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An inverse finance problem for estimating volatility in American option pricing under jump-diffusion dynamics

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Abstract. This study attempts to estimate the volatility of the American options pricing model under jump-diffusion underlying asset model. Therefore, the problem is formulated then inverted, and afterward, direct finance problems are defined. It is demonstrated, then, that the price of this type of options satisfies a free boundary Partial Integral Differential Equation (PIDE). The inverse method for estimating the volatility and the American options price is also described in three phases: first, transformation of the direct problem to a non-linear initial and boundary value problem. Second, finding the solution by using the method of lines and the fourth-order Runge-Kutta method.Third, presenting a minimization function with Tikhonov regularization.

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1 Introduction

Analyzing securities is considered truly important for researchers and analysts of finance and economics, auditors, and risk managers in financial institutions. These types of securities, including stocks and derivatives,

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without using a proper mathematical model, usually, some slight mistakes, unrecognizable errors, and misunderstandings will occur, that none of which should be ignored by financial analysts and economists. Fortunately, financial and risk analysts, using advanced mathematical techniques, have recently developed new models that may react to market volatilities quite well. These attempts have led to solving some of market problems.

The Black-Scholes model is one of the most famous models in financial markets. In financial modeling, the Black-Scholes model plays an important role in pricing risky assets and also provides an efficient foundation for option-pricing modeling frameworks. The usefulness of this famous model as a theoretical base in financial markets is proven. It, however, has some weaknesses in predicting important features of underlying asset's revenues, and implicit market volatility. For this reason, a lot of reasoning is made in favor of developing alternative models, and some works have been done in this respect.

Under the Black-Scholes model, the price of an underlying asset follows the geometric Brownian motion, which considers constant volatility and drifts; so that, it cannot predict or explain the dynamic or the stochastic behavior of the price variations. To solve this problem, considering underlying asset model as a combination of diffusion and jump terms, we construct an American option pricing model.

Many researchers have studied this type of modeling, all of which have assumed in their option pricing studies that volatility and drift are to be known. This assumption is due to the lack of market data in most of the financial markets, and volatility is an unknown function. So many researchers, including Rubinstein (1994), Kani and Derman (1994), Dupire [7], Andersen and Brotherton-Ratcliffe [1], Bouchouev and Isakov [3], Verma (1999), Coleman and Avellaneda (1997) Lagnado and Osher [12], Jackson et al. [9], and Neisy and Salmani [13], have concluded that volatility might depend on the underlying asset price (S) and time (t). Most of the researches, in this area, have focused on European options and American options, which the diffusion underlying asset models were used for their pricing; however; this paper attempts to study the volatility in American option pricing model under jump-diffusion underlying asset model, by using the inverse problem in the PDE. In this study, we also utilize new numerical methods with high accuracy, which will be explained in the next parts.

This article is organized as follows: in the second section, we study the problem of American option pricing under jump-diffusion underlying asset model. In the third section, the direct finance problem for American option pricing is defined. The direct problem is an initial and boundary value problem with free boundary, that by adding a penalty term to the PIDE is transformed to I& BVP with fix boundary. The direct finance problem, then, can be solved by combining it with the method of lines, and Runge-Kutta method. In the fourth section, the inverse problem for American option pricing is defined and using Tikhonov regularization method, and the Euler-Lagrange equations we will show that volatility satisfies in the Poisson problem. In the fifth section, an algorithm will be developed for the above mentioned process. Finally, in the last section, an example will be solved using Mathematica software, to test our proposed method.

2 The mathematical formulation

In this section, we try to develop a pricing model which its underlying asset model is a combination of diffusion and jump terms. Other alternatives such as the combination of the stochastic volatility and jumps or even the simple models without considering the jumps in the market are noteworthy. However, due to the great importance of the effects of the jumps in the pricing models and also due to their applicability in Iran's market, our main focus in this paper is on the models which consider the jumps in the market price. Considering the fact that, in general, the stock price volatility process does not only depend on the diffusion, but also it is affected by big jumps in some cases; and also it is noteworthy that the stock's return does not follow a log-normal distribution in most of the times, as a result, the Brownian motion model cannot provide a real image of the underlying asset. Therefore, we utilize an alternative model for the underlying asset. For this purpose, we present a model, which is a combination of both diffusion and jump variables, by regarding the American option price as the underlying asset:

$$dS = (\mu - \gamma\kappa)Sdt + \sigma dW + (J-1)Sdp = dS_{BM} + dS_{JM}, \qquad 0 < t < T,$$

where S denotes the underlying asset price, μ is the drift rate, σ is the volatility, dW is the standard Winner process, J is the jump size. dS_{BM} and dS_{JM} represent the change in the stock price due to the geometric Brownian motion and the Jump, respectively. dp is the independent Poisson process with density of $\gamma > 0$:

$$dp = \begin{cases} 0, & \text{with probablity } 1-\gamma dt, \\ 1, & \text{with probablity } \gamma dt, \end{cases}$$

and $(J-1)Sdp$ is due to jump. Assume that the jump size has some known probability density $f(J)$. Given that a jump occurs, the probability of a jump in $[J, J+dJ]$. Also,

$$\int_{-\infty}^{+\infty} f(J)p(J)dJ = \int_{0}^{+\infty} f(J)p(J)dJ = 1.$$

The strategy taken in this paper, is to consider only the positive jumps such that $p(J) = 0$ if $J \leq 0$. If $f = f(J)$, then the expected value of f is $E(f) = \int_0^{+\infty} f(J)p(J)dJ$. So that we will have:

$$\kappa = E(J-1) = \int_{0}^{+\infty} (J-1)p(J)dJ.$$

It is noteworthy to mention that many researchers, such as Bates [2] and Kou [11] have studied on development and application of jump term in pricing. Now we consider a portfolio that consists of an option with a price of $V = V(S,t)$, and $-\Lambda$ of the underlying asset with the abovementioned model. If $\mathfrak{T}$ is the value of this portfolio then

$$\mathfrak{T} = V - \Lambda S.$$

Since we have considered the change in the underlying asset price dS to be dependent on the change in the stock price due to both the geometric Brownian motion and the Jump, we consider the dynamic of our portfolio to be consisted of the dynamics of two portfolios one due to geometric Brownian motion and one due to the Jump:

$$d\mathfrak{T} = d\mathfrak{T}_{BM} + d\mathfrak{T}_{JM},$$

where

$$\begin{aligned} d\mathfrak{T}_{BM} &= \text{Variations in value of portfolio for underlying asset with} \\ &\quad\ \text{geometrical Brownian motion model,} \\ d\mathfrak{T}_{JM} &= \text{Variations due to pure jump.} \end{aligned}$$

So, considering the Ito's lemma we have:

$$\begin{aligned} \mathfrak{T}_{BM} &= dV(S,t) - \Lambda dS_{BM} \\ &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS_{BM} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}dt - \Lambda dS_{BM} \\ &= (\frac{\partial V}{\partial t} + (\mu - \gamma\kappa)S(\frac{\partial V}{\partial S} - \Lambda) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt \\ &\quad +\sigma S(\frac{\partial V}{\partial S} - \Lambda)dW. \end{aligned}$$

Black and Scholes showed that in order to eliminate the stochastic term in the portfolio model, we can assume $\frac{\partial V}{\partial S} = \Lambda$ condition (an assumption which is impossible in perfect markets). We, therefore, have:

$$d\mathfrak{T}_{BM} = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt,$$

and also:

$$d\mathfrak{T}_{JM} = [V(JS,t) - V(S,t)]dq - \frac{\partial V}{\partial S}(J-1)Sdq.$$

Now we assume that the jumps for these portfolios are uncorrelated and the variance of the total portfolio is small (there is little risk). The expected return should be

$$r\mathfrak{T}dt = E[d\mathfrak{T}] = E[\mathfrak{T}_{BM}] + E[\mathfrak{T}_{JM}],$$

where r is the risk free interest rate.

Now by substituting the above-mentioned equations, we will find that:

$$\begin{aligned} r(V - \frac{\partial V}{\partial S}S)dt = E[(\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt] \\ + E[V(JS,t) - V(S,t)]E[dq] - \frac{\partial V}{\partial S}E[(J-1)]SE[dq]. \end{aligned}$$

Since, $[dq] = \gamma dt.1 + (1-\gamma dt).0 = \gamma dt$, so we have:

$$\begin{aligned} r(V - \frac{\partial V}{\partial S}S)dt = (\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt \\ + E[V(JS,t))]\gamma dt - V(S,t)\gamma dt - \frac{\partial V}{\partial S}E[(J-1)]S\gamma dt. \end{aligned}$$

And finally we can eliminate dt from each side of equation and reach to:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \gamma\kappa)S\frac{\partial V}{\partial S} - (r+\gamma)V + \gamma E[V(JS,t)] = 0, \quad (1)$$

where

$$E[V(JS,t)] = \int_0^{+\infty} V(JS,t)p(J)dJ. \quad (2)$$

Eq. (1) is a PIDE, and solving it requires the initial and boundary conditions, so that knowing these conditions, the option that includes this equation, will be revealed.

3 The direct finance problem

We suppose that S is the price of the underlying asset (stock), with jump-diffusion model introduced in Section 2, and $V = V(S,t)$ is the price of the American put-option under S (underlying asset). We consider the variable r as the risk neutral interest rate, μ represents the drift, σ represents the volatility, t represents the time, T represents the maturity date, and K represents the strike price. Since an important feature of American options is that they can be exercised anytime up to the maturity date, by considering $\mathcal{G} = \mathcal{G}(t)$ as the boundary of exercising; therefore, considering what is described in Section 2, the price of American option in no-arbitrage condition should satisfy in the following PIDE, for $0 < t < T$,

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \gamma\kappa) S \frac{\partial V}{\partial S} - (r+\gamma)V \\ &\qquad\qquad\qquad\qquad + \gamma E[V(JS,t)] = 0, \qquad \text{if } S > \mathcal{G}(t), \\ &V(S,t) = K - S, \qquad\qquad\qquad\qquad\qquad \text{if } 0 \le S \le \mathcal{G}(t), \end{aligned}$$

with the following initial and boundary conditions:

$$\begin{aligned} &V(S,T) = \max(K - S, 0), \\ &V(0,t) = 0, \\ &\lim_{S \to +\infty} V(S,t) = 0, \\ &V(\mathcal{G}(t), t) = K - \mathcal{G}(t), \\ &\frac{\partial V(\mathcal{G}(t), t)}{\partial S} = -1, \\ &\mathcal{G}(T) = K. \end{aligned} \tag{3}$$

The above problem is an initial and free boundary value problem. If all of functions and parameters in this problem are known except V=V(S,t), then the problem is called a Direct Finance Problem (DFP).

3.1 Penalty method to solve the DFP

To solve the DFP, we first fix the boundary of the function by adding a penalty term to the PIDE, and then rewrite the problem. Nielsen et al. [14], and Zvan et al. [17] proposed the following penalty term for the American option:

$$\frac{\epsilon C}{V_\epsilon(S,t) + \epsilon - K + S}, \tag{4}$$

where $C > rK$ is a constant, and $0 < \epsilon \leq 1$. The problem (3), by adding this term to the PIDE, it is transformed as follows:

$$\frac{\partial V_\epsilon}{\partial t}dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_\epsilon}{\partial S^2} + (r - \gamma\kappa)S\frac{\partial V_\epsilon}{\partial S} - (r+\gamma)V_\epsilon + \gamma \int_0^{+\infty} V_\epsilon(JS,t)p(J)dJ$$
$$+ \frac{\epsilon C}{V_\epsilon(S,t) + \epsilon - K + S} = 0,$$

In which we have

$$\begin{aligned} &0 < S < \infty, \qquad 0 < t < T, \\ &V_\epsilon(S,T) = \max(K - S, 0), \\ &V_\epsilon(0,t) = 0, \\ &\lim_{S \to +\infty} V_\epsilon(S,t) = 0, \end{aligned} \tag{5}$$

In order to solve the problem by using the method that will be described in the next section, we consider the following changes of variables:

$$x = \log S, \qquad \zeta = \log J, \qquad \tau = T - t,$$

and

$$V_\epsilon(\exp(x), T - \tau) = u_\epsilon(x,\tau),$$
$$\mathfrak{p} = p(\exp(\zeta))\exp(\zeta).$$

Therefore, we have

$$V_\epsilon(\exp(x)\exp(\zeta), T-\tau) = V_\epsilon(\exp(x+\zeta), T-\tau) = u_\epsilon(x+\zeta,\tau),$$

with this substitution of variables, the problem (5) is transformed to

$$\begin{aligned} \frac{\partial u_\epsilon}{\partial \tau} =& \frac{1}{2}\sigma^2 \frac{\partial^2 u_\epsilon}{\partial x^2} + (r - \kappa\gamma - \frac{1}{2}\sigma^2)\frac{\partial u_\epsilon}{\partial x} - (r+\gamma)u_\epsilon \\ &+ \gamma \int_{-\infty}^{\infty} u_\epsilon(x+\zeta)\mathfrak{p}d\zeta - \frac{\epsilon C}{u\epsilon(x,\tau) + \epsilon - K + exp(x)} = 0, \end{aligned}$$

with the following terms and conditions,

$$\begin{aligned} -\infty < x < \infty,& \quad 0 < \tau < T, \\ u_\epsilon(S,0) =& \max\{K - \exp(x), 0\}, && -\infty < x < \infty, \\ u_\epsilon(0,\tau) =& 0, && 0 \leq \tau \leq T, \\ \lim_{x\to+\infty} u_\epsilon(x,\tau) =& 0, && 0 \leq \tau \leq T, \\ \lim_{x\to-\infty} u_\epsilon(x,\tau) =& K\exp(-r\tau), && 0 \leq \tau \leq T. \end{aligned} \tag{6}$$

3.2 The method of lines

In order to apply the method of lines to the problem (6), we first need to obtain an acceptable approximation of the integral term. For this purpose, we must calculate $a_{\min}$ and $b_{\max}$ that satisfies following relation:

$$|\int_{-\infty}^{\infty} u_\epsilon(x+\zeta,\tau)\mathfrak{p}(\zeta)d\zeta - \int_{a_{\min}}^{b_{\max}} u_\epsilon(x+\zeta,\tau)\mathfrak{p}(\zeta)d\zeta| < \epsilon_{int},$$

by choosing the following probabilistic density function:

$$\mathfrak{p}(\zeta) = \frac{1}{\sqrt{2\Pi}\delta}\exp(-\frac{\zeta^2}{2\delta^2}),$$

we will need to have [6]

$$\mathfrak{p}(\zeta) \geq \epsilon_{int},$$

and therefore:

$$a_{\min} = +\sqrt{-2\delta^2\log(\delta\epsilon_{int}\sqrt{2\pi})}$$

$$b_{\max} = -a_{\min}.$$

In this method, an acceptable approximation of the integral term can be stated as follows

$$\int_{-\infty}^{\infty} u_\epsilon(x+\zeta,\tau)\mathfrak{p}(\zeta)d\zeta \cong I_N(u_\epsilon(x,\tau),\tau) = \int_{a_{\min}}^{b_{\max}} u_\epsilon(x+\zeta,\tau)\mathfrak{p}(\zeta)d\zeta.$$

Now the interval $[a_{\min}, b_{\max}]$ can be partitioned into N_{int} subintervals with length of $\Delta x_{int} = \frac{b_{\max}-a_{\min}}{N_{int}}$, for each, and using the Newton-Cotes integration method, we have:

$$\begin{aligned} I_N(u_\epsilon(x,\tau),\tau) &\cong \frac{b_{\max}-a_{\min}}{N_{int}}\sum_{k=0}^{N_{int}} \mathcal{W}_{k=0}^{NG} u_\epsilon(x,\zeta_k),\tau) \\ &= \Delta x_{int}\sum_{k=0}^{N_{int}} \mathcal{W}_{k=0}^{NG}(u_\epsilon(x,\tau)\zeta_k\frac{\partial u_\epsilon(x,\tau)}{\partial x} + \frac{\zeta_k^2}{2!}\frac{\partial^2 u_\epsilon(x,\tau)}{\partial x^2} + \cdots), \end{aligned}$$

where, $\{\mathcal{W}_k^{NG}\}$ is a set of weights so that $\sum_{k=0}^{N_{int}} \mathcal{W}_k^{NG} = 1$.

In order to design an acceptable approximation of the derivatives in the PIDE, we can transform $-\infty < x < \infty$ to $x_{\min} < x < x_{\max}$. Then the interval $[x_{\min}, x_{\max}]$ is partitioned into N_{dif} subintervals with length of $\Delta x = \frac{x_{\max}-x_{\min}}{N_{dif}}$ for each, and apply approximation of the centered-difference formula for second order derivative, and the Euler method for first order derivative of x. Therefore, we have:

$$\begin{aligned} I_N(u_\epsilon(x_i,\tau),\tau) &\cong \mathfrak{a}u_\epsilon(x_i,\tau) + \mathfrak{b}\frac{\partial u_\epsilon(x,\tau)}{\partial x} + \mathfrak{c}\frac{\partial^2 u_\epsilon(x,\tau)}{\partial x^2} \\ &= \mathfrak{a}u_\epsilon^i + \mathfrak{b}(\frac{u_\epsilon^{i+1} - u_\epsilon^i}{\Delta x}) + \mathfrak{c}(\frac{u_\epsilon^{i-1} - 2u_\epsilon^i + u_\epsilon^{i+1}}{\Delta x^2}), \end{aligned}$$

also discretize the PIDE for the problem (6) as follows:

$$\begin{aligned} \frac{du_\epsilon}{d\tau} &= \frac{1}{2}\sigma^2(\frac{u_\epsilon^{i-1} - 2u_\epsilon^i + u_\epsilon^{i+1}}{\Delta x^2}) + (r - \kappa\gamma - \frac{1}{2}\sigma^2)(\frac{u_\epsilon^{i+1} - u_\epsilon^i}{\Delta x}) \\ &\quad - (r+\gamma)u_\epsilon^i + \gamma I_N(u_\epsilon(x_i,\tau)) - \frac{\epsilon C}{u_\epsilon^i + \epsilon - K + \exp(x_i)} \\ &\quad = \mathfrak{d}u_\epsilon^{i-1} + \mathfrak{e}u_\epsilon^i + \mathfrak{f}u_\epsilon^{i+1} - \frac{\epsilon C}{u_\epsilon^i(S,t) + \epsilon - K + \exp(x_i)}, \end{aligned}$$

for $i = 1, 2, \ldots, N_{dif}$, where

$$\begin{aligned} &u_\epsilon^i = u_\epsilon^i(\tau) = u_\epsilon(x_i,\tau), \qquad \mathfrak{a} = \Delta x_{int}, \qquad \mathfrak{b} = \Delta x_{int}\sum_{k=1}^{N_{int}} \mathcal{W}_k^{NG}\zeta_k, \\ &\mathfrak{c} = \Delta x_{int}\sum_{k=1}^{N_{int}} \mathcal{W}_k^{NG}\frac{\zeta_k^2}{2!}, \qquad \mathfrak{d} = \frac{1}{2\Delta x^2}(\sigma^2 + \gamma\mathfrak{c}), \\ &\mathfrak{e} = -\frac{1}{\Delta x^2}(\sigma^2 + 2\gamma\mathfrak{c}) - \frac{1}{\Delta x}(r - \kappa\gamma - \frac{1}{2}\sigma^2 + \gamma\mathfrak{b}) - (r + \gamma(1+\mathfrak{a})) \\ &\mathfrak{f} = \frac{1}{2\Delta x^2}(\sigma^2 + 2\gamma\mathfrak{c}) + \frac{1}{\Delta x}(r - \kappa\gamma - \frac{1}{2}\sigma^2 + \gamma\mathfrak{b}), \end{aligned}$$

In addition, the boundary conditions are discretized as below:

$$u_\epsilon^0 = K\exp(-r\tau), \qquad u_\epsilon^{N_{dif}+1} = 0.$$

Therefore, the problem (6) is transformed to the following ordinary non-linear system of differential equations:

$$\begin{aligned} &\frac{du_\epsilon}{d\tau} = A\mathbf{u}_\epsilon + F(\mathbf{u}_\epsilon), \quad 0 < \tau < T, \\ &\mathbf{u}_\epsilon(0) = \mathbf{u_0}, \end{aligned} \tag{7}$$

where

$$A = \begin{pmatrix} \mathfrak{e} & \mathfrak{f} & & & \\ \mathfrak{d} & \mathfrak{e} & \mathfrak{f} & & \\ & \ddots & \ddots & \ddots & \\ & & \mathfrak{d} & \mathfrak{e} & \mathfrak{f} \\ & & & \mathfrak{d} & \mathfrak{e} \end{pmatrix}, \quad \mathbf{u}_\epsilon = \begin{pmatrix} u_\epsilon^1 \\ u_\epsilon^2 \\ \vdots \\ u_\epsilon^{N_{dif}} \end{pmatrix}, \quad \frac{du_\epsilon}{d\tau} = \begin{pmatrix} \frac{du_\epsilon^1}{d\tau} \\ \frac{du_\epsilon^2}{d\tau} \\ \vdots \\ \frac{du_\epsilon^{N_{dif}}}{d\tau} \end{pmatrix},$$

$$\mathbf{u}_0 = \begin{pmatrix} \max\{K - \exp(x_1), 0\} \\ \max\{K - \exp(x_2), 0\} \\ \vdots \\ \max\{K - \exp(x_{N_{dif}}), 0\} \end{pmatrix},$$

$$\mathbf{F}(\mathbf{u}_\epsilon) = \begin{pmatrix} K\exp(-r\tau) - \frac{\epsilon C}{u_\epsilon^1 + \epsilon - K + \exp(x_1)} \\ -\frac{\epsilon C}{u_\epsilon^2 + \epsilon - K + \exp(x_2)} \\ \vdots \\ -\frac{\epsilon C}{u_\epsilon^{N_{dif}} + \epsilon - K + \exp(x_{N_{dif}})} \end{pmatrix}.$$

Finally, we complete the model by solving the initial value problem (7) using the fourth-order Runge-Kutta method.

4 The inverse finance problem

Since market variations depend on σ parameter, it is quite natural that the market volatility, σ, cannot remain constant in long-term. Furthermore, the volatility parameter is one the principle variables in the pricing models which is indeterminate in most cases. Moreover, proper pricing of derivatives requires information about the volatility in the underlying asset's price; finding an efficient method for recognition and calculation of the volatility, therefore, is considerably important in the stock and derivatives markets. Many researchers have focused on finding an efficient way to estimate the volatility parameter, and a lot of investigation has been done in this area, including studies on estimation of the volatility parameter using extra-data, or implicit method which derives the volatility by applying the derivatives pricing model (e.g. Black-Scholes). Dupire [7], Lagnado [12], Neisy [13], Jackson [9], Andersen et al. [1] and Deng et al. [5] amongst others, have done useful studies in this respect. In these studies, some methods have been proposed for the estimation of the volatility, each of which has some pros and cons. For instance, most of these studies do not consider jumps in the underlying asset's process or just focus on estimating a specific case of the volatility parameter. In this study, without any further analysis of the advantages and disadvantages of these methods, by using the inverse problem in the PDEs, we attempt to estimate the volatility in American option pricing models. For this purpose, we first define the inverse finance problem.

4.1 Mathematical model for the inverse finance problem

As explained in the introduction section, volatility in markets can be considered in various forms; but in this article, we attempt to focus on an unknown stochastic volatility. To this end, we assume that S denotes the price of the underlying asset (stock), with jump-diffusion model introduced in section 2, that its volatility $-\sigma = \sigma(S,t)-$ is an unknown function. Also we consider $V = V(S,t)$ as the price of the American option under the underlying asset S. Therefore, based on what was described in section 2, and assumptions posited in section 3, the price of this option in no-arbitrage conditions should satisfy the following PIDE for $0 < t < T$:

$$
\begin{aligned}
&\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2(S,t)S^2\frac{\partial^2 V}{\partial S^2} + (r-\gamma)S\frac{\partial V}{\partial S} - (r+\gamma)V \\
&\qquad\qquad\qquad\qquad + \gamma E[V(JS,t)] = 0, && \text{if } S > \mathcal{G}(t), \\
&V(S,t) = K - S, && \text{if } 0 \le S \le \mathcal{G}(t),
\end{aligned}
$$

where, the initial and boundary conditions are the same as the problem (5). The above-mentioned problem is an initial and free boundary value problem, which is referred to as the inverse problem in partial differential equations literature; and since we apply it to the financial markets, we call it The Inverse Finance Problem (IFP) hereafter.
Since the inverse problem has two unknown factors, solving it requires determination of extra-data as described below.

There are different methods for determination of the extra-data. Thereupon,we assume that we have the parameter N_B with free boundary $\mathcal{G}_1(t)$, $\mathcal{G}_2(t), \mathcal{G}_3(t), \ldots, \mathcal{G}_{N_B}(t)$, and that we have N_i-number of strike prices K_{i1}, $K_{i2}, K_{i3}, \ldots, K_{iN_i}$ for each $\mathcal{G}_i(t)$. We have, therefore, a set of empirical option prices $\{V_{ij}\}$ for each American option for $K_{ij}(i = 1, 2, \ldots, N_B, j = 1, 2, \ldots, N_i)$ strike prices.

So we can consider a set of empirical option prices $\{V_{ij}\}$ with the domain S and t values.

4.2 The minimization function

Based on extra-data V_{ij} ,the minimization function can be defined as follows [8,13,15]:

$$\Pi(\sigma)=\sum_{i=1}^{N_B}\sum_{j=1}^{N_i}\int_0^{\infty}\int_0^{\infty}(V(S,t,K,\mathcal{G}_i(t),\sigma)-V_{ij})^2dSdt, \tag{8}$$

Considering the fact that there are not sufficient data to meet the expectation of the extra-data in the market, using the minimization function (8) we cannot calculate a unique value for the parameter σ. Furthermore, the value of a typical function like (8) is not continuously dependent on the data. We, therefore, face a difficult problem of determining σ using this method [8], so by applying regularization strategies, we try to somehow mitigate the difficulty of the problem.

Considering the principles of regularization strategies, we cannot expect to find an accurate solution for the problem. Hence, we attempt to reach an approximation which is the closest to the correct solution. Many investigators have had useful attempts in this area, including Oshler and Lagnado [12], Neisy and salmani [13] who based on Tikhonov strategy introduced a typical regularization strategy. They calibrated their model for a constant value of S_0 in a specific point of time $t=0$. Therefore, there is no guarantee that the value determined for σ using their method will also explain other values of the index or for any time in the future [3,10].

Here we solve this problem with extra-data, by choosing the following Tikhonov's minimization function [5,8]:

$$\begin{aligned}\mathbb{J}_\beta(\sigma)&=|\,\|\Delta\sigma\||^2+\beta\Pi(\sigma)\\&=\int_0^{\infty}\int_0^{\infty}\Big((\frac{\partial\sigma}{\partial S})^2+(\frac{\partial\sigma}{\partial t})^2+\beta\sum_{i=1}^{N_B}\sum_{j=1}^{N_i}(V(S,t,K_{ij},\mathcal{G}_i(t),\sigma)\\&\quad-V_{ij})^2\Big)dSdt,\end{aligned} \tag{9}$$

and for the minimization of the previously mentioned function, we use the Euler-Lagrange equations [4]. For this purpose we set:

$$H=((\frac{\partial\sigma}{\partial S})^2+(\frac{\partial\sigma}{\partial t})^2+\beta\sum_{i=1}^{N_B}\sum_{j=1}^{N_i}(V(S,t,K_{ij},\mathcal{G}_i(t),\sigma)-V_{ij})^2,$$

so that a σ that minimizes $\mathbb{J}_\beta(\sigma)$ function can be derived by solving the following equation [4]:

$$\frac{d}{dS}\frac{\partial H}{\partial\sigma_s}+\frac{d}{dt}\frac{\partial H}{\partial\sigma_t}-\frac{\partial H}{\partial\sigma}=0, \tag{10}$$

since,

$$\frac{\partial H}{\partial \sigma_s} = 2\frac{\partial \sigma}{\partial S}, \qquad \frac{\partial H}{\partial \sigma_t} = 2\frac{\partial \sigma}{\partial t},$$
$$\frac{\partial H}{\partial \sigma} = 2\beta \sum_{i=1}^{N_B} \sum_{j=1}^{N_i} \frac{\partial V}{\partial \sigma}(V(S,t,K_{ij},\mathcal{G}_i(t),\sigma) - V_{ij}).$$

Then, Eq. (10), is transformed to the following Poisson problem:

$$\frac{\partial^2 \sigma}{\partial S^2} + \frac{\partial^2 \sigma}{\partial t^2} = \beta\psi(V, \frac{\partial V}{\partial \sigma}, \sigma),$$
$$\sigma(0,t) = 0, \qquad \sigma(S,0) = \sigma_0(S), \quad \text{(known)} \tag{11}$$

where

$$\psi(V, \frac{\partial V}{\partial \sigma}, \sigma) = 2\sum_{i=1}^{N_B} \sum_{j=1}^{N_i} \frac{\partial V}{\partial \sigma}(V(S,t,K_{ij},\mathcal{G}_i(t),\sigma) - V_{ij}).$$

Consequently, the IFP of volatility - σ - determination led to solving a Poisson problem. This is a very practical technique since there are a lot of analytical and numerical methods for solving the above-mentioned Poisson problem.

It is clear that the process of determining the volatility -σ - just by solving the Poisson problem is not finished yet, but it rather requires solving the DFP and applying other methods mentioned in this article. Here we summarize all necessary steps for estimating this parameter as the following algorithm.

5 The inverse finance algorithm

- First step: Initial Forecast For The Volatility
  Finding an initial approximation of $\sigma_0(S,t)$ for σ as follows:

$$m = 0,$$
$$\sigma(S,t) = \sigma_m(S,t). \tag{12}$$

  There are many methods for predicting, one of which, uses the historical data of the volatility parameter.

- Second Step: Solving The FDP
  Substituting $\sigma(S,t) = \sigma_m(S,t)$, so that the problem (12) is transformed to a direct problem. Now, we are able to solve the direct

problem by using the method introduced in section 2, and to calculate the following terms in greed-points:

$$V(S,t,K_{ij},\mathcal{G}_i^*(t),\sigma_m(S,t)),\quad \frac{\partial V}{\partial \sigma}(S,t,K_{ij},\mathcal{G}_i^*(t),\sigma_m(S,t)),$$
$$\psi_m=\psi(V,\frac{\partial V}{\partial \sigma},\sigma_m(S,t)).$$

- Third Step: Calculating a Better Approximation of Volatility
  Calculating $\sigma(S,t)$ by solving the Poisson problem (11) for $\psi=\psi_m$, and setting

$$m=m+1,$$
$$\sigma_m(S,t)=\sigma(S,t).$$

- Fourth Step: Stopping Condition
  If, $|\sigma_m(S,t)-\sigma_{m-1}(S,t)|/|\sigma_m(S,t)|$ is smaller than a favorable accuracy, then go to the fifth step. Otherwise, return to the second step.

- Fifth Step: Stopping
  We Solve the direct problem for $\sigma=\sigma_m(S,t)$. Then $V(S,t)$, which is derived from solving the direct problem, and $\sigma_m(S,t)$, are acceptable approximations of American put-option price and volatility, respectively.

6 Numerical Results

To test our proposed method, let us consider the following function:

$$\sigma=\sigma(S)=0.4+0.2e^{-0.35},$$

for an American put-option with maturity date of $T=1$ year, and 8 strike prices

$$K_1=50,K_2=60,K_3=80,K_4=90,K_5=110,K_6=120,$$
$$K_7=150,K-8=160,$$

on free boundaries. Also, we consider r=0.05, and $\epsilon=0.001$ in the penalty term. Therefore, the direct problem (6) is discretized by choosing $N_{int}=N_{dif}=100$, and the initial value problem (10) is solved by using the fourth-order Runge-Kutta method and also by setting $\Delta t=0.02$. Furthermore,

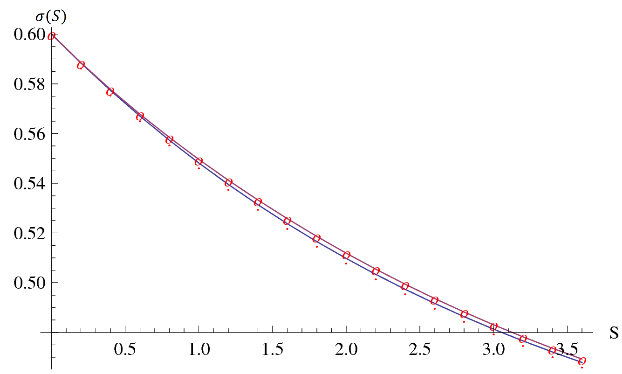

Figure 1: Approximation of American put-options volatility function for initial approximation of $\sigma_0(s) = 0.4$ and Tikhonov parameter of $\beta = 1$. (-: Exact Solution and .: Numerical Solution).

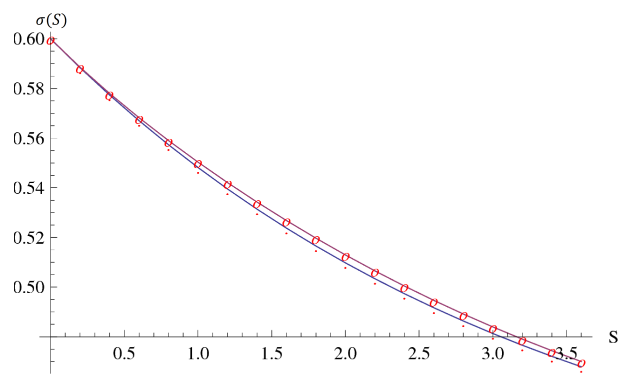

Figure 2: Approximation of American put-options volatility function for initial approximation of $\sigma_0(s) = 0.2$ and Tikhonov parameter of $\beta = 0.1$. (-: Exact Solution and .: Numerical Solution).

the resulting system of non-linear algebraic equations from the fourth-order Runge-Kutta method is solved, by applying the Jacobi and Gauss-Seidel method.

Now we run the inverse algorithm in section 5 in Mathematica software by considering the following Stopping condition:

$$\max |\sigma(s) - \sigma_m(s)| < 0.008.$$

Consequently, the following figures show the results with two initial approximations and three arbitrary Tikhonov parameters.

As can be seen in the above figures, in all of the three cases, applying

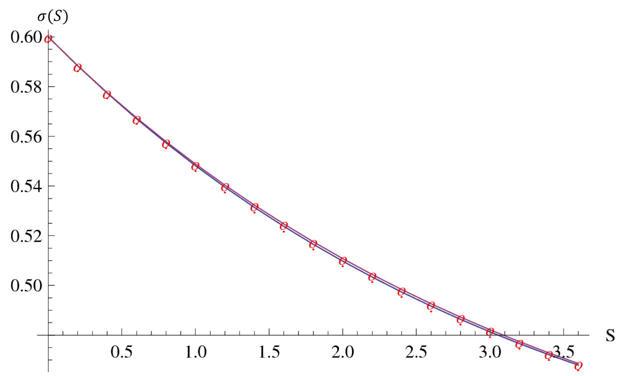

Figure 3: Approximation of American put-options volatility function for initial approximation of $\sigma_0(s) = 0.4$ and Tikhonov parameter of $\beta = 0.04$. (-: Exact Solution and .: Numerical Solution).

this method results in finding an efficient approximations of the volatility parameter. Figure 1, after 8 rounds results in an approximation with maximum error of

$$\max |\sigma(s) - \sigma_8(s)| < 0.0065.$$

Figure 2, after 7 rounds results in an approximation with maximum error of

$$\max |\sigma(s) - \sigma_7(s)| < 0.0071,$$

and finally, Figure 3, after 5 rounds results in an approximation with maximum error of

$$\max |\sigma(s) - \sigma_5(s)| < 0.0021.$$

So we can conclude than this method is somehow fast in obtaining appropriate results. But with increase in sophisticated volatility functions, and complicated free boundaries, complexity and, consequently, time of calculations is expected to increase. Also, in cases where free boundaries are more than one, complexity of calculations will increase.

7 Conclusions and suggestions for future research

In this article we have attempted to develop a model for American options, a direct solving method, and an inverse solving method for estimation of the volatility in this type of options. So the future researches can further explore following areas. Frist, American option pricing models can be derived from underlying asset models with known volatility but unknown and stochastic drift. In this case, resulting models can be used for the purposes of pricing oil futures with unknown convenience yields. Therefore, with some slight changes in our proposed method, resulting models can be solved. Second, by obtaining a closed form of the direct problem, and using extra-data and a minimization function instead of the Poisson problem, one can reach to an integral equation for estimating volatility, which seems to result in a more accurate solution. Finally, one can use entropy regularization instead of the inverse method. Amongst other useful applications not mentioned here that can be subject for future researches, is studying co-integration and consistency with methods discussed in this article.

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