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Solving the general form of the Emden-Fowler equations with the Moving Least Squares method

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Abstract. In the present paper, we have used moving least squares (MLS) method to solve the integral form of the Emden-Fowler equations with initial conditions. The Volterra integral form of the Emden-Fowler equations overcomes their singular behavior at $x = 0$, and the MLS method leads to a satisfactory solution for the equation. The convergence of the method is investigated and finally its applicability is displayed through numerical examples.

Keywords: Emden-Fowler equations, Volterra integral equation, moving least squares method.
AMS Subject Classification: 34K28, 65L05.

1 Introduction

The Emden-Fowler equation with the general following form

$$u''(x) + \frac{k}{x}u'(x) + \alpha f(u(x))g(x) = 0, \;\; u(0) = \beta, \;\; u'(0) = 0, \tag{1}$$

is a model for various phenomena in physics, chemistry, mechanics, etc. in which $f(u)$ and $g(x)$ are functions of u and x respectively, k is a shape factor and α and β are constants. For $\alpha = 1$ and $g(x) \equiv 1$ the equation in (1)

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with different $f(u)$s will be changed into Lane-Emden equation which can be observed in a number of phenomena in mathematical physics specially astrophysics and quantum mechanics. For details on these models, their physical structure and the calculation of their answers refer to [1, 5, 7, 8, 11, 13, 14, 16].

Our aim in this research is to transform the Emden-Fowler equation to its Volterra integral form in order to overcome its singular point in $x = 0$ and then to solve it with the MLS method.

A number of ways have been proposed to solve the Emden-Fowler equation including Adomian decomposition method (Wazwaz et al.) [33], Variational iteration method (Shang et al.) [26], homotopy perturbation method (Chowdhury and Hashim) [8] and Hybrid function method (Tabrizidooz et al.) [28].

2 The integral form of Emden-Fowler equation

In this section, in order to overcome the singularity of the equation, its Volterra integral form will be presented. This form has been offered by Wazwaz et al. to solve the Lane-Emden equation in their article [36].

According to Eq. (1), if $k \neq 1$, in order to transform it to integral form we set

$$u = \beta - \frac{\alpha}{k-1} \int_0^x t(1 - \frac{t^{k-1}}{x^{k-1}}) f(u(t)) g(t) dt. \tag{2}$$

By two times derivating of Eq. (2) and applying the Leibniz's rule we will have

$$u'(x) = -\alpha \int_0^x (\frac{t^k}{x^k}) f(u(t)) g(t) dt,$$
$$u''(x) = -\alpha f(u(x)) g(x) + \alpha \int_0^x k(\frac{t^k}{x^{k+1}}) f(u(t)) g(t) dt.$$

By multiplying the $u'(x)$ in $\frac{k}{x}$ and adding the result to $u''(x)$ the general form of Emden-Fowler equation is achieved. If $k = 1$ the integral form will be as the following

$$u = \beta + \alpha \int_0^x t \ln(\frac{t}{x}) f(u(t)) g(t) dt, \tag{3}$$

which Eq. (3) can be obtained in the limit as $k \to 1$ where the L' Hopital's rule has been used. Therefore, the integral form of Emden-Fowler equation is as the following

$$u(x) = \begin{cases} \beta + \alpha \int_0^x t \ln(\frac{t}{x}) f(u(t)) g(t) dt, & k = 1, \\ \\ \beta - \frac{\alpha}{k-1} \int_0^x t(1 - \frac{t^{k-1}}{x^{k-1}}) f(u(t)) g(t) dt, & k > 0, \quad k \neq 1. \end{cases} \tag{4}$$

3 A summary of MLS method

The MLS, as an approximation method, has been introduced by Shepard [27] in the lowest order case and has been generalized to a higher degree by Lancaster and Salkauskas [13]. The use of MLS in solving PDEs was pioneered by several authors [5, 20, 21].

Suppose that the discrete values of a function u are given at certain data sites $X = \{x_1, x_2, \ldots, x_N\} \subseteq \Omega \subseteq \mathbb{R}$. In the MLS method, approximating function in each certain data site $x \in \Omega$ is written according to the value of functions which are local data sites, and in order to determine the influence of each data point, a weight function $\omega : \Omega \times \Omega \to \mathbb{R}$ is used which the further it goes from data site x, the more the value tends towards zero and for data sites $x, y \in \Omega$ which $|x - y|$ is greater than a certain threshold, it is zero. Let $\mathcal{P}_q$ be the space of polynomials of maximum degree q, $q \ll N$ and $q \leq s$, and suppose $\{p_0, p_1, \ldots, p_m\}$ are basis for $\mathcal{P}_q$, where $m = q$ [37]. The MLS approximation $\hat{u}(x)$ of $u(x)$, $\forall x \in \overline{\Omega}$, can be defined as

$$\hat{u}(x) = \mathbf{p}^T(x)\mathbf{a}(x), \quad \forall x \in \overline{\Omega}, \tag{5}$$

where $\mathbf{p}^T(x) = [p_0(x), p_1(x), \ldots, p_m(x)]$ and $\mathbf{a}(x)$ is a vector with components $a_i(x), i = 0, 1, \ldots, m$. We can use different basis such as monomials, Chebyshev and Legendre polynomials for this method, but the MLS approximation can be implemented in a more stable fashion, if a shifted and scaled polynomial basis function is used as a basis for $\mathcal{P}_q$. In this paper, we use the basis

$$p_\eta(x) = \{\frac{(x_i - x)^\eta}{h^\eta_{X,\Omega}}\}, \quad \eta = 0, 1, \ldots, m,$$

where the fill distance is $h_{X,\Omega} = \sup_{x\in\Omega} \min_{1\leq i\leq n} |x - x_i|$. Components of the vector $\mathbf{a}(x)$ are achieved by functional minimizing as follows

$$\begin{aligned} \mathcal{J}(x) &= \sum_{i=1}^{n} \omega_{h_i}(x - x_i)(\mathbf{p}^T(x_i)\mathbf{a}(x) - u_i)^2 \\ &= [\mathbf{p}.\mathbf{a}(x) - \mathbf{u}]^T.\mathcal{W}.[\mathbf{p}.\mathbf{a}(x) - \mathbf{u}], \end{aligned} \tag{6}$$

where $\omega_{h_i}(x-x_i)$ is the weight function associated with the node i, n is the number of nodes in $\overline{\Omega}$ for which the weight function $\omega_{h_i}(x-x_i) > 0$ and u_i are the fictitious nodal values, but not the nodal values of the unknown trial function $\hat{u}(x)$, i.e. $\hat{u}(x_i) \neq u_i$. In this study, we use the Gaussian weight functions

$$\omega_{h_i}(x-x_i) = \begin{cases} \frac{\exp(-(\frac{d_i}{\gamma})^2)-\exp(-(\frac{h_i}{\gamma})^2)}{1-\exp(-(\frac{h_i}{\gamma})^2)}, & 0 \leq d_i \leq h_i, \\ 0, & d_i > h_i, \end{cases}$$

where $d_i = |x - x_i|$, γ is a constant controlling the shape of the weight function $\omega_{h_i}(x-x_i)$ and h_i is the size of the support domain. The matrices P and $\mathcal{W}$ are defined

$$P = \begin{bmatrix} \mathbf{p}^T(x_1) \\ \mathbf{p}^T(x_2) \\ \vdots \\ \mathbf{p}^T(x_n) \end{bmatrix}, \quad \mathcal{W} = diag(\omega_{h_1}(x-x_1), \omega_{h_2}(x-x_2), \ldots, \omega_{h_n}(x-x_n)).$$

From functional solution of $\mathcal{J}$ with regard to $\mathbf{a}(x)$, the linear relation between $\mathbf{a}(x)$ and $\mathbf{u}$ is achieved as follows

$$A(x)\mathbf{a}(x) = B(x)\mathbf{u}, \tag{7}$$

where matrices $A(x)$ and $B(x)$ are defined as follows

$$\begin{aligned} A(x) &= P^T\mathcal{W}P = B(x)P = \sum_{i=1}^{n} \omega_{h_i}(x-x_i)\mathbf{p}(x_i)\mathbf{p}^T(x_i), \\ B(x) &= P^T\mathcal{W} \\ &= [\omega_{h_1}(x-x_1)\mathbf{p}(x_1), \omega_{h_2}(x-x_2)\mathbf{p}(x_2), \ldots, \omega_{h_n}(x-x_n)\mathbf{p}(x_n)]. \end{aligned}$$

The matrix A is often called the moment matrix and its size equals $(m+1) \times (m+1)$. If we select the nodal points such that $A(x)$ is non-singular, then Eq. (7) has the unique solution

$$\mathbf{a}(x) = A^{-1}(x)B(x)\mathbf{u}, \tag{8}$$

by putting $\mathbf{a}(x)$ in Eq. (5) we get

$$\hat{u}(x) = \mathbf{p}^T(x)A^{-1}(x)B(x)\mathbf{u} = \sum_{i=1}^{n} \varphi_i(x)u_i, \quad x \in \overline{\Omega}, \tag{9}$$

where

$$\varphi_i(x) = \sum_{j=0}^{m} p_j(x)[A^{-1}(x)B(x)]_{ji},$$

are called the shape functions of the MLS approximation, corresponding to nodal point x_i. If $\omega_{h_i}(x - x_i) \in C^r(\Omega)$ and $p_j(x) \in C^s(\Omega)$, $\quad i = 1, 2, \ldots, n$, $j = 0, 1, \ldots, m$, then $\varphi_i(x) \in C^{min(r,s)}(\Omega)$.

4 The details of the suggested method

As already introduced in Section 1, and with regard to the information in Section 3, the Volterra integral form of the Eq. (1) for $k > 0,\ k \neq 0$ is written as follows

$$u(x) = \beta - \frac{\alpha}{k-1} \int_0^x t(1 - \frac{t^{k-1}}{x^{k-1}}) f(u(t))g(t)dt, \quad x \in [0, 1].$$

Now, by changing the variable $t \longrightarrow \mu x$ the above Volterra integral form is transformed to Fredholm integral form

$$u(x) = \beta - \frac{\alpha x^2}{k-1} \int_0^1 \mu(1 - \mu^{k-1}) f(u(\mu x))g(\mu x)d\mu, \quad x \in [0, 1]. \qquad (10)$$

To apply the MLS method, at first N evaluation points $\{x_i\}$ are selected on the interval $[0, 1]$ where $0 \leq x_1 < x_2 < \cdots < x_N \leq 1$. The distribution of nodes could be selected regularly or randomly. Then we can replace u by $\hat{u} = \sum_{i=1}^{N} \varphi_i(x)u_i$. So Eq. (10) becomes

$$\sum_{i=1}^{N} \varphi_i(x)u_i = \beta - \frac{\alpha x^2}{k-1} \int_0^1 \mu(1 - \mu^{k-1}) f(\sum_{i=1}^{N} \varphi_i(\mu x)u_i)g(\mu x)d\mu, \qquad (11)$$

for $x \in [0, 1]$. Since for $N - n$ nodes $\varphi_i(x) = 0$, n is replaced by N in Eq. (11). Now, by replacing x with the evaluation points of $x_1, x_2, \ldots, x_N$ in the Eq. (11), the following system of equations will be achieved

$$\sum_{i=1}^{n} \varphi_i(x_j)u_i + \frac{\alpha x_j^2}{k-1} \int_0^1 \mu(1 - \mu^{k-1}) f(\sum_{i=1}^{n} \varphi_i(\mu x_j)u_i)g(\mu x_j)d\mu = \beta, \quad (12)$$

using an n_1-point quadrature formula with the coefficients $\{\tau_\ell\}$ and weights $\{\omega_\ell\}$ in the interval $[0, 1]$ for numerically solving the integration in Eq. (12) yields

$$\sum_{i=1}^{n} \varphi_i(x_j)\hat{u}_i + \frac{\alpha x_j^2}{k-1} \sum_{\ell=1}^{n_1} \tau_\ell(1 - {\tau_\ell}^{k-1}) f(\sum_{i=1}^{n} \varphi_i(\tau_\ell x_j)\hat{u}_i)g(\tau_\ell x_j)\omega_\ell = \beta.$$

Here, $\hat{u}_i$s are an estimate for the u_is and by using the Levenberg-Marquardt algorithm, $\hat{u}$ is finally achieved. Then the values of $u(x)$ at any point of $x \in [0,1]$ can be approximated by Eq. (9) as

$$u(x) \approx u_N(x) = \sum_{i=1}^{N} \varphi_i(x)\hat{u}_i, \quad x \in [0,1]. \tag{13}$$

5 Convergence analysis

This section covers the convergence analysis of the proposed method. At first, the error estimate of MLS approximation is presented in terms of the parameter R which plays the role of the mesh-size. In [15] Levin analyzed the MLS method for a particular weight function obtaining error estimate in the uniform norm for the approximation of a regular function in N dimensions. In [3] Armentano and Duran proved error estimates in L^∞ for the function and its derivatives in the one-dimensional case. In [2] Armentano obtained the error estimates in L^∞ and L^2 norms for one and higher dimensions. In [41] Zuppa proved error estimates for approximation of the function and the first and second order derivatives in L^∞ norm. The error estimate of the method, proposed in this work is based on those obtained in [16, 41] for the one-dimensional cases.

Let Ω be an open bounded domain in $\mathbb{R}$ and $\mathcal{Q}_N$ denotes an arbitrarily chosen set of N points $x_i \in \overline{\Omega}$ referred to as nodes $\mathcal{Q}_N = \{x_1, x_2, \dots, x_N\}$, $x_i \in \overline{\Omega}$. Let $\mathcal{I}_N := \{\Upsilon_i\}_{i=1}^N$ denotes a finite open covering of $\overline{\Omega}$ consisting N clouds Υ_i such that $x_i \in \Upsilon_i$ and Υ_i is centered around x_i in some way, and $\overline{\Omega} \subset \bigcup_{i=1}^N \Upsilon_i$. Define the radius h_i of Υ_i as $\max_{x \in \partial\Upsilon_i}\{|x - x_i|\}$.

A function u is said of class $C^{q,1}$ in $\overline{\Omega}$ if and only if u is of class C^q in $\overline{\Omega}$ and the partial derivatives $D^s u$ of u of order $q(|s| = q)$ are Lipschitz continuous in $\overline{\Omega}$. The semi-norm $| \,.\, |_{q,1}$ is defined as [41]

$$|u|_{q,1} = \sup\left\{\frac{|D^s u(x) - D^s u(y)|}{|x - y|} : \, x, y \in \overline{\Omega}, \; x \neq y, |s| = q\right\}.$$

In order to have the MLS approximation well defined we need the minimization problem to have a unique solution at every point $x \in \overline{\Omega}$ and this is equivalent to the non-singularity of matrix $A(x)$. In [41] the error estimate was obtained with the following assumption about the system of nodes and weight functions $\{\mathcal{Q}_N, \mathcal{S}_N = \{\omega_{h_i}\}_{i=1}^N\}$.

Proposition 1. *([41]) For any $x \in \overline{\Omega}$, the matrix $A(x)$ defined in (7) is non-singular.*

Definition 1. Given $x \in \overline{\Omega}$, the set $\mathcal{ST}(x) = \{i : \omega_{h_i}(x - x_i) \neq 0\}$ will be called the star of x.

Theorem 1. *([41]) A necessary condition for the Property 1 to be satisfied is that for any $x \in \overline{\Omega}$,*

$$n = card(\mathcal{ST}(x)) \geq card(\mathcal{P}_q) = m + 1.$$

For a sample point $c \in \overline{\Omega}$, if $\mathcal{ST}(c) = \{i_1, i_2, \ldots, i_s\}$, the mesh-size of the star $\mathcal{ST}(c)$ is defined by the number $h(\mathcal{ST}(c)) := \max\{h_{i_1}, h_{i_2}, \ldots, h_{i_s}\}$.

Assumptions 1. Consider the following global assumptions about parameters. There exist

1. An upper bound of the overlap of clouds:

$$E = \sup_{c \in \overline{\Omega}} \Big\{ card\Big(\mathcal{ST}(c)\Big)\Big\}.$$

2. Upper bounds of the condition number:

$$CB_q = \sup_{c \in \overline{\Omega}} \Big\{ CN_q\Big(\mathcal{ST}(c)\Big)\Big\}, \quad q = 1, 2,$$

where the numbers $CN_q(\mathcal{ST}(c))$ are computable measures of the quality of the star $\mathcal{ST}(c)$ which is defined in theorem 7 of [41].

3. An upper bound of the mesh-size of stars:

$$R = \sup_{c \in \overline{\Omega}} \Big\{ h\Big(\mathcal{ST}(c)\Big)\Big\}.$$

4. A uniform bound of the derivatives of $\{\omega_{h_i}\}$, that is the constant $G_q > 0,\ q = 1, 2$, such that

$$\|D^s \omega_{h_i}\|_{L^\infty} \leq \frac{G_q}{R^{|s|}}, \quad 1 \leq |s| \leq q.$$

5. There exists the number $\gamma \geq 0$ such that any two points $x, y \in \overline{\Omega}$ can be jointed by a rectifiable curve Γ in $\overline{\Omega}$ with length $|\Gamma| \leq \gamma |x - y|$.

Assuming all these conditions, Zuppa [41] proved:

Theorem 2. *There exist constants $C_q,\ q = 1$ or 2,*

$$C_1 = C_1(\gamma, d, E, G_1, CB_1), \qquad C_2 = C_2(\gamma, d, E, G_2, CB_1, CB_2),$$

such that for each $u \in C^{q,1}(\overline{\Omega})$,

$$\|D^s u - D^s \hat{u}\|_{L^\infty(\Omega)} \leq C_q R^{q+1-|s|} |u|_{q,1}, \quad 0 \leq |s| \leq q. \tag{14}$$

As highlighted in [41], the number $CN_2(\mathcal{ST}(c))$ can be very high near the boundary points. This drawback can degrade appreciatively the global error estimate when $q = 2$.

Now we write Eq. (10) in an abstract form as

$$(\lambda - \mathcal{F})u = g, \tag{15}$$

where

$$\mathcal{F}u = \int_\Omega k(x, \mu) f(u(\mu x)) d\mu.$$

Similarly, Eq. (11) can be written as

$$(\lambda - \mathcal{F})\hat{u} = g, \tag{16}$$

Assume that $\mathcal{F}$ is a compact operator (for more details about the compact integral operators see chapter 1 of [4]).

Lemma 1. *If Eq. (15) is uniquely solvable and $\|u - \hat{u}\| \to 0$ then Eq. (16) is uniquely solvable.*

Proof. See [16, lemma 4.1]. □

Suppose that

$$\mathcal{F}_M u = \sum_{i=1}^{M} k(x, \tau_i) u(x\tau_i) \omega_i,$$

then

$$\|\mathcal{F}_M\| = \max_{x \in \Omega} \sum_{i=1}^{M} |k(x, \tau_i) \omega_i|.$$

At the first view, the error analysis of method depends on showing $\|\mathcal{F} - \mathcal{F}_M\| \to 0$ as M increases. This can not be done here; and in fact [4] $\|\mathcal{F} - \mathcal{F}_M\| \geq \|\mathcal{F}\|$.We begin by looking at quantities which do converge to zero as $M \to \infty$.

Lemma 2. *Let Ω be a closed, bounded set in $\mathbb{R}$, and let $k(x, \mu)$ be continuous for $x, \mu \in \Omega$. Let the quadrature scheme*

$$\int_{\Omega} f(y)dy = \sum_{i=1}^{M} f(y_i)\omega_i,$$

be convergent for all continuous functions on Ω. Define

$$e_M(x, \mu) = \int_{\Omega} k(x, \nu)k(\nu, \mu)d\nu - \sum_{i=1}^{M} k(x, \nu_i)k(\nu_i, \mu)\omega_i, \quad x, \mu \in \Omega,\ M \geq 1,$$

as the numerical integration error for the integrand $k(x, .)k(., \mu)$. Then for $u \in C(\Omega)$,

$$(\mathcal{F} - \mathcal{F}_M)\mathcal{F}u(x) = \int_{\Omega} e_M(x, \mu)u(\mu x)d\mu,$$
$$(\mathcal{F} - \mathcal{F}_M)\mathcal{F}_M u(x) = \sum_{i=1}^{M} e_M(x, \tau_i)u(x\tau_i)\omega_i.$$

In addition,

$$\|(\mathcal{F} - \mathcal{F}_M)\mathcal{F}\| = \max_{x\in\Omega} \int_{\Omega} |e_M(x, \mu)|d\mu,$$
$$\|(\mathcal{F} - \mathcal{F}_M)\mathcal{F}_M\| = \max_{x\in\Omega} \sum_{i=1}^{M} |e_M(x, \tau_i)\omega_i|.$$

Finally, the numerical integration error converges to zero uniformly on Ω,

$$\lim_{M\to\infty} \max_{x,\mu\in\Omega} |e_M(x, \mu)| = 0,$$

and thus

$$\|(\mathcal{F} - \mathcal{F}_M)\mathcal{F}\|,\ \|(\mathcal{F} - \mathcal{F}_M)\mathcal{F}_M\| \to 0 \text{ as } M \to \infty. \tag{17}$$

Proof. See [4, chapter 4, lemma 4.1.1]. □

To carry out an error analysis, we need the following perturbation theorem.

Theorem 3. *Let $\mathcal{X}$ be a Banach space, let $\mathcal{S}$, $\mathcal{T}$ be bounded operators on $\mathcal{X}$ to $\mathcal{X}$, and let $\mathcal{S}$ be compact. For given $\lambda \neq 0$, assume $\lambda - \mathcal{T} : \mathcal{X} \to \mathcal{X}$ is* *one to one and onto, which implies* $(\lambda - \mathcal{T})^{-1}$ *exists as a bounded operator on* $\mathcal{X}$ *to* $\mathcal{X}$*. Finally, assume that*

$$\|(\mathcal{T} - \mathcal{S})\mathcal{S}\| < \frac{|\lambda|}{\|(\lambda - \mathcal{T})^{-1}\|}.$$

Then $(\lambda - \mathcal{S})^{-1}$ *exists and is bounded on* $\mathcal{X}$ *to* $\mathcal{X}$*, with*

$$\|(\lambda - \mathcal{S})^{-1}\| \leq \frac{1 + \|(\lambda - \mathcal{T})^{-1}\|\|\mathcal{S}\|}{|\lambda| - \|(\lambda - \mathcal{T})^{-1}\|\|(\mathcal{T} - \mathcal{S})\mathcal{S}\|}.$$

If $(\lambda - \mathcal{T})\omega = g$ *and* $(\lambda - \mathcal{S})z = g$*, then*

$$\|\omega - z\| \leq \|(\lambda - \mathcal{S})^{-1}\|\|\mathcal{T}\omega - \mathcal{S}\omega\|.$$

Proof. See [4, chapter 4, theorem 4.1.1]. □

Finally the following theorem completes the convergence analysis of the method in L^∞ norm. Before that, we note that the approximation scheme in one dimension could be written in compact form as

$$(\lambda - \mathcal{F}_M)u_N = g.$$

Theorem 4. *Let* $u \in C^{q,1}(\overline{\Omega})$ $(q = 1, 2)$ *where* Ω *be a closed, bounded set in* $\mathbb{R}$*; and let* $k(x, \mu)$ *be continuous for* $x, \mu \in \Omega$*. Assume the quadrature scheme is convergent for all continuous functions on* Ω*. Further, assume that the integral equation (15) is uniquely solvable for given* $g \in C(\Omega)$ *with* $\lambda \neq 0$*. Moreover take a suitable approximation* $\hat{u}$ *of* u*. Then for all sufficiently large* M*, the approximate inverses* $(\lambda - \mathcal{F}_M)^{-1}$ *exist and are uniformly bounded,*

$$\|(\lambda - \mathcal{F}_M)^{-1}\| \leq \frac{1 + \|(\lambda - \mathcal{F})^{-1}\|\|\mathcal{F}_M\|}{|\lambda| - \|(\lambda - \mathcal{F})^{-1}\|\|(\mathcal{F} - \mathcal{F}_M)\mathcal{F}_M\|} \leq \mathcal{Q},$$

with a suitable constant $\mathcal{Q} < \infty$*. For the equations* $(\lambda - \mathcal{F})u = g$ *and* $(\lambda - \mathcal{F}_M)u_N = g$*, we have*

$$\begin{aligned}\|u - u_N\|_{L^\infty(\Omega)} \leq\ & C_q R^{q+1}|u|_{q,1}(1 + \mathcal{Q}\|(\mathcal{F} - \mathcal{F}_M)\|_{L^\infty(\Omega)}) \\ & + \|(\mathcal{F} - \mathcal{F}_M)\|_{L^\infty(\Omega)}\|u\|_{L^\infty(\Omega)}.\end{aligned}$$

where $\hat{u}$*,* R *and* C_q *are introduced in Theorem 2.*

Proof. See [16, theorem 4.4]. □

Remark. Note that in this paper the mesh-sizes h_i are taken constant for all nodes ($\equiv R$). The size of support domain in the MLS approximation should be large enough to accommodate a sufficiently large number of nodes covered in the domain of definition of point x to ensure the regularity of the matrix $A(x)$. A necessary condition is presented in Theorem (1) as $n = card(\mathcal{ST}(x)) \geq card(\mathcal{P}_q) = m + 1$. The role of this size is played by R in Theorem (1) Otherwise for convergence of $\hat{u}$ to u from (14), R must be a small real number. To overcome these two opposite phenomena, we must take R as a small real number and increase the density of nodes in the support domain.

6 Numerical experiments

In this section, some examples are provided to show the strength of the proposed method in approximating the solution of Emden-Fowler equations. For computational details and the numerical implementation of the method we take $h_i = 2/(N-1)$ and $\gamma = 0.6/(N-1)$ for all of them to ensure the invertibility of the matrix A in MLS method. Also in our computations we use the 7-point Gauss-Legendre quadrature rule for numerical integration.

First, a special form of the Emden-Fowler equation which has been well studied before, will be considered

$$u''(x) + \frac{2}{x}u'(x) + \alpha u^{\nu} x^{\xi} = 0, \;\; u(0) = 1, \;\; u'(0) = 0. \tag{18}$$

As was mentioned before, for $\xi = 0$ the above equation will transform to Lane-Emden with the index of ν. It was phisically shown that interesting values of ν lie in the interval $[0, 5]$, moreover, the exact answer has been available only for $\nu = 0, 1, 5$ and for the other values of ν series solutions are available [9, 31].

It is important to note that Eq. (18) for $\nu = 0, 1$ is linear and for the other values of ν it is nonlinear. According to Section 2 since $k = 2$ the integral form of Eq. (18) will be as follows

$$u(x) = 1 - \alpha \int_0^x t(1 - \frac{t}{x}) u^{\nu} t^{\xi} dt,$$

after replacing $\nu = \xi = 0$ the above equation will be transformed to the following

$$u(x) = 1 - \alpha \int_0^x t(1 - \frac{t}{x}) dt,$$

Table 1: Maximum error for different m (*number of basis)*, N (*points number)*, $\alpha = 1$.

N	$m = 1$	$m = 2$	$m = 3$
11	7.4×10^{-3}	3.08×10^{-4}	9.91×10^{-5}
21	2.3×10^{-3}	4.01×10^{-5}	7.50×10^{-6}
41	2.6×10^{-4}	9.47×10^{-7}	4.05×10^{-7}
201	8.0×10^{-6}	1.7×10^{-8}	1.27×10^{-10}

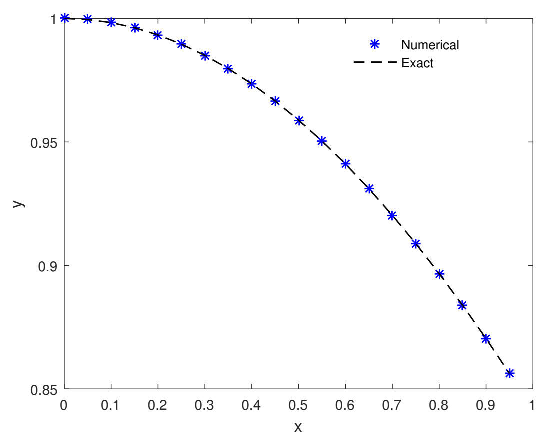

Figure 1: Numerical approximation with 201 points.

after solving that equation, we will come up with the exact answer of the $u(x) = 1 - \frac{\alpha x^2}{3!}$.

For $\nu = 1$ and $\xi = 0$ we have

$$u(x) = 1 - \alpha \int_0^x t(1 - \frac{t}{x})u(t)dt,$$

which has been solved with the MLS method. The results of the process and the comparison of its answer with the exact answer of $u(x) = sin(\sqrt{\alpha}x)/\sqrt{\alpha}x$ can be found in Table 1 and Figures 1 and 2. Also for $\nu = 5$ and $\xi = 0$ we have:

$$u(x) = 1 - \alpha \int_0^x \mu(1 - \frac{\mu}{x})u^5(\mu)d\mu,$$

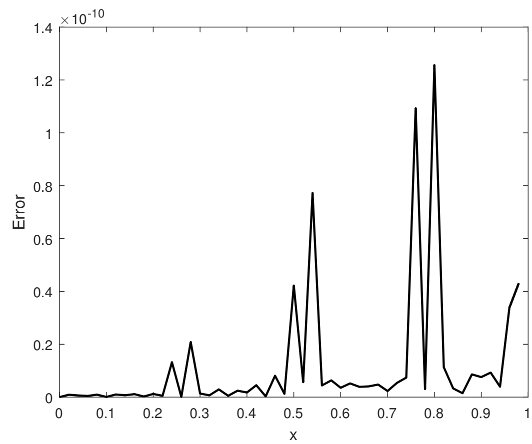

Figure 2: The MLS approximation error of degree 3 with 201 points.

Table 2: Error calculation with Euclidean norm for different m and N.

N	$m = 1$	$m = 2$	$m = 3$
11	2.08×10^{-2}	8.94×10^{-3}	3.52×10^{-5}
21	1.03×10^{-2}	4.56×10^{-3}	2.32×10^{-6}
51	4.19×10^{-3}	1.43×10^{-3}	7.54×10^{-7}

Its exact answer is $u(x) = (1 + \frac{\alpha x^2}{3})^{\frac{-1}{2}}$. We have achieved the following answers using the MLS method through comparisons with the exact answer. Refer to Table 2 and Fiqure 3 for details. In general, for $\nu = 0$, we can obtain the exact solution of the problem for every real ξ and $\xi \neq -2, -3$ by solving the following integral equation [33]

$$y(x) = 1 - \alpha \int_0^x t(1 - \frac{t}{x}) t^{\xi} dt.$$

Now, consider the following Emden-Fowler equation:

$$u''(x) + \frac{2}{x} u'(x) + e^{u(x)} = 0, \;\; u(0) = 0, \;\; u'(0) = 0. \tag{19}$$

This equation models the distribution of mass in an isothermal sphere [17]. In order to solve Eq. (19), first its integral form is considered,

$$u(x) = -\int_0^x t(1 - \frac{t}{x}) e^{u(t)} dt.$$

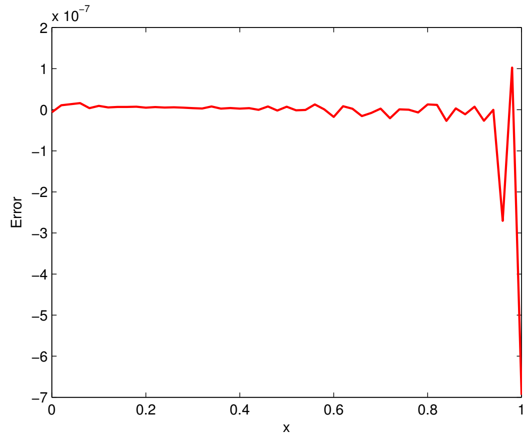

Figure 3: The MLS approximation error of degree 3 with 51 points.

Table 3: Error calculation with Euclidean norm for different m and N.

N	$m=1$	$m=2$	$m=3$
11	2.51×10^{-2}	1.21×10^{-2}	1.01×10^{-5}
21	1.32×10^{-2}	6.01×10^{-3}	1.32×10^{-6}
51	5.55×10^{-3}	1.88×10^{-3}	6.71×10^{-7}

Now, by applying the MLS method, we will solve it and compare the results with the following series answer achieved through the Adomian Decomposition Method (ADM) [31]. See Table 3 and Figure 4.

$$u(x) = -\frac{x^2}{3!} + \frac{x^4}{5!} - \frac{8x^6}{3.7!} + \frac{122x^8}{9.9!} - \frac{4087x^{10}}{45.11!} + e(O^{12}).$$

Similarly, acceptable results can be achieved for the following equation:

$$u''(x) + \frac{2}{x}u'(x) + e^{-u(x)} = 0, \;\; u(0) = 0, \;\; u'(0) = 0,$$

that is used in Richardson's theory [25] of thermionic currents which is related to the emission of electricity from hot bodies.

Now, consider the following equation as another example:

$$u''(x) + \frac{k}{x}u'(x) - \alpha(6+4x^2)u(x) = 0, \;\; u(0) = \beta, \;\; u'(0) = 0,$$

here we will solve a special form of the equation as referred to in [26]

$$u''(x) + \frac{2}{x}u'(x) + (6+4x^2)u(x) = 0, \;\; u(0) = 1, \;\; u'(0) = 0,$$

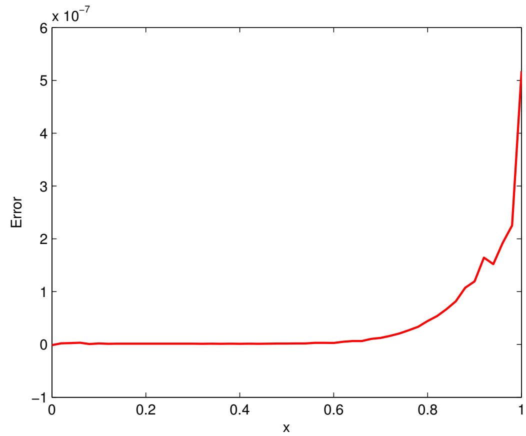

Figure 4: The MLS approximation error of degree 3 with 51 points compared to solution of ADM.

Table 4: Error calculation with Euclidean norm for different m and N.

N	$m=1$	$m=2$	$m=3$
21	1.81×10^{-1}	9.95×10^{-2}	3.81×10^{-4}
51	8.61×10^{-2}	3.64×10^{-2}	2.54×10^{-5}
101	3.81×10^{-2}	8.61×10^{-3}	7.54×10^{-6}

its integral form equals

$$u(x) = 1 + x^2 \int_0^1 \mu(1-\mu)(6+4(\mu x)^2)u(\mu x)d\mu.$$

After comparing the answer of the above equation with MLS, the result will be compared with the exact solution $u(x) = e^{x^2}$. The results can be seen in Table 4 and Figure 5.

And the last example is expressed in [26]

$$u''(x) + \frac{2}{x}u'(x) - 6u(x) - 4u(x)\ln(u(x)) = 0, \;\; u(0) = 1, \;\; u'(0) = 0.$$

the integral form of which is as follows

$$u(x) = 1 + x^2 \int_0^1 \mu(1-\mu)(6u(\mu x) + 4u(\mu x)\ln(u(\mu x)))d\mu.$$

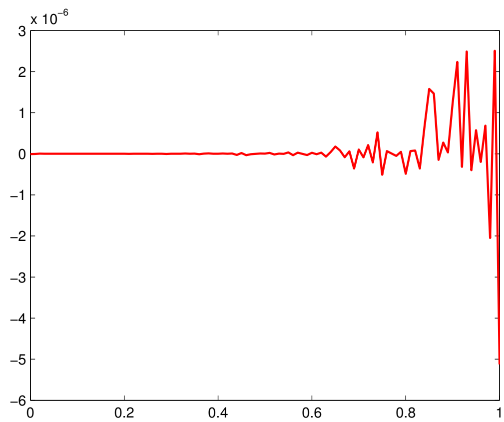

Figure 5: The MLS approximation error of degree 3 with 101 points compared to the exact solution.

Table 5: Error calculation with Euclidean norm for different m and N.

N	$m=1$	$m=2$	$m=3$
21	1.83×10^{-1}	9.92×10^{-2}	3.81×10^{-4}
51	8.54×10^{-2}	3.69×10^{-2}	2.49×10^{-5}
101	3.89×10^{-2}	7.75×10^{-3}	7.83×10^{-6}

In the following, the result achieved through MLS for this example will be compared with the exact solution $u(x) = e^{x^2}$. The results are shown in Table 5 and Figure 6.

7 Conclusion

In this paper, the MLS method was used to solve the integral form of the Emden-Fowler equations in various forms and at the end an acceptable level of accuracy in answering the equation was achieved. First, the Emden-Fowler equation has been investigated in multiple occasions, then different approaches have been offered to solve it. In this article, we came to a new approach to solve it more efficiently.

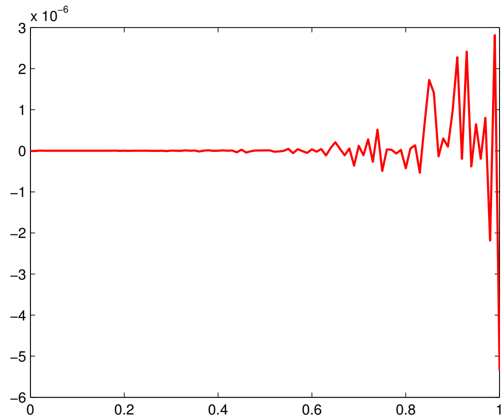

Figure 6: The MLS approximation error of degree 3 with 101 points compared to Exact solution.

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