Thus the stabilized IMLS based EFG method is presented for stochastic elliptic equations.

Algorithm 1. *This algorithm computes the solution of (36) by the presented method for arbitrary points in the domain of the solution.*

i) Determine scattered points.

ii) Compute ϕ_i and $\nabla\phi_i, i = 1, 2, \ldots, N$ for scattered points by equation (13).

iii) Assemble matrices K, F via equations (41) and (42).

iv) Enforce boundary conditions into the system (40).

v) Solving the modified system $Ku = F$.

vi) Compute the solution in arbitrary points via equation (9).

7 Numerical results

Before we start with our numerical experiments, let us briefly explain how we approximate the noise present in the above stochastic partial differential equation. From (17) we can write

$$\dot{W}^Q(\mathbf{x}, \omega) = \sum_{j=1}^{\infty} \sigma_j^{\frac{1}{2}} \psi_j(\mathbf{x}) \eta_j(\omega), \quad \mathbf{x} \in \mathscr{D}, \omega \in \Omega,$$

Then we have

$$\int_{\mathscr{D}} \Phi_I^s(\mathbf{x}) \cdot \dot{W}^Q(\mathbf{x}, \omega) \mathrm{d}\mathscr{D} = \sum_{j=1}^{\infty} \sigma_j^{\frac{1}{2}} \eta_j(\omega) \int_{\mathscr{D}} \Phi_I^s(\mathbf{x}) \cdot \psi_j(\mathbf{x}) \mathrm{d}\mathscr{D}.$$

One can approximate the above expansion with

$$\int_{\mathscr{D}} \Phi_I^s(\mathbf{x}) \cdot \dot{W}^Q(\mathbf{x}, \omega) \mathrm{d}\mathscr{D} \approx \sum_{j=1}^{J} \sigma_j^{\frac{1}{2}} \eta_j(\omega) \int_{\mathscr{D}} \Phi_I^s(\mathbf{x}) \cdot \psi_j(\mathbf{x}) \mathrm{d}\mathscr{D}.$$

For the below numerical experiments, we will consider two kinds of noise: a Gaussian noise with covariance operator $Q = I$ and a correlated one. For correlated noise we choose $Q = \Lambda^{-s}$ with $s \in \mathbb{R}$, where $\Lambda = -\nabla^2$.

Let $\mathbb{E}u$ and $Dev(u)$ denote the mean solution and the standard deviation, respectively. For evaluation of $\mathbb{E}u$ and $Dev(u)$ we used the Monte-Carlo method:

$$\mathbb{E}(u) \approx \frac{1}{m} \sum_{k=1}^{m} \hat{u}(w_k), \quad Dev(u) \approx \sqrt{\frac{1}{m} \sum_{l=1}^{m} \left(\hat{u}(w_l) - \frac{1}{m} \sum_{k=1}^{m} \hat{u}(w_k) \right)^2}. \tag{43}$$

Consider (36) and (37). Since $\mathbb{E}\dot{W}^Q = 0$, it is easy to check that $\mathbb{E}u$ is the exact solution of the related deterministic problem i.e.,

$$\begin{cases} -\nabla^2 \mathbb{E}u(\mathbf{x}) + b\mathbb{E}u(\mathbf{x}) = g(\mathbf{x}), & \forall \mathbf{x} \in \mathscr{D}, \\ \mathbb{E}u(\mathbf{x}) = \bar{u}(\mathbf{x}), & \forall \mathbf{x} \in \partial\mathscr{D}. \end{cases} \tag{44}$$

Table 1: Errors estimate values using presented method for Example 1.

$q_{\mathbf{x}}$	$\mathrm{L}_\infty - error$	$RMS - error$
0.25	1.5198e-03	9.2583e-04
0.125	1.0052e-03	7.0809e-04
0.0625	7.4290e-04	4.6395e-04
0.0313	1.5356e-04	7.6647e-05

To see the convergence of the proposed scheme we also employ the following errors:

$$\mathrm{L}_\infty - error = \|\mathbb{E}u - \mathbb{E}(u)\|_\infty, \quad RMS - error = \frac{1}{\sqrt{N}}\|\mathbb{E}u - \mathbb{E}(u)\|_2 \; . \quad (45)$$

Note that $\mathbb{E}u$ is the solution in the absence of Gaussian noise, i.e. the related deterministic solution of (44), but $\mathbb{E}(u)$, that is appeared in (45) is the mean solution that was defined in (43).

In the forthcoming examples we show that $\mathbb{E}(u)$ approaches to $\mathbb{E}u$ by increasing the number of nodes. In the following examples, we assume boundary condition and function g calculated from the exact solution in the absence of Gaussian noise. in the following test problems, $q_{\mathbf{x}} = \max_{1\le i\le N} \min_{1\le j\le N, j\ne i} \|\mathbf{x}_i - \mathbf{x}_j\|_2$.

Example 1. Consider the one-dimensional version of (36) with $b = 0$ on $\mathscr{D} = [0, 1]$. We choose $\hat{m} = 1$ and $Q = \Lambda^{-1}$. The exact solution of this example is $\bar{u} = \sin(\pi x)$ in the absence of the Gaussian noise. The error estimate obtained for this problem are reported in Table 1.

We can see in Table 1 and the Log-Error figure in Fig. 1 that the presented method in this paper is convergent. Also the approximation of the mean solution, absolute error and the standard deviation value of this example are shown in Fig. 1.

Example 2. As another example of the one-dimensional problem suppose that $\bar{u} = (1 - x^2)$ is the solution of (36) in the absence of Gaussian noise on $\mathscr{D} = [0, 1]$. We also assume that $b = 1$, choose $\hat{m} = 3$ and $Q = \Lambda^{-1/2}$.

Table 2 and the Log-Error figure in Fig. 2 shows the presented method is convergent. Also the approximation of the mean solution, absolute error and the standard deviation value of this example are shown in Fig. 2.

Example 3. Consider (36) on $\mathscr{D} = [-1, 1]$. Suppose, $b = 1$ and the exact mean solution of this problem is $\bar{u} = (1 - x^2)\cos(\pi x)$. We choose $\hat{m} = 4$ and $Q = I$.

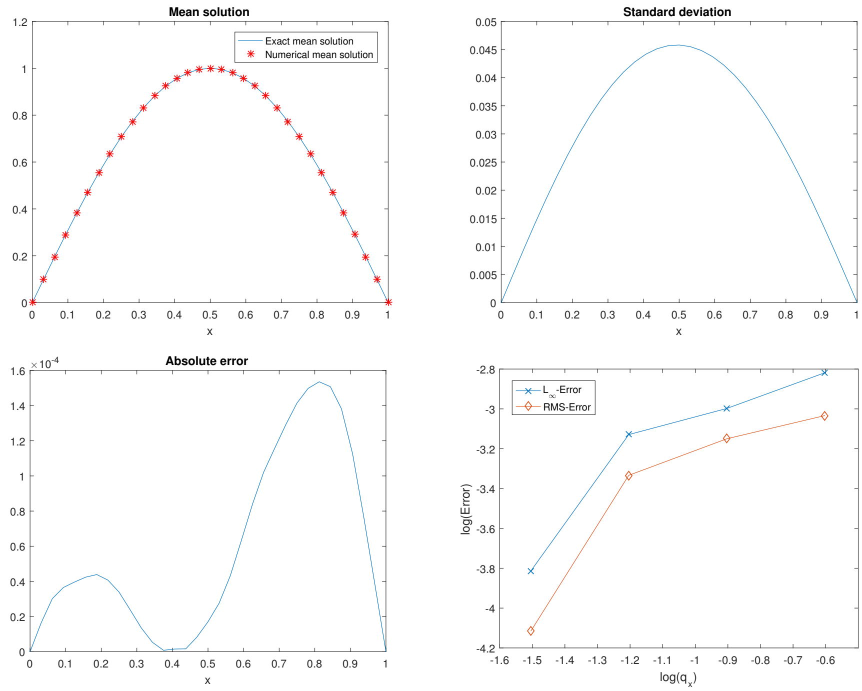

Figure 1: The mean solution, absolute error, standard deviation and the Log-Error for Example 1 with $q_{\mathbf{x}} = 0.0313$.

Table 2: Errors estimate values using presented method for Example 2.

$q_{\mathbf{x}}$	$\mathrm{L}_\infty - error$	$RMS - error$
0.25	5.5810e-03	3.6095e-03
0.125	1.6822e-03	9.1262e-04
0.0625	6.7063e-04	3.8281e-04
0.0313	3.8489e-04	2.1324e-04

Table 3: Errors estimate values using presented method for Example 3.

$q_{\mathbf{x}}$	$\mathrm{L}_\infty - error$	$RMS - error$
0.5	1.5318e-01	1.0271e-01
0.25	4.4953e-02	2.9091e-02
0.125	3.9099e-03	2.2389e-03
0.0625	1.1241e-03	5.3623e-04
0.0313	2.6423e-04	1.2292e-04

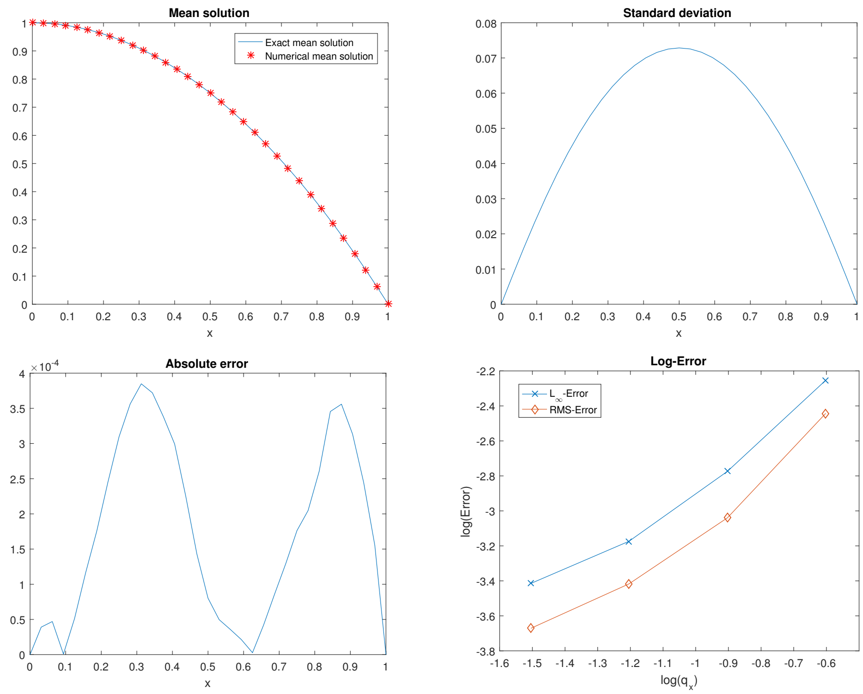

Figure 2: The mean solution, absolute error, standard deviation and the Log-Error for Example 2 with $q_{\mathbf{x}} = 0.0313$.

Table 4: Errors estimate values using presented method for Example 4.

$q_{\mathbf{x}}$	$\mathrm{L}_\infty - error$	$RMS - error$
0.5	5.1970e-02	1.7323e-02
0.25	9.2556e-03	3.6959e-03
0.125	6.1659e-04	2.9699e-04
0.0625	1.8609e-04	8.1140e-05

Table 5: Errors estimate values using presented method for Example 5.

$q_{\mathbf{x}}$	$\mathrm{L}_\infty - error$	$RMS - error$
0.5	1.9470e-02	6.4901e-03
0.25	6.6476e-03	3.7409e-03
0.125	1.9912e-03	6.4659e-04
0.0625	4.1162e-04	2.0494e-04

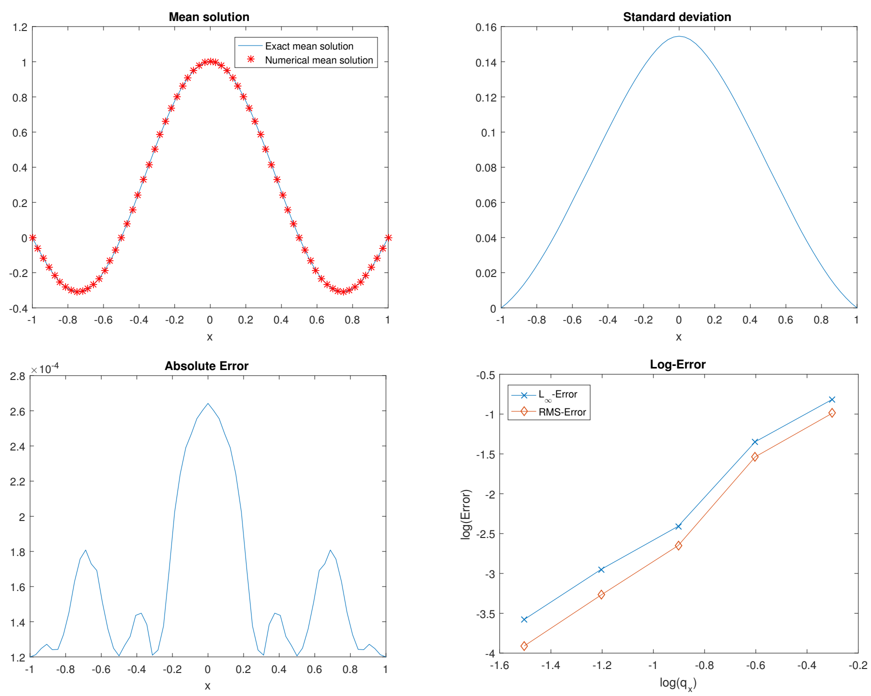

Figure 3: The mean solution, absolute error, standard deviation and the Log-Error for Example 3 with $q_{\mathbf{x}} = 0.0313$.

The convergence of the method is shown in Table 3 and the Log-Error figure in Fig. 3. Also the approximation of the mean solution, absolute error and the standard deviation value of this example are shown in Fig. 3.

Example 4. Consider the two-dimensional version of (36) on $\mathscr{D} = [0,1] \times [0,1]$. Suppose $b = 0$ then the exact mean solution of this problem is $\bar{u} = \sin(\pi x)\sin(\pi y)$. We choose $\hat{m} = 1$ and $Q = \Lambda^{-1}$.

The convergence of this method can be seen in Table 4 and the Log-Error in Fig. 5. Also the approximation of the mean solution, absolute error and the standard deviation value of this example are shown in Fig. 4.

Example 5. Consider the two-dimensional version of (36) on $\mathscr{D} = [-1,1] \times [-1,1]$. Suppose, $b = 0$ and the exact solution of this problem in the absence of Gaussian noise is $\bar{u} = (1-x^2)(1+y^2)$. We choose $\hat{m} = 1$ and $Q = \Lambda^{-1/2}$.

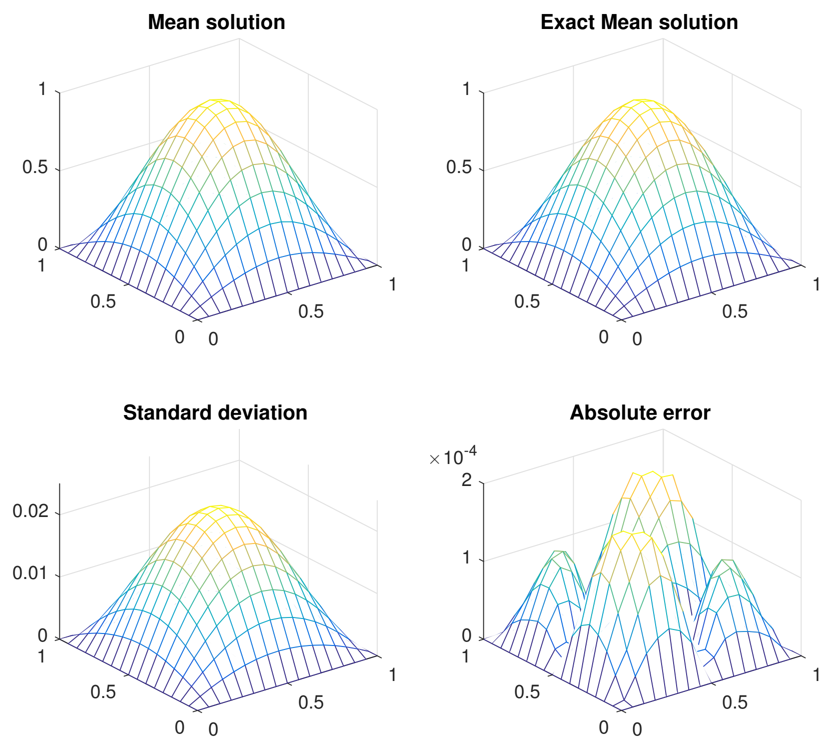

Figure 4: The mean solution, absolute error and standard deviation for Example 4 with $q_{\mathbf{x}} = 0.0625$.

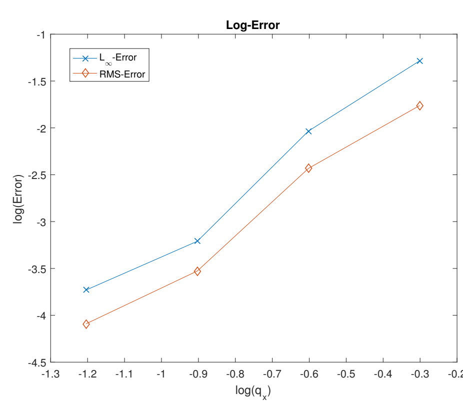

Figure 5: The Log-Error for Example 4.

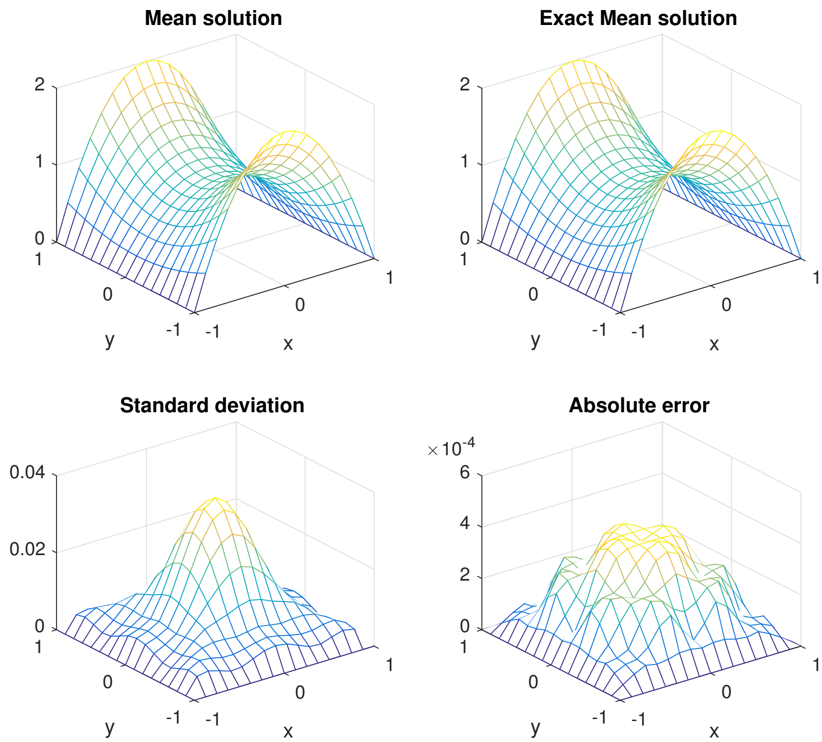

Figure 6: The mean solution, absolute error and standard deviation for example 5 with $q_{\mathbf{x}} = 0.0625$

Table 5 and Log-Error figure in Fig. 7 shows the presented method in this paper is convergent. Also the approximation of the mean solution, absolute error and the standard deviation value of this example are shown in Fig. 6.

8 Conclusion

A stabilized interpolating moving least square based element free Galerkin method have been proposed for the numerical solution of the linear elliptic stochastic partial differential equations (SPDEs). The numerical method employ scattered nodes in the domain and approximate the solution using stabilized IMLS and compute integrals numerically. The numerical simulations show the accuracy of the proposed method. An advantage of the pro-

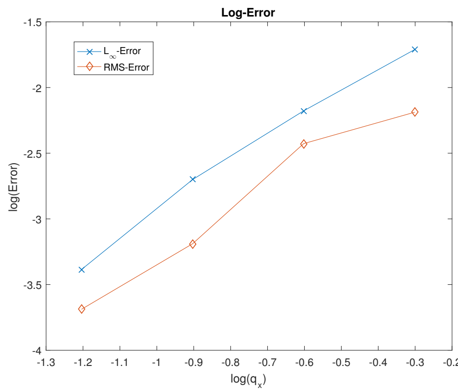

Figure 7: The Log-Error for Example 5.

posed method for solving elliptic SPDEs (even though for the deterministic problems) is that the IMLS method does not require mesh to discretize the domain of the problem and the approximate solution is constructed based on a set of nodes. Another advantage of the proposed method is interpolating property of IMLS shape functions that we can apply essential boundary conditions directly.

Acknowledgements

The authors are very grateful to referees for their useful comments and suggestions that helped us improve the quality of the paper.

References

[1] E.J. Allen, *Modeling with Itô stochastic differential equations*, Math. Model. Theory Appl. Springer Netherlands, 2007

[2] E.J. Allen, S.J. Novosel and Z. Zhang, *Finite element and difference approximation of some linear stochastic partial differential equations*, Stochastics **64(1-2)** (1998) 117-142.

[3] E.J. Allen, *Derivation of Stochastic Partial Differential Equations*, Stoch. Anal. Appl. **26(2)** (2008) 357-378.

[4] T. Belytschko, Y.Y. Lu and L. Gu, *Element-free Galerkin methods*, Internat. J. Numer. Methods Engrg. **37(2)** (1994) 229-256.

[5] Y. Cao, J. Hong and Z. Liu, *Well-posedness and finite element approximations for elliptic SPDEs with Gaussian noises*, arXiv preprint arXiv:1510.01873v5, 2018

[6] Y. Cao, H. Yang and L. Yin, *Finite element methods for semilinear elliptic stochastic partial differential equations*, Numer. Math. **106(2)** (2007) 181-198.

[7] R. Cheng and Y. Cheng, *Boundary element-free method for elasticity problems*, Chin. J. Theor. Appl. Mech. **35** (2003) 181-186.

[8] R. Cheng and Y. Cheng, *Error estimates for the finite point method*, Appl. Numer. Math. **58(6)** (2008) 884-898.

[9] R. Cheng and Y. Cheng, *Error estimates of element-free Galerkin method for potential problems*, Acta Phys. Sinica **57(10)** (2008) 6037-6046.

[10] I. Cialenco, G.E. Fasshauer and Q. Ye, *Approximation of stochastic partial differential equations by a kernel-based collocation method*, Int. J. Comput. Math. **89** (2012) 2543-2561.

[11] D. Cohen, L. Quer-Sardanyons, *A fully discrete approximation of the one-dimensional stochastic wave equation*, IMA J. Numer. Anal. **36(1)** (2016) 400-420.

[12] G.D. Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Encyclopedia Math. Appl. Cambridge University Press, 2014.

[13] M. Dehghan and M. Shirzadi, *Meshless simulation of stochastic advectiondiffusion equations based on radial basis functions*, Eng. Anal. Bound. Elem. **53** (2015) 18-26.

[14] M. Dehghan and M. Shirzadi, *The modified dual reciprocity boundary elements method and its application for solving stochastic partial differential equations*, Eng. Anal. Bound. Elem. **58** (2015) 99-111.

[15] M. Dehghan and M. Shirzadi, *Numerical solution of stochastic elliptic partial differential equations using the meshless method of radial basis functions*, Eng. Anal. Bound. Elem. **50** (2015) 291-303.

[16] M. Dehghan and M. Shirzadi, *A meshless method based on the dual reciprocity method for one-dimensional stochastic partial differential equations*, Numer. Methods Partial Differential Equations, **32(1)** (2016) 292-306.

[17] G.E. Fasshauer and Q. Ye, *Kernel-based Collocation methods versus Galerkin finite element methods for approximating elliptic stochastic partial differential equations*, Springer, Berlin, 2013.

[18] G.E. Fasshauer and Q. Ye, *A kernel-based collocation method for elliptic partial differential equations with random coefficients*, MCQMC 2012. Springer, Berlin, 2014.

[19] H.-Y. Hu, Z.-C. Li and A. H.-D. Cheng, *Radial basis collocation methods for elliptic boundary value problems*, Comput. Math. Appl. **50(1)** (2005) 289-320.

[20] A. Jentzen and P. Kloeden, *Taylor approximations for stochastic partial differential equations*, SIAM 2011.

[21] I. Kaljević and S. Saigal, *An improved element free Galerkin formulation*, Internat. J. Numer. Methods Engrg. **40(16)** (1997) 2953-2974.

[22] M. Kamrani and S.M. Hosseini, *Spectral collocation method for stochastic Burgers equation driven by additive noise*, Math. Comput. Simulation **82(9)** (2012) 1630-1644.

[23] P. Lancaster and K. Salkauskas, *Surfaces generated by moving least squares methods*, Math. Comp. **37** (1981) 141-158.

[24] X. Li, H. Chen and Y. Wang, *Error analysis in Sobolev spaces for the improved moving least-square approximation and the improved element-free Galerkin method*, Appl. Math. Comput. **262** (2015) 56-78.

[25] X. Li and Q. Wang, *Analysis of the inherent instability of the interpolating moving least squares method when using improper polynomial bases*, Eng. Anal. Bound. Elem. **73** (2016) 21-34.

[26] K.M. Liew, Y. Cheng and S. Kitipornchai, *Boundary element-free method (BEFM) and its application to two-dimensional elasticity problems*, Internat. J. Numer. Methods Engrg. **65(8)** (2006) 1310-1332.

[27] D. Mirzaei, *Analysis of moving least squares approximation revisited*, J. Comput. Appl. Math. **282** (2015) 237-250.

[28] D. Mirzaei, R. Schaback and M. Dehghan, *On generalized moving least squares and diffuse derivatives*, IMA J. Numer. Anal. **32(3)** (2012) 983-1000.

[29] Y.X. Mukherjee and S. Mukherjee, *On boundary conditions in the element-free Galerkin method*, Comput. Mech. **19(4)** (1997) 264-270.

[30] H.-P. Ren and Y.-M. Cheng, *The interpolating element-free Galerkin (IEFG) method for two-dimensional elasticity problems*, Chinese Physics B **3(4)** (2011) 735-758.

[31] H.-P. Ren, Y.-M. Cheng and W. Zhang, *An improved boundary element-free method (IBEFM) for two-dimensional potential problems*, Chinese Physics B **18(10)** (2009) 4065-4073.

[32] H.-P. Ren, Y.-M. Cheng and W. Zhang, *An improved boundary element-free method (IBEFM) for two-dimensional elasticity problems*, Science China Physics **53** (2010) 758-766.

[33] H.-P. Ren and Y.-M. Cheng, *The interpolating element-free Galerkin (IEFG) method for two-dimensional potential problems*, Eng. Anal. Bound. Elem. **36(5)** (2012) 873-880.

[34] R. Scitovski, Š. Ungar and D. Jukić, *Approximating surfaces by moving total least squares method*, Appl. Math. Comput. **93(2)** (1998) 219-232.

[35] T. Shardlow, *Numerical methods for stochastic parabolic PDEs*, Numer. Funct. Anal. Optim. **20(1-2)** (1999) 121-145.

[36] T. Shardlow, *Weak convergence of a numerical method for a stochastic heat equation*, BIT **43(1)** (2003) 179-193.

[37] J.F. Wang, S.Y. Hao and Y.M. Cheng, *The error estimates of the interpolating element-free Galerkin method for two-point boundary value problems*, Math. Probl. Eng. **2014** (2014) 12 pages.

[38] J.F. Wang, F.X. Sun, Y.M. Cheng and A.X. Huang, *Error estimates for the interpolating moving least-squares method*, Appl. Math. Comput. **245** (2014) 321-342.

[39] X. Wang, S. Gan and J. Tang, *Higher order strong approximations of semilinear stochastic wave equation with additive space-time white noise*, SIAM J. Sci. Comput. **36(6)** (2014) A2611-A2632.