

A new iteration method for solving non-Hermitian positive definite linear systems

Hamideh Nasabzadeh*

Department of Mathematics, Faculty of Basic Sciences, University of Bojnord,

P. O. Box 9453155111, Bojnord, Iran

Email: h.nasabzadeh @ ub.ac.ir

Abstract. In this paper, based on the single-step Hermitian and Skew-Hermitian (SHSS) iteration method [C.-X. Li, S.-L. Wu, A single-step method for non-Hermitian positive definite linear systems, *Appl. Math. Lett.* 44 (2015) 26-29] and by using the generalized Taylor expansion method for solving linear systems [F. Toutounian, H. Nasabzadeh, A new method based on the generalized Taylor expansion for computing a series solution of linear systems, *Appl. Math. Comput.* 248 (2014) 602-609], a new method (GT-SHSS) is introduced to solve non-Hermitian positive definite linear systems. The convergence properties of the new method are discussed. We show that by using suitable parameters, the GT-SHSS iteration method is faster than the corresponding SHSS iteration method. The numerical examples confirm the effectiveness of the new method.

Keywords : Non-Hermitian, HSS method, convergence Analysis, iterative Method.

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1 Introduction

Many problems in scientific computing require to solve a large and sparse non-Hermitian positive definite linear system

$$Ax = b, \tag{1}$$

*Corresponding author.

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where $A \in \mathbb{C}^{n \times n}$ and $x, b \in \mathbb{C}^n$. For solving (1), Bai et al. [6] proposed the Hermitian and skew-Hermitian splitting (HSS) iteration method as follows.

The HSS Iteration Method: Let $x^{(0)}$ be an initial guess and $\alpha > 0$. For $k = 0, 1, \dots$, until the sequence $\{x^{(k)}\}$ converges to the unique solution of (1), compute,

$$\begin{aligned}(\alpha I + H)x^{(k+\frac{1}{2})} &= (\alpha I - S)x^{(k)} + b, \\(\alpha I + S)x^{(k+1)} &= (\alpha I - H)x^{(k+\frac{1}{2})} + b,\end{aligned}$$

where, $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$, in which A^* stands for the conjugate transpose of the matrix A .

Many studies have been done on the convergence analysis of the HSS iteration method and applications of the HSS iteration method (see [1–4,10]). Also, several variants of the HSS method were developed, such as preconditioned Hermitian and skew-Hermitian splitting (PHSS) iteration method [9], block triangular and skew-Hermitian splitting (BTSS) iteration method [5], normal and skew-Hermitian splitting (NSS) method [8], modified Hermitian and skew-Hermitian splitting (MHSS) iteration method [11], inexact HSS iteration method [7], accelerated HSS iteration (AHSS) [3] and the LHSS iteration method [13].

Li and Wu [12] proposed the single-step HSS method to solve the non-Hermitian positive definite linear systems. Since the HSS method requires to solve two linear subsystems,

$$(\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b, \quad (2)$$

and

$$(\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+\frac{1}{2})} + b, \quad (3)$$

they introduced the SHSS method to avoid solving shifted skew-Hermitian linear subsystem (3).

The SHSS Iteration Method: Let $x^{(0)}$ be an initial guess. For $k = 0, 1, \dots$ until the sequence $\{x^{(k)}\}$ converges to the exact solution of (1), compute

$$x^{(k+1)} = T_\alpha x^{(k)} + (\alpha I + H)^{-1}b, \quad k = 0, 1, \dots \quad (4)$$

where $T_\alpha = (\alpha I + H)^{-1}(\alpha I - S)$.

They showed that the SHSS method converges to the exact solution of (1) for a loose restriction of the iteration parameter α .

Theorem 1. (see [12]) Let $A \in \mathbb{C}^{n \times n}$ be a positive definite matrix, and $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$ be the Hermitian and skew-Hermitian parts of A . Also let α be a positive constant and λ_{\min} and σ_{\max} be the smallest eigenvalue of the matrix H and the largest singular value of the matrix S , respectively. Then for the spectral radius of the iteration matrix of the SHSS iteration method we have the following relation,

$$\rho(T_\alpha) \leq \|T_\alpha\|_2 \leq \delta_\alpha = \frac{\sqrt{\alpha^2 + \sigma_{\max}^2}}{\alpha + \lambda_{\min}}. \quad (5)$$

Moreover,

(i) if $\lambda_{\min} \geq \sigma_{\max}$ then $\delta_\alpha < 1$ for any $\alpha > 0$.

(ii) if $\lambda_{\min} < \sigma_{\max}$ then $\delta_\alpha < 1$ if and only if

$$\alpha > \frac{\sigma_{\max}^2 - \lambda_{\min}^2}{2\lambda_{\min}}. \quad (6)$$

Now, in this paper, by using the Taylor expansion method for solving linear systems [14], we propose a new method that is more efficient than the SHSS iteration method. The new method is called GT-SHSS. Numerical experiments show the new method is superior to the SHSS iteration method under certain conditions.

The rest of this paper is organized as follows. In Section 2, we present the GT-SHSS method and study its convergence properties and also we present suitable parameter to achieve a new method that is faster than the SHSS iterative method. Numerical example is given to demonstrate the theoretical results and the effectiveness of the GT-SHSS method in Section 3.

2 The GT-SHSS method

From Eq. (4), Eq. (1) can be written as

$$x = Gx + c, \quad (7)$$

where $G = T_\alpha$ and $c = (\alpha I + H)^{-1}b$. Take

$$G_{\gamma, h} = \frac{hG - \gamma(h+1)I}{h - \gamma(h+1)}, \quad (8)$$

where γ and $\hbar \in \mathbb{R}$, see [14]. Let u_0 be an arbitrary initial guess to the exact solution u of (7), and

$$u_1 = \frac{\hbar}{\hbar - \gamma(\hbar + 1)}[(G - I)u_0 + c], \quad (9)$$

$$u_{i+1} = G_{\gamma, \hbar} u_i, \quad i = 1, 2, \dots \quad (10)$$

Then $u = \sum_{i=0}^{\infty} u_i$ is the exact solution of (7) if and only if $\rho(G_{\gamma, \hbar}) < 1$ (see [14]). Now, put $\beta = -\frac{\hbar}{\hbar - \gamma(\hbar + 1)}$. Then, from (8), $G_{\gamma, \hbar}$ can be written as

$$G_{\gamma, \hbar} = T_{\alpha, \beta} = (\alpha I + H)^{-1}(\alpha I + \beta S + (\beta + 1)H). \quad (11)$$

Therefore, the GT-SHSS iteration method can be written as following.

The GT-SHSS Iteration Method: Let u_0 be an initial approximation for the unique solution u of (7). For $k = 0, 1, 2, \dots$ until $\sum_{i=0}^{i=k} u_i$ converges, compute

$$(\alpha I + H)u_1 = \beta[(H + S)u_0 - b], \quad (12)$$

$$(\alpha I + H)u_{i+1} = (\alpha I + \beta S + (\beta + 1)H)u_i, \quad (13)$$

for $i = 1, 2, 3, \dots$

We can see that if spectral radius of

$$T_{\alpha, \beta} = (\alpha I + H)^{-1}(\alpha I + \beta S + (\beta + 1)H), \quad (14)$$

is less than one then

$$\sum_{i=0}^{\infty} u_i = u_0 + \sum_{i=1}^{\infty} T_{\alpha, \beta}^{i-1} u_1, \quad (15)$$

converges to the exact solution of (1), (see [15] Theorem 3.15).

When $\beta = -1$, it is straightforward to see that the GT-SHSS method reduces to the SHSS method. In the sequel, the convergence of the GT-SHSS iteration method is discussed.

In the rest of this paper, we suppose that $A \in \mathbb{C}^{n \times n}$ is a positive definite matrix, also $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$ are Hermitian and skew-Hermitian parts of A , α is a positive constant and λ_{\min} and λ_{\max} are the smallest and largest eigenvalue of the matrix H , respectively, and σ_{\max} is

the largest singular value of matrix S . Also, we define

$$\begin{aligned}\gamma_\alpha &= 2\lambda_{\max}(\alpha + \lambda_{\min} - \lambda_{\max}), \\ \mu_\alpha &= -(\alpha + \lambda_{\min} - \lambda_{\max})^2, \\ \beta' &= \frac{-\gamma_\alpha - \sqrt{\gamma_\alpha^2 - 4\mu_\alpha(\sigma_{\max}^2 - \lambda_{\max}^2)}}{2(\sigma_{\max}^2 - \lambda_{\max}^2)}, \\ \beta'' &= \frac{-\gamma_\alpha + \sqrt{\gamma_\alpha^2 - 4\mu_\alpha(\sigma_{\max}^2 - \lambda_{\max}^2)}}{2(\sigma_{\max}^2 - \lambda_{\max}^2)}.\end{aligned}$$

Theorem 2. *If $\beta > -1$, then for the spectral radius of the iteration matrix of the GT-SHSS iteration method we have the following relation*

$$\rho(T_{\alpha,\beta}) \leq \|T_{\alpha,\beta}\|_2 \leq \delta_{\alpha,\beta} = \frac{\sqrt{(\alpha^2 + \beta^2\sigma_{\max}^2)} + (\beta + 1)\lambda_{\max}}{\alpha + \lambda_{\min}}. \quad (16)$$

Moreover,

(i) let $\sigma_{\max}^2 - \lambda_{\max}^2 > 0$, then $\delta_{\alpha,\beta} < 1$ if and only if

$$\max\{-1, \beta_1\} < \beta < \beta_2;$$

(ii) let $\sigma_{\max}^2 - \lambda_{\max}^2 = 0$, then $\delta_{\alpha,\beta} < 1$ if and only if

$$\gamma_\alpha > 0 \quad \text{and} \quad -1 < \beta < \frac{-\mu_\alpha}{\gamma_\alpha}$$

or

$$\gamma_\alpha < 0 \quad \text{and} \quad \beta > \max\left\{\frac{-\mu_\alpha}{\gamma_\alpha}, -1\right\};$$

(iii) let $\sigma_{\max}^2 - \lambda_{\max}^2 < 0$,

(a) if $\gamma_\alpha^2 - 4\mu_\alpha(\sigma_{\max}^2 - \lambda_{\max}^2) > 0$ then $\delta_{\alpha,\beta} < 1$ if and only if

$$\beta \in (-1, \beta') \cup (\beta'', +\infty);$$

(b) if $\gamma_\alpha^2 - 4\mu_\alpha(\sigma_{\max}^2 - \lambda_{\max}^2) < 0$ then $\delta_{\alpha,\beta} < 1$ if and only if $\beta > -1$;

(c) if $\gamma_\alpha^2 - 4\mu_\alpha(\sigma_{\max}^2 - \lambda_{\max}^2) = 0$ then $\delta_{\alpha,\beta} < 1$ if and only if $\beta > -1$

$$\text{and } \beta \neq \frac{-\gamma_\alpha}{2(\sigma_{\max}^2 - \lambda_{\max}^2)}.$$

Proof. It follows from Eq. (14) that

$$\begin{aligned} \rho(T_{\alpha,\beta}) &\leq \|T_{\alpha,\beta}\|_2 \\ &\leq \|(\alpha I + H)^{-1}(\alpha I + \beta S)\|_2 + \|(\alpha I + H)^{-1}(\beta + 1)H\|_2 \\ &\leq \frac{\sqrt{(\alpha^2 + \beta^2 \sigma_{\max}^2)} + (\beta + 1)\lambda_{\max}}{\alpha + \lambda_{\min}} = \delta_{\alpha,\beta}. \end{aligned} \quad (17)$$

(i) By some computations, it is easy to see that $\delta_{\alpha,\beta} < 1$ is equivalent to

$$f(\beta) = (\sigma_{\max}^2 - \lambda_{\max}^2)\beta^2 + \gamma_{\alpha}\beta + \mu_{\alpha} < 0.$$

Since for all α , $\mu_{\alpha} \leq 0$ and $\sigma_{\max}^2 - \lambda_{\max}^2 > 0$, we deduce that β' and β'' are two real simple roots of $f(\beta)$ and $\beta' < 0 < \beta''$. Hence, it is easy to see that $\forall \beta \in (\max\{-1, \beta'\}, \beta'')$, $f(\beta) < 0$ and $\delta_{\alpha,\beta} < 1$.

(ii), (iii) By discussing about the roots of $f(\beta)$ the proof of these parts are trivial. \square

The next theorem provides conditions on β , under which the spectral radius of the iteration matrix of the GT-SHSS method is smaller than that of the SHSS iteration method.

Theorem 3. *Let the assumptions of Theorem 1 hold and the SHSS iteration method be convergent. Suppose that $\mu_i = \text{Re}(\mu_i) + i\text{Im}(\mu_i)$ and θ_i , $i = 1, 2, \dots, n$ are the eigenvalues of T_{α} and $T_{\alpha,\beta}$, respectively, and $z_i = \text{Re}(z_i) + i\text{Im}(z_i) = (1 - \text{Re}(\mu_i)) + i\text{Im}(\mu_i)$, $i = 1, 2, \dots, n$, also $N = \{i \mid |\mu_i| = \rho(T_{\alpha})\}$ and*

$$\gamma_1^{(i)} = \frac{-\text{Re}z_i - \sqrt{|z_i|^2 \rho(T_{\alpha})^2 - \text{Im}^2 z_i}}{|z_i|^2}, \quad (18)$$

$$\gamma_2^{(i)} = \frac{-\text{Re}z_i + \sqrt{|z_i|^2 \rho(T_{\alpha})^2 - \text{Im}^2 z_i}}{|z_i|^2}. \quad (19)$$

Then, $\forall \beta \in [\max_i \gamma_1^{(i)}, \min_i \gamma_2^{(i)}]$, $\rho(T_{\alpha,\beta}) \leq \rho(T_{\alpha}) < 1$. Moreover

(i) if $\forall i \in N$, $\frac{\text{Re}z_i}{|z_i|^2} \geq 1$, then, $\min_i \gamma_2^{(i)} = -1$ so, $\forall \beta \in [\max_i \gamma_1^{(i)}, -1]$, $\rho(T_{\alpha,\beta}) \leq \rho(T_{\alpha}) < 1$.

(ii) if $\forall i \in N$, $\frac{\text{Re}z_i}{|z_i|^2} \leq 1$, then, $\max_i \gamma_1^{(i)} = -1$ so, $\forall \beta \in [-1, \min_i \gamma_2^{(i)}]$, $\rho(T_{\alpha,\beta}) \leq \rho(T_{\alpha}) < 1$.

Proof. By (14) we have $T_{\alpha,\beta} = (\beta + 1)I - \beta T_\alpha$. Hence

$$\theta_i = (\beta(1 - \operatorname{Re}(\mu_i)) + 1) - i\beta \operatorname{Im}(\mu_i). \quad (20)$$

Define,

$$g_i(\beta) = |\theta_i|^2 - \rho(T_\alpha)^2 = |z_i|^2\beta^2 + 2\operatorname{Re}(z_i)\beta + (1 - \rho(T_\alpha)^2), \quad (21)$$

so $\forall i \notin N$, we have $g_i(0) = 1 - \rho(T_\alpha)^2 > 0$, $g_i(-1) = |\mu_i|^2 - \rho(T_\alpha)^2 < 0$ and $\operatorname{sign} g_i(-\infty) = 1$ so $g_i(\beta)$ has two real simple roots $\gamma_1^{(i)}$ and $\gamma_2^{(i)}$ where,

$$\gamma_1^{(i)} < -1 < \gamma_2^{(i)} < 0. \quad (22)$$

From (21), we can see that

$$\gamma_1^{(i)} = \frac{-\operatorname{Re}z_i - \sqrt{|z_i|^2\rho(T_\alpha)^2 - \operatorname{Im}^2z_i}}{|z_i|^2},$$

and

$$\gamma_2^{(i)} = \frac{-\operatorname{Re}z_i + \sqrt{|z_i|^2\rho(T_\alpha)^2 - \operatorname{Im}^2z_i}}{|z_i|^2},$$

and $\forall \beta \in [\gamma_1^{(i)}, \gamma_2^{(i)}]$, $g_i(\beta) \leq 0$ and $|\theta_i| \leq \rho(T_\alpha)$. Also, $\forall i \in N$, we have $g(-1) = 0$ and

$$g\left(\frac{|z_i|^2 - 2\operatorname{Re}(z_i)}{|z_i|^2}\right) = 0.$$

If $\frac{\operatorname{Re}(z_i)}{|z_i|^2} \geq 1$, then

$$\gamma_1^i = \frac{|z_i|^2 - 2\operatorname{Re}(z_i)}{|z_i|^2} \leq -1 = \gamma_2^{(i)},$$

and if $\frac{\operatorname{Re}(z_i)}{|z_i|^2} \leq 1$ then

$$\gamma_1^i = -1 \leq \frac{|z_i|^2 - 2\operatorname{Re}(z_i)}{|z_i|^2} = \gamma_2^{(i)},$$

and $\forall \beta \in [\gamma_1^{(i)}, \gamma_2^{(i)}]$, $g_i(\beta) < 0$ and $|\theta_i| \leq \rho(T_\alpha)$. These imply that $\forall \beta \in [\max_i \gamma_1^{(i)}, \min_i \gamma_2^{(i)}]$, $\max_i |\theta_i| = \rho(T_{\alpha,\beta}) \leq \rho(T_\alpha) < 1$.

It is easy to see from (22) that if $\forall i \in N$, $\frac{\operatorname{Re}(z_i)}{|z_i|^2} \geq 1$ then $\min_i \gamma_2^{(i)} = -1$ and $\forall \beta \in [\max_i \gamma_1^{(i)}, -1]$, $\rho(T_{\alpha,\beta}) \leq \rho(T_\alpha) < 1$ and if $\forall i \in N$, $\frac{\operatorname{Re}(z_i)}{|z_i|^2} \leq 1$ then $\max_i \gamma_1^{(i)} = -1$ and $\forall \beta \in [-1, \min_i \gamma_2^{(i)}]$, $\rho(T_{\alpha,\beta}) \leq \rho(T_\alpha) < 1$. \square

Now we compare the upper bounds of the spectral radius of T_α and $T_{\alpha,\beta}$, i.e., δ_α and $\delta_{\alpha,\beta}$. For fixed $\alpha > 0$, define

$$g_\alpha(\beta) = \delta_{\alpha,\beta} - \delta_\alpha, \quad (23)$$

clearly, $g_\alpha(\beta)$ is continuous, and also

$$\frac{\partial g_\alpha}{\partial \beta} = \frac{1}{\alpha + \lambda_{\min}} \left(\frac{\sigma_{\max}^2 \beta}{\sqrt{\alpha^2 + \beta^2 \sigma_{\max}^2}} + \lambda_{\max} \right). \quad (24)$$

If $\sigma_{\max}^2 - \lambda_{\max}^2 > 0$, then the extremum point β^* of $g_\alpha(\beta)$ satisfies the relations

- (i) $\beta^* < 0$
- (ii) $\beta^{*2} = \Gamma^2 \alpha^2$ where $\Gamma = \frac{\lambda_{\max}}{\sigma_{\max} \sqrt{\sigma_{\max}^2 - \lambda_{\max}^2}}$,

and also

$$\lim_{\alpha \rightarrow +\infty} g_\alpha(\beta) = +\infty \quad \text{and} \quad \lim_{\alpha \rightarrow -\infty} g_\alpha(\beta) = \begin{cases} +\infty, & \sigma_{\max}^2 - \lambda_{\max}^2 > 0, \\ -\infty, & \sigma_{\max}^2 - \lambda_{\max}^2 < 0. \end{cases}$$

So if $\sigma_{\max}^2 - \lambda_{\max}^2 > 0$, then β^* is the minimum point of g_α and this function has two real roots $\beta_1 = -1$ and

$$\beta_2 = \frac{\sigma_{\max}^2 + \lambda_{\max}^2 - 2\lambda_{\max} \sqrt{\sigma_{\max}^2 + \alpha^2}}{\sigma_{\max}^2 - \lambda_{\max}^2},$$

otherwise, $g_\alpha(\beta)$ has not a minimum point, and g_α has only one real root $\beta_1 = -1$ and g_α is a strictly increasing function.

Theorem 4. Let $\alpha > 0$ and $\beta \geq -1$.

- (i) If $\sigma_{\max}^2 - \lambda_{\max}^2 \leq 0$ then $\forall \beta \geq -1$, $\delta_{\alpha,\beta} \geq \delta_\alpha$.
- (ii) If $\sigma_{\max}^2 - \lambda_{\max}^2 > 0$ and $\alpha \geq \frac{1}{\Gamma}$, then $\forall \beta \geq -1$, $\delta_{\alpha,\beta} \geq \delta_\alpha$.
- (iii) If $\sigma_{\max}^2 - \lambda_{\max}^2 > 0$ and $\alpha \leq \frac{1}{\Gamma}$, if $-1 \leq \beta \leq \beta_2$, then $\delta_{\alpha,\beta^*} \leq \delta_{\alpha,\beta} \leq \delta_\alpha$.

Proof. (i) If $\sigma_{\max}^2 - \lambda_{\max}^2 \leq 0$ then g_α has not minimum point and it is a strictly increasing function so, $\forall \beta \geq -1$, $g_\alpha(\beta) \geq g_\alpha(-1) = 0$ and then $\delta_{\alpha,\beta} \geq \delta_\alpha$.

(ii) If $\sigma_{\max}^2 - \lambda_{\max}^2 > 0$ then g_α has two real roots,

$$\beta_1 = -1 \quad \text{and} \quad \beta_2 = \frac{\sigma_{\max}^2 + \lambda_{\max}^2 - 2\lambda_{\max} \sqrt{\sigma_{\max}^2 + \alpha^2}}{\sigma_{\max}^2 - \lambda_{\max}^2}$$

by some computations we can see that if $\alpha \geq \frac{1}{\Gamma}$, then $\beta_2 \leq \beta_1 = -1$. So $\forall \beta \geq -1, g_\alpha(\beta) \geq g_\alpha(-1) = 0$, and then $\delta_{\alpha,\beta} \geq \delta_\alpha$.

(iii) If $\alpha \leq \frac{1}{\Gamma}$, then $-1 = \beta_1 \leq \beta_2$, so for $\beta_1 = -1 \leq \beta \leq \beta_2, g_\alpha(\beta) \leq 0$ and $\delta_{\alpha,\beta} \leq \delta_\alpha$. Since β^* is minimum point for g_α , we have

$$\delta_{\alpha,\beta^*} \leq \delta_{\alpha,\beta} \leq \delta_\alpha.$$

Hence, the proof of theorem is completed. □

3 Numerical Experiments

In this section, we present an example to test the effectiveness of the GT-SHSS iteration method for solving the non-Hermitian positive definite linear system (1). All of the computation results are shown in MATLAB R2015a and performed on a PC with Intel(R) Core(TM) i3-2330M Processor/ 2.20 GHz and 4GBz RAM.

Example 1. (See [5]) For the system of linear equation (1), let

$$A = \begin{pmatrix} W & FM \\ -F^T & N \end{pmatrix}$$

where $W \in \mathbb{R}^{q \times q}$ and $N, M \in \mathbb{R}^{n-q \times n-q}, 2q \geq n$ and the elements of W, N, F and M are defined as follows:

$$w_{kj} = \begin{cases} k+1, & \text{for } j = k, \\ 1, & \text{for } |k-j| = 1, \\ 0, & \text{otherwise,} \end{cases} \quad k, j = 1, \dots, q,$$

$$n_{kj} = \begin{cases} k+1, & \text{for } j = k, \\ 1, & \text{for } |k-j| = 1, \\ 0, & \text{otherwise,} \end{cases} \quad k, j = 1, \dots, n-q,$$

$$f_{kj} = \begin{cases} j, & \text{for } k = j + 2q - n, \\ 0, & \text{otherwise,} \end{cases} \quad k = 1, \dots, q,$$

$$m_{kj} = \begin{cases} \frac{1}{k}, & \text{for } k = j, \\ 0, & \text{otherwise.} \end{cases} \quad k, j = 1, \dots, n-q,$$

We choose the right-hand side vector $b = Ae$, with $e = (1, 1, \dots, 1)^T$ and the initial guesses $x^{(0)}$ and u_0 are chosen to be zero. The stopping criterion

for the GT-SHSS and the SHSS methods is

$$\frac{\|b - Ax^{(k)}\|_2}{\|b\|_2} \leq 10^{-6}.$$

In Tables 1 and 2, we present the number of iterations (denoted by IT), the CPU time in seconds (denoted by CPU), the corresponding $\rho(T_{\alpha,\beta})$ and $\rho(T_\alpha)$ for $n = 1000$ and $n = 2000$.

Table 1: $n = 1000, q = 501, \alpha = 0.02$

β	-0.75	-0.6	-0.55	-0.5	SHSS	HSS
IT	20	21	23	24	26	-
CPU(s)	4.87	5.09	5.50	5.71	8.47	-
$\rho(T_{\alpha,\beta})$	0.5935	0.5897	0.5893	0.6180	$\rho(T_\alpha) = 0.7133$	$\rho = 0.999894$

Table 2: $n = 2000, q = 1001, \alpha = 0.008$

β	-0.7	-0.6	-0.55	-0.5	SHSS	HSS
IT	20	21	23	24	26	-
CPU(s)	27.69	30.03	35.87	36.81	63.79	-
$\rho(T_{\alpha,\beta})$	0.5864	0.5892	0.6001	0.6170	$\rho(T_\alpha) = 0.7175$	$\rho = 0.999978$

Tables 1-2 indicate that the GT-SHSS method is much more effective than the SHSS and HSS iteration methods, since the GT-SHSS requires much less iteration steps and CPU times than them.

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