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On the moments of order statistics from the standard two-sided power distribution

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Abstract. In this paper, we obtain new explicit expressions for the single and product moments of order statistics from the standard two-sided power (STSP) distribution. These expressions can be used to compute the means, variances and the covariances of order statistics from the STSP distribution. We also have a glance at the application of the results to the lifetimes of the coherent systems. Two real data examples are given to illustrate the flexibility of the STSP distribution.

Keywords: Coherent systems, explicit expressions, product moments, standard two-sided power distribution.
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1 Introduction

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a distribution, then the corresponding order statistics are obtained by arranging the random sample in nondecreasing order, denoted by $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$. Order statistics play a key role in statistics and provide the most suitable and appropriate solution for several problems arising in life-testing experiments, reliability studies, breaking strength analysis, detection of outliers and so

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on. For an elaborate treatment on the theory, methods and applications of order statistics, one may refer to [12].

The moments of order statistics are quite well known in the literature and play an important role in many inferential problems. For example, they are applied in deriving best linear unbiased estimators (BLUEs) for the location-scale families of distributions based on complete and type-II censored samples. They are also useful for developing goodness of fit tests, point prediction of future observations and nonparametric inference. There has been a large amount of work related to the moments of order statistics, see for example [1, 2, 4–7, 13, 15, 17, 19, 24, 26, 27] for excellent accounts.

Van Dorp and Kotz [28] proposed the standard two-sided power (STSP) distribution with the probability density function (pdf)

$$f(x;p,\theta) = \begin{cases} p\Big(\dfrac{x}{\theta}\Big)^{p-1}, & \text{for } 0 < x \le \theta, \\ p\Big(\dfrac{1-x}{1-\theta}\Big)^{p-1}, & \text{for } \theta \le x < 1, \end{cases} \tag{1}$$

and the corresponding cumulative distribution function (cdf),

$$F(x;p,\theta) = \begin{cases} \theta\Big(\dfrac{x}{\theta}\Big)^{p}, & \text{for } 0 < x \le \theta, \\ 1-(1-\theta)\Big(\dfrac{1-x}{1-\theta}\Big)^{p}, & \text{for } \theta \le x < 1, \end{cases} \tag{2}$$

where $\theta \in [0,1]$ and $p > 0$ which is not necessarily integer.

Let X be a random variable whose cdf is given by (2), then we write $X \sim \text{STSP}(\theta, p)$. The STSP distribution is defined on a bounded support and it can be used as an alternative to the beta and some other bounded distributions. Some features of the STSP distribution are given as follows (see Van Dorp and Kotz [28])

1. For $0 \le \theta \le 1$ and $p > 0$, (1) is unimodal with the mode at θ.
2. For $0 < \theta < 1$ and $0 < p < 1$, (1) is U-shaped with a mode at 0 or 1.
3. For $p = 1$, the STSP distribution reduces to the standard uniform distribution.
4. For $p = 2$, the STSP distribution reduces to the triangular distribution on $[0, 1]$.
5. For $\theta = 1$, the STSP distribution simplifies to the power function distribution.

For more discussions related to the properties and applications of the STSP distribution and its extension (the two-sided power distribution), one may refer to [8, 14, 18, 22, 28, 29].

The rest of the paper is organized as follows. In Section 2, we derive explicit expressions for the single moments of order statistics from the STSP distribution. Section 3 is devoted to an application of the theoretical results of Section 2 to the lifetimes of the coherent systems. In Section 4, we focus on the product moments. Section 5 deals with the computational aspect of the results of the paper and two tables are provided to present the computed means, variances and covariances of order statistics for selected parameter combinations. In Section 6, we present two real data examples to emphasize the flexibility and applicability of the STSP distribution. Finally, some concluding remarks are given in Section 7.

2 Single moments

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the STSP distribution with the pdf $f(x)$ and cdf $F(x)$ given in (1) and (2), respectively, and let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ denote the corresponding order statistics. Then the pdf of the r^{th} order statistic, $X_{r:n}$, denoted by $f_{r:n}(x)$ for $1 \leq r \leq n$ is given by [3, 12]

$$f_{r:n}(x) = C_{r:n}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x), \qquad 0<x<1, \tag{3}$$

where

$$C_{r:n} = \frac{n!}{(r-1)!(n-r)!}.$$

Let $\mu_{r:n}^{(k)} = E(X_{r:n}^k)$; $1 \leq r \leq n$; $k \in \mathbb{N}$, namely $\mu_{r:n}^{(k)}$ is the k^{th} moment of the r^{th} order statistic. In the next theorem, we provide an explicit expression for $\mu_{r:n}^{(k)}$.

Theorem 1. *For $1 \leq r \leq n$ and $k \in \mathbb{N}$, we have*

$$\mu_{r:n}^{(k)} = C_{r:n}\left[\sum_{\alpha=0}^{n-r}(-1)^{\alpha}\binom{n-r}{\alpha}\frac{\theta^{k+\alpha+r}}{\left(\frac{k}{p}+\alpha+r\right)} + \sum_{\alpha=0}^{k}\sum_{\beta=0}^{r-1}(-1)^{\alpha+\beta}\binom{k}{\alpha}\binom{r-1}{\beta}\frac{(1-\theta)^{\alpha+\beta+n-r+1}}{\left(\frac{\alpha}{p}+\beta+n-r+1\right)}\right], \tag{4}$$

where $C_{r:n}$ is defined as before.

Proof. From (3), we have

$$\mu_{r:n}^{(k)} = C_{r:n} \int_0^1 x^k [F(x)]^{r-1} [1-F(x)]^{n-r} f(x)\,\mathrm{d}x$$

$$= C_{r:n} \int_0^\theta x^k [F(x)]^{r-1} [1-F(x)]^{n-r} f(x)\,\mathrm{d}x$$

$$+ C_{r:n} \int_\theta^1 [1-(1-x)]^k [1-(1-F(x))]^{r-1} [1-F(x)]^{n-r} f(x)\,\mathrm{d}x$$

$$= C_{r:n} \sum_{\alpha=0}^{n-r} (-1)^\alpha \binom{n-r}{\alpha} \int_0^\theta x^k [F(x)]^{\alpha+r-1} f(x)\,\mathrm{d}x \tag{5}$$

$$+ C_{r:n} \sum_{\alpha=0}^{k} \sum_{\beta=0}^{r-1} (-1)^{\alpha+\beta} \binom{k}{\alpha} \binom{r-1}{\beta} \int_\theta^1 (1-x)^\alpha [1-F(x)]^{\beta+n-r} f(x)\,\mathrm{d}x \tag{6}$$

$$= I_1 + I_2,$$

where I_1 and I_2 are the expressions given in (5) and (6), respectively.

Using the transformations $u = F(x) = \theta \left(\frac{x}{\theta}\right)^p$ for I_1 and

$$v = 1 - F(x) = (1-\theta) \left(\frac{1-x}{1-\theta}\right)^p,$$

for I_2, we get

$$I_1 = \theta^{k-\frac{k}{p}}\ C_{r:n} \sum_{\alpha=0}^{n-r} (-1)^\alpha \binom{n-r}{\alpha} \int_0^\theta u^{\frac{k}{p}+\alpha+r-1} \mathrm{d}u$$

$$= C_{r:n} \sum_{\alpha=0}^{n-r} (-1)^\alpha \binom{n-r}{\alpha} \frac{\theta^{k+\alpha+r}}{\frac{k}{p}+\alpha+r}, \tag{7}$$

and

$$I_2 = C_{r:n} \sum_{\alpha=0}^{k} \sum_{\beta=0}^{r-1} (-1)^{\alpha+\beta} \binom{k}{\alpha} \binom{r-1}{\beta} \int_0^{1-\theta} v^{\frac{\alpha}{p}+\beta+n-r} \mathrm{d}v$$

$$= C_{r:n} \sum_{\alpha=0}^{k} \sum_{\beta=0}^{r-1} (-1)^{\alpha+\beta} \binom{k}{\alpha} \binom{r-1}{\beta} \frac{(1-\theta)^{\alpha+\beta+n-r+1}}{\frac{\alpha}{p}+\beta+n-r+1}. \tag{8}$$

Considering (7) and (8), we reach the required result. □

Remark 1. *Setting $n = r = 1$ in (4), we get*

$$\mu_{1:1}^{(k)} = \frac{p\ \theta^{k+1}}{p+k} + \sum_{\alpha=0}^{k} (-1)^{\alpha} \binom{k}{\alpha} \frac{p}{p+\alpha} (1-\theta)^{\alpha+1} = E(X^k).$$

Consequently, if we set $k = n = r = 1$ in (4), we get

$$\mu_{1:1}^{(1)} = \frac{(p-1)\theta + 1}{p+1} = E(X),$$

which is the mean of the STSP distribution and it coincides with the expression obtained by [28]. Similarly, if we set $r = 1$ in (4), we get

$$\mu_{1:n}^{(k)} = n \left[\sum_{\alpha=0}^{n-1} (-1)^{\alpha} \binom{n-1}{\alpha} \frac{\theta^{k+\alpha+1}}{\frac{k}{p} + \alpha + 1} + \sum_{\alpha=0}^{k} (-1)^{\alpha} \binom{k}{\alpha} \frac{(1-\theta)^{\alpha+n}}{\frac{\alpha}{p} + n} \right], \quad (9)$$

and if we set $r = n$ in (4), we get

$$\mu_{n:n}^{(k)} = n \left[\frac{\theta^{k+n}}{\frac{k}{p} + n} + \sum_{\alpha=0}^{k} \sum_{\beta=0}^{n-1} (-1)^{\alpha+\beta} \binom{k}{\alpha} \binom{n-1}{\beta} \frac{(1-\theta)^{\alpha+\beta+1}}{\frac{\alpha}{p} + \beta + 1} \right]. \quad (10)$$

The expressions in (9) and (10) are the k^{th} moments of the sample minimum and the sample maximum, respectively.

3 Application to the lifetimes of coherent systems

The results of Section 2 can be used to evaluate the expected lifetimes of coherent systems as well. Coherent systems play a key role in reliability analysis. Samaniego [25] discussed the concept of signature of a coherent system, that is widely applied to computation of the expected lifetimes of these systems. Let a system involve n components whose lifetimes, denoted as $X_1, \ldots, X_n$, are independent identically distributed (*iid*) random variables and T be the lifetime of the whole system. Then the i-th signature is defined as $s_i = Pr(T = X_{i:n})$ for $i = 1, \ldots, n$. The expected lifetime of the whole system is then obtained to be

$$E(T) = \sum_{i=1}^{n} s_i E(X_{i:n}). \quad (11)$$

The vector $\boldsymbol{s} = (s_1, \ldots, s_n)$ is called the signature vector. An immediate consequence of (11) is given by (see [25])

$$E(T^k) = \sum_{i=1}^{n} s_i E(X_{i:n}^k), \tag{12}$$

where $k \in \mathbb{N}$.

Navarro and Rychlik [21] emphasized that (12) is valid for coherent systems with components whose lifetimes possess an exchangeable absolutely continuous joint distribution. For more details pertaining to the lifetimes of coherent systems and the concept of signature, see [11, 20].

Now, suppose that a coherent system with n components is considered to be analyzed and suppose further that the component lifetimes, $X_1, \ldots, X_n$, are *iid* random variables from the two-sided power distribution. Then we can apply the result of Theorem 1 to evaluate the expected lifetime of the system. For instance, for $k \in \mathbb{N}$, we have

$$\begin{aligned} E(T^k) \quad = \quad & \sum_{i=1}^{n} s_i C_{i:n} \Bigg[\sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \frac{\theta^{k+j+i}}{\left(\frac{k}{p}+i+j\right)} \\ & + \sum_{j=0}^{k} \sum_{m=0}^{i-1} (-1)^{j+m} \binom{k}{j} \binom{i-1}{m} \frac{(1-\theta)^{j+m+n-i+1}}{\left(\frac{j}{p}+m+n-i+1\right)} \Bigg]. \end{aligned}$$

4 Product moments

The joint pdf of $X_{r:n}$ and $X_{s:n}$, the r^{th} and s^{th} order statistics, denoted by $f_{r,s:n}(x, y)$, for $1 \leq r < s \leq n$ is given by [3, 12]

$$f_{r,s:n}(x,y) = C_{r,s:n}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s} f(x) f(y), \quad 0 < x < y < \infty, \tag{13}$$

where

$$C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}.$$

Let $\mu_{r,s:n}^{(k,l)} = E(X_{r:n}^k X_{s:n}^l)$, $1 \leq r < s \leq n$; $k, l \in \mathbb{N}$, denote the $(k,l)^{th}$ product moment of the r^{th} and s^{th} order statistics from the STSP distribution. Çetinkaya and Genç [9] derived an explicit expression for the simple product moment of the r^{th} and s^{th} order statistics for the STSP distribution by means of taking $l = k = 1$. i.e. $\mu_{r,s:n} = \mu_{r,s:n}^{(1,1)} = E(X_{r:n} X_{s:n})$. Here, an attempt has been made to derive a general expression for the $(k,l)^{th}$ product moment for the STSP distribution, which includes the simple product moment as well. The result is presented in the following theorem.

Theorem 2. *For $1 \leq r < s \leq n$ and $k, l \in \mathbb{N}$, we have*

$$\mu_{r,s:n}^{(k,l)} = C_{r,s:n}\Bigg[\sum_{\alpha=0}^{n-s}\sum_{\beta=0}^{s-r-1}\binom{n-s}{\alpha}\binom{s-r-1}{\beta}\frac{(-1)^{\alpha+\beta}\;\theta^{k+l+\alpha+s}}{\left(\frac{k}{p}+\beta+r\right)\left(\frac{k+l}{p}+\alpha+s\right)}$$

$$+\sum_{\alpha=0}^{s-r-1}\sum_{\beta=0}^{s-r-1-\alpha}\sum_{\gamma=0}^{l}(-1)^{\alpha+\beta+\gamma}\binom{s-r-1}{\alpha}\binom{s-r-1-\alpha}{\beta}\binom{l}{\gamma}$$

$$\times\frac{\theta^{k+\beta+r}(1-\theta)^{\alpha+\gamma+n-s+1}}{\left(\frac{k}{p}+\beta+r\right)\left(\frac{\gamma}{p}+\alpha+n-s+1\right)}$$

$$+\sum_{\alpha=0}^{r-1}\sum_{\beta=0}^{s-r-1}\sum_{\gamma=0}^{l}\sum_{\delta=0}^{k}(-1)^{\alpha+\beta+\gamma+\delta}\binom{r-1}{\alpha}\binom{s-r-1}{\beta}\binom{l}{\gamma}\binom{k}{\delta}$$

$$\times\frac{(1-\theta)^{\alpha+\gamma+\delta+n-r+1}}{\left(\frac{\gamma}{p}+\beta+n-s+1\right)\left(\frac{\gamma+\delta}{p}+\alpha+n-r+1\right)}\Bigg], \tag{14}$$

where $C_{r,s:n}$ is defined as before.

Proof. From (13), we have

$$\begin{aligned}\mu_{r,s:n}^{(k,l)} = {} & C_{r,s:n}\int_0^\theta\int_x^\theta x^k y^l[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}\\ & \times f(x)f(y)\,\mathrm{d}y\,\mathrm{d}x\\ & + C_{r,s:n}\int_0^\theta\int_\theta^1 x^k y^l[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}\\ & \times f(x)f(y)\,\mathrm{d}y\,\mathrm{d}x\\ & + C_{r,s:n}\int_\theta^1\int_x^1 x^k y^l[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}\\ & \times f(x)f(y)\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Using the binomial expansion, we have

$$\begin{aligned}
\mu_{r,s:n}^{(k,l)} &= C_{r,s:n} \sum_{\alpha=0}^{n-s} \sum_{\beta=0}^{s-r-1} (-1)^{\alpha+\beta} \binom{n-s}{\alpha} \binom{s-r-1}{\beta} \\
&\times \int_0^\theta x^k [F(x)]^{\beta+r-1} \left\{ \int_x^\theta y^l [F(y)]^{\alpha-\beta+s-r-1} f(y) \mathrm{d}y \right\} f(x)\, \mathrm{d}x \\
&+ C_{r,s:n} \sum_{\alpha=0}^{s-r-1} \sum_{\beta=0}^{s-r-1-\alpha} (-1)^{\alpha+\beta} \binom{s-r-1}{\alpha} \binom{s-r-1-\alpha}{\beta} \\
&\times \int_0^\theta x^k [F(x)]^{\beta+r-1} \left\{ \int_\theta^1 y^l [1-F(y)]^{n-s+\alpha} f(y) \mathrm{d}y \right\} f(x)\, \mathrm{d}x \\
&+ C_{r,s:n} \sum_{\alpha=0}^{r-1} \sum_{\beta=0}^{s-r-1} (-1)^{\alpha+\beta} \binom{r-1}{\alpha} \binom{s-r-1}{\beta} \\
&\times \int_\theta^1 x^k [1-F(x)]^{\alpha-\beta+s-r-1} \left\{ \int_x^1 y^l [1-F(y)]^{\beta+n-s} f(y) \mathrm{d}y \right\} f(x)\, \mathrm{d}x.
\end{aligned}$$

Evaluating the above integrals, we arrive at the required result. □

5 Computational results

In this section, the results, established in the Sections 2 and 4, are utilized to compute the means, variances and the covariances of the order statistics coming from the STSP distribution. R software (see [23]) has been used to compute the means, variances and covariances.

Setting $k = 1$ in (4), a table of the computed means of the order statistics from the STSP distribution has been prepared for $n = 1(1)5$ and selected parameter values. The computed values of the means to seven decimal places are reported in Table 1. It can be seen that the condition $\sum_{r=1}^{n} \mu_{r:n} = nE(X)$, (see [12]) is satisfied.

We can find the variance of $X_{r:n}$, $1 \leq r \leq n$ using the relation

$$Var(X_{r:n}) = \mu_{r:n}^{(2)} - \left[\mu_{r:n}^{(1)}\right]^2,$$

where $\mu_{r:n}^{(1)}$ and $\mu_{r:n}^{(2)}$ can be calculated by setting $k = 1$ and $k = 2$ in (4), respectively. In addition, the covariance of $X_{r:n}$ and $X_{s:n}, 1 \leq r < s \leq n$, can be found by using the relation

$$Cov(X_{r:n}, X_{s:n}) = \mu_{r,s:n}^{(1,1)} - \mu_{r:n}^{(1)} \mu_{s:n}^{(1)},$$

Table 1: Means of order statistics.

n	r	$\theta = 0.75, p = 0.5$	$\theta = 0.75, p = 1.5$	$\theta = 0.50, p = 4$
1	1	0.4166667	0.5500000	0.5000000
2	1	0.2187500	0.409375	0.4277778
	2	0.6145833	0.690625	0.5722222
3	1	0.1328125	0.3345881	0.3916667
	2	0.3906250	0.5589489	0.5000000
	3	0.7265625	0.7564631	0.6083333
4	1	0.0888021	0.2867390	0.3677790
	2	0.2648438	0.4781351	0.4633296
	3	0.5164062	0.6397626	0.5366704
	4	0.7966146	0.7953632	0.6322210
5	1	0.0634766	0.2529968	0.3500031
	2	0.1901042	0.4217079	0.4388826
	3	0.3769531	0.5627760	0.5000000
	4	0.6093750	0.6910869	0.5611174
	5	0.8434245	0.8214323	0.6499969

where $\mu_{r,s:n}^{(1,1)}$ can be obtained by setting $k = l = 1$ in (14). The variances and covariances of the order statistics coming from the STSP distribution are computed to seven decimal places for selected parameter combinations and $n = 1(1)5$ [for the covariances $n = 2(1)5$] and the results are reported in Table 2. It can be seen that the condition $\sum_{r=1}^{n}\sum_{s=1}^{n}\sigma_{r,s:n} = n\sigma^2$ (see [12]), is satisfied, where $\sigma_{r,s:n} = Cov(X_{r:n}, X_{s:n})$ and $\sigma^2 = Var(X)$.

6 Real data examples

In this section, we provide two real data examples to show the flexibility and applicability of the STSP distribution. To this end, we compare the fit of the STSP distribution with those of some well-known bounded distributions, listed as follows:

- The power function (PF) distribution as a special case of the STSP distribution with $\theta = 1$.
- The triangular (Tr) distribution as a special case of the STSP distribution with $p = 2$.
- The beta distribution with the following pdf

$$f_B(x) = \frac{\Gamma(p+\theta)}{\Gamma(\theta)\Gamma(p)}\, x^{\theta-1}(1-x)^{p-1}, \qquad 0 < x < 1,\, p > 0,\, \theta > 0.$$

Table 2: Variances and covariances of order statistics.

n	s	r	$\theta=0.75, p=0.5$	$\theta=0.75, p=1.5$	$\theta=0.50, p=4$
1	1	1	0.1222222	0.0600000	0.0166670
2	1	1	0.0636067	0.0455817	0.0114506
	2	1	0.0391710	0.0197754	0.0052160
		2	0.1024957	0.0348675	0.0114507
3	1	1	0.0321376	0.0351272	0.0097840
	2	1	0.0288679	0.0195107	0.0044781
		2	0.0822335	0.0329323	0.0069597
	3	1	0.0154454	0.0093084	0.0027801
		2	0.0443301	0.0157354	0.0044781
		3	0.0750087	0.0228310	0.0097841
4	1	1	0.0173467	0.0280153	0.0088913
	2	1	0.0181015	0.0173400	0.0041074
		2	0.0532674	0.0289885	0.0056150
	3	1	0.0144564	0.0104387	0.0025847
		2	0.0426694	0.0174567	0.0036086
		3	0.0795579	0.0238145	0.0056150
	4	1	0.0070585	0.0052685	0.0018343
		2	0.0208761	0.0088133	0.0025846
		3	0.0392650	0.0120485	0.0041074
		4	0.0538632	0.0164504	0.0088912
5	1	1	0.0100514	0.0230081	0.0082698
	2	1	0.0113115	0.0151527	0.0038938
		2	0.0337001	0.0252735	0.0050574
	3	1	0.0106405	0.0101057	0.0024203
		2	0.0317361	0.0168567	0.0031761
		3	0.0616708	0.0226207	0.0042103
	4	1	0.0078693	0.0063583	0.0017796
		2	0.0234898	0.0106068	0.0023507
		3	0.0458214	0.0142417	0.0031761
		4	0.0698746	0.0180251	0.0050574
	5	1	0.0035906	0.0033387	0.0013441
		2	0.0107237	0.0055698	0.0017796
		3	0.0209754	0.0074826	0.0024204
		4	0.0322964	0.0094946	0.0038938
		5	0.0389044	0.0126587	0.0082697

- The Kumaraswamy (Kw) distribution with the following pdf

$$f_K(x) = \theta\, p\, x^{\theta-1}\left(1-x^{\theta}\right)^{p-1}, \qquad 0<x<1,\, p>0,\, \theta>0.$$

Table 3: The data of Example 1.

0.3389363	0.4319150	0.7599317	0.7246257	0.7575833
0.8115556	0.7853385	0.7836599	0.8156268	0.8474128
0.7680072	0.8434851	0.7874084	0.8498682	0.6959701
0.8423163	0.8286891	0.5801935	0.4306806	0.7425630

- The Topp-Leone (TL) distribution with the following pdf

$$f_T(x) = 2\theta(1-x)\left(2x - x^2\right)^{\theta-1}, \qquad 0 < x < 1,\ \theta > 0.$$

We calculate the maximum likelihood (ML) estimates of the unknown parameters of all the considered distributions. We also obtain some well-known goodness-of-fit measures such as the Akaike information criterion (AIC), Bayesian information criterion (BIC), the Kolmogorov-Smirnov test statistic (K-S TS) for the purpose of comparison. Smaller values of these statistics signal a better fit. In addition, we obtain the p-value of the K-S test for all the considered distributions.

Example 1. The first real data set includes the transformed water capacity data for the month of February from 1991 to 2010 taken from the Shasta Reservoir in California, USA, see http://cdec.water.ca.gov/reservoir_map .html and [16]. The maximum capacity of the reservoir is 4552000 Acro-Foot (AF). The actual data do not lie in the interval $[0, 1]$ and therefore we divided the original data by the maximum capacity of the reservoir. The transformed data are given in Table 3.

The ML estimates of the parameters of the considered models, as well as the computed goodness-of-fit measures, for the data of Table 3 are given in Table 4. From Table 4, we see that the STSP distribution possesses the smallest values of the considered goodness-of-fit statistics and the largest K-S test p-value, therefore it provides the best fit among all the considered distributions. We also provided the histogram of the data, as well as the fitted pdfs of the considered models, in Figure 1 and the probability-probability (P-P) plots in Figure 2 for the data of Example 1 to compare the models visually. Figures 1 and 2 also confirm that the STSP distribution can be considered as the most suitable model among the considered distributions for the data of Example 1.

Example 2. The second data set deals with the measurements on petroleum rock samples, see [10]. The data are given in Table 5.

Table 4: The ML estimates of the parameters and the computed goodness-of-fit measures for Example 1.

Model	θ	p	AIC	BIC	K-S TS	K-S p-value
STSP	0.8423163	5.2976	−30.24096	−28.24950	0.13199	0.8328
PF	—	2.8163	−13.61988	−12.62415	0.36754	0.0063
Tr	0.8423163	—	−18.17535	−17.17962	0.37505	0.0049
Beta	7.31572	2.90989	−21.12384	−19.13238	0.23594	0.1834
Kw	6.34757	4.48939	−22.94942	−20.95795	0.22088	0.2447
TL	8.66677	—	−21.17524	−20.17951	0.25486	0.1241

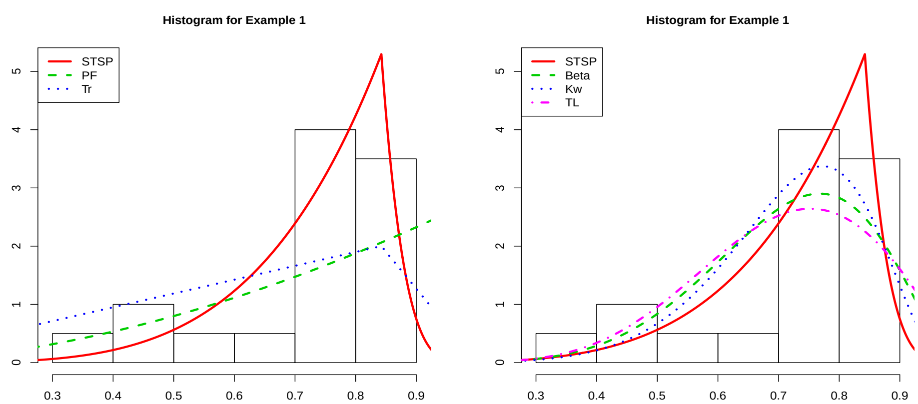

Figure 1: Histogram of the data and the fitted pdfs of the considered models for Example 1.

Table 6 contains the ML estimates of the parameters of the considered models and the calculated goodness-of-fit measures for the data of Table 5.

From Table 6, we observe that the STSP distribution outperforms all the other considered distributions in the sense of the considered criteria. Figures 3 and 4 display the histogram of the data, as well as the fitted pdfs of the considered models, and the P-P plots for the data of Example 2, respectively. Figures 3 and 4 verify that the STSP model is superior to the

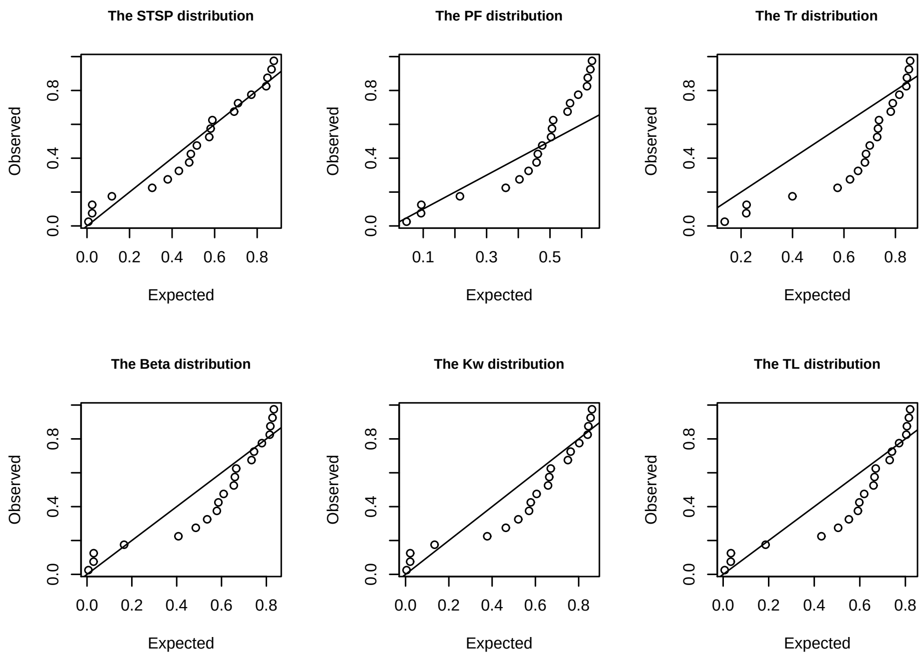

Figure 2: The P-P plots for the data of Example 1.

Table 5: The data of Example 2.

0.0903296	0.148622	0.183312	0.117063	0.122417	0.167045
0.189651	0.164127	0.203654	0.162394	0.150944	0.148141
0.228595	0.231623	0.172567	0.153481	0.204314	0.262727
0.200071	0.144810	0.113852	0.291029	0.240077	0.161865
0.280887	0.179455	0.191802	0.133083	0.225214	0.341273
0.311646	0.276016	0.197653	0.326635	0.154192	0.276016
0.176969	0.438712	0.163586	0.253832	0.328641	0.230081
0.464125	0.420477	0.200744	0.262651	0.182453	0.200447

other considered models in Example 2.

7 Concluding remarks

In this paper, we focused on finding explicit expressions for the single and product moments of order statistics coming from the STSP distribution.

Table 6: The ML estimates of the parameters and the computed goodness-of-fit measures for Example 2.

Model	θ	p	AIC	BIC	K-S TS	K-S p-value
STSP	0.148622	8.11149	-112.7901	-109.0477	0.09598	0.7323
PF	—	0.63002	-10.02375	-8.152545	0.42952	0.0000
Tr	0.148622	—	-52.70707	-50.83587	0.44717	0.0000
Beta	5.941767	21.20572	-107.20044	-103.45804	0.14276	0.2565
Kw	2.718741	44.66038	-100.98307	-97.24067	0.15331	0.1886
TL	0.989418	—	-40.33193	-38.46073	0.3680	0.0000

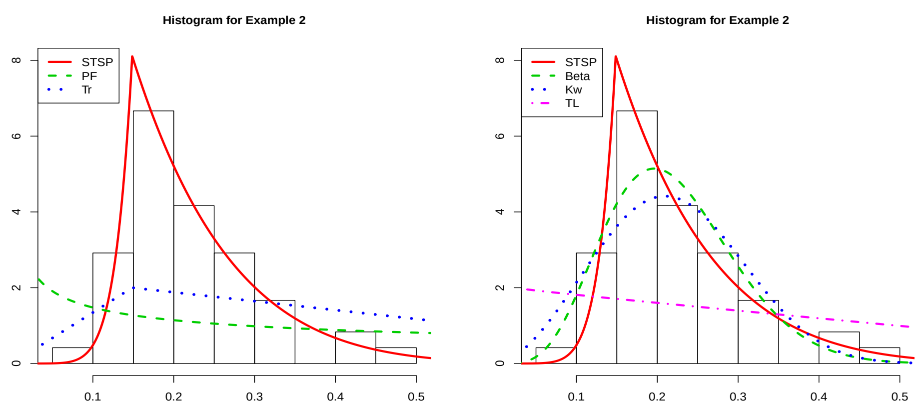

Figure 3: Histogram of the data and the fitted pdfs of the considered models for Example 2.

The results of the paper enable one to calculate the means, variances and covariances of order statistics extracted from a sample of size n, where $n \in \mathbb{N}$. One application of our results refers to the expected lifetimes of coherent systems. The flexibility and applicability of the STSP distribution were also explored using two real data sets. All the computations of the paper were performed by using R software [23] and Maple 16.

Though many properties of the STSP distribution have been discussed

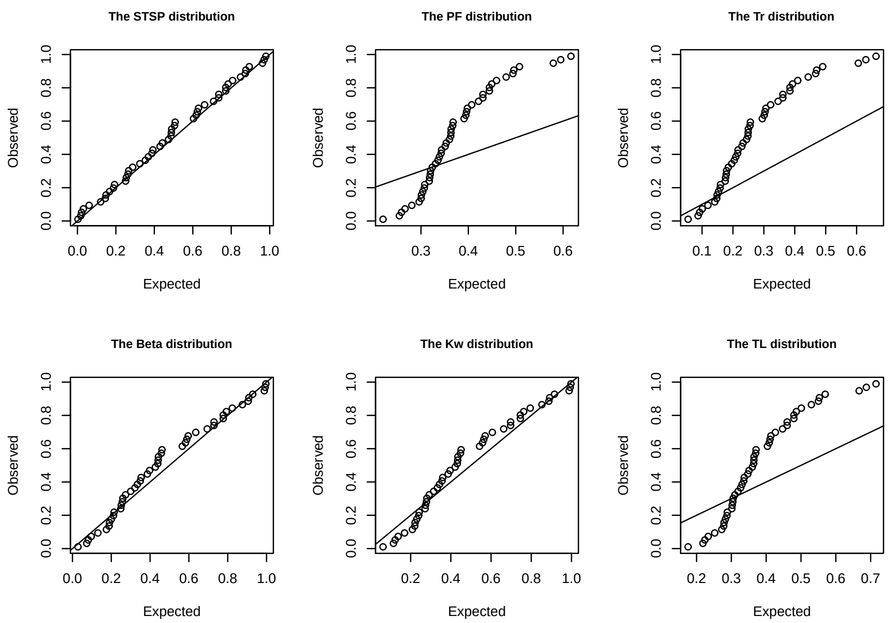

Figure 4: The P-P plots for the data of Example 2.

in the literature, the estimation of the parameters and prediction of future observations based on some types of ordered data and similar topics for this distribution have not been discussed in detail yet and we hope that the results of this paper will be useful for inferential investigations in the future.

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