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Inverse eigenvalue problem of interval nonnegative matrices of order ≤ 3

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Abstract. In this paper for a given set of real or complex interval numbers σ satisfying special conditions, we find an interval nonnegative matrix C such that for each point set δ of given interval spectrum σ, there exists a point matrix A of C such that δ is its spectrum. We also study some conditions for the solution existence of the problem.

Keywords: Interval matrix, inverse eigenvalue problem, nonnegative matrices.
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1 Introduction

An interval matrix is a matrix whose elements are interval numbers. Interval matrices were used by mathematicians in the late twentieth century and in the present century.

In 1965, Zadeh presented fuzzy logic for the first time and introduced interval logic [11]. In 1993 Rohn found the inverse of interval matrices [9]. The eigenvalue problem of real and symmetric interval matrices studied by Hladik and Daney and they found some bounds for interval eigenvalues of these interval matrices [4]. In 2006, again Rhon obtained some results for the spectral radius of irreducible nonnegative interval matrix and found the Perron eigenvector of them [10]. In 2011, Hladik et al. by iterative

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filtering method found the approximate eigenvalues of interval matrices [2]. In the recent paper [3], a cheap and tight formula for bounding real and imaginary parts of eigenvalues of real or complex interval matrices has been presented by Hladik. His method generalized and improved the results presented by Rohn [8] and Hertz [1]. Although finding the prescribed interval eigenvalues of interval matrices has some problems, we try to study the inverse eigenvalue problem.

The summation, subtraction, multiplication and division of two interval numbers $\mathbf{b} = [\underline{b}, \overline{b}]$ and $\mathbf{a} = [\underline{a}, \overline{a}]$ are respectively defined as:

- $\mathbf{a} + \mathbf{b} = [\underline{a} + \underline{b}, \overline{a} + \overline{b}]$;
- $\mathbf{a} - \mathbf{b} = [\underline{a} - \overline{b}, \overline{a} - \underline{b}]$;
- $\mathbf{a} \cdot \mathbf{b} = [\min\{\underline{a} \cdot \underline{b}, \underline{a} \cdot \overline{b}, \overline{a} \cdot \underline{b}, \overline{a} \cdot \overline{b}\}, \max\{\underline{a} \cdot \underline{b}, \underline{a} \cdot \overline{b}, \overline{a} \cdot \underline{b}, \overline{a} \cdot \overline{b}\}]$;
- $\frac{\mathbf{a}}{\mathbf{b}} = \mathbf{a} \cdot \mathbf{b}'$, $\mathbf{b}' = [\frac{1}{\overline{b}}, \frac{1}{\underline{b}}]$ and $0 \notin \mathbf{b}$;
- $\mathbf{a}^2 = \begin{cases} [\underline{a}^2, \overline{a}^2], & \text{if} \quad 0 \leq \underline{a} \leq \overline{a}, \\ \left[\overline{a}^2, \underline{a}^2\right], & \text{if} \quad \underline{a} \leq \overline{a} \leq 0, \\ \left[0, \max\{[\underline{a}^2, \overline{a}^2\}\right], & \text{if} \quad \underline{a} \leq 0 \leq \overline{a}. \end{cases}$

Definition 1. Let $\underline{A}$ and $\overline{A}$ be $n \times n$ real matrices. The following set

$$A^I = [\underline{A}, \overline{A}] = \{A : \underline{A} \leq A \leq \overline{A}\},$$

is called an $n \times n$ real interval matrix. The midpoint and the radius of A^I are denoted respectively by

$$A_c = \frac{\underline{A} + \overline{A}}{2}, \qquad A_\Delta = \frac{\underline{A} - \overline{A}}{2}.$$

If all interval entries of a real interval matrix are nonnegative, then A^I is called nonnegative interval matrix. The set of all real interval matrices and the set of all nonnegative interval matrices are denoted by $\mathbb{IR}^{n\times n}$ and $\mathbb{NIR}^{n\times n}$, respectively.

Definition 2. Let A^I be an interval square matrix. Then the set of eigenvalues of A^I is defined as follows

$$\Lambda(A^I) = \{\lambda \in \mathbb{R}; Ax = \lambda x, x \neq 0, A \in A^I\}.$$

The eigenvalue of nonnegative interval matrix A^I is called Perron interval eigenvalue of A^I if it is nonnegative and greater than or equal to all absolute value of eigenvalues of A^I and denoted by $\lambda_1 = [\underline{\lambda_1}, \overline{\lambda_1}]$, i.e.,

$$[\underline{\lambda_1}, \overline{\lambda_1}] \geq |[\underline{\lambda_i}, \overline{\lambda_i}]|, \quad i = 2, 3, \ldots, n,$$

where $[\underline{\lambda_i}, \overline{\lambda_i}] = [\min\{|\underline{\lambda_i}|, |\overline{\lambda_i}|\}, \max\{|\underline{\lambda_i}|, |\overline{\lambda_i}|\}]$.

The nonnegative inverse eigenvalue problem (NIEP) asks for necessary and sufficient conditions on a list $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of real or complex numbers in order that it be the spectrum of a nonnegative matrix A, and in this case we will say that σ is realizable and that it is realization of σ.

Some necessary conditions on the list of real number $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ to be the spectrum of a nonnegative matrix are listed below.
1. The Perron eigenvalue $\max\{|\lambda_i|; \lambda_i \in \sigma\}$ belongs to σ (Perron-Frobenius theorem);
2. $s_k = \sum_{i=1}^{n} \lambda_i^k \geq 0$;
3. $s_k^m \leq n^{m-1} s_{km}$ for $k, m = 1, 2, \ldots$ (JLL (for Johnson, Lowey and London) inequality) [5,6].

In this paper by using some theorems of [7], for a given set of interval numbers σ with special conditions, we find interval nonnegative matrices C^I such that for every point set σ_1 of interval set of eigenvalues (for each interval one point), there exists a point nonnegative matrix such that this set is its spectrum. The necessary and sufficient conditions for the solutions existence of the problem will be studied.

Theorem 1. *[7] Let B be an $m \times m$ nonnegative matrix, $M_1 = \{\mu_1, \mu_2, \ldots, \mu_m\}$, be its eigenvalues and μ_1 be the Perron eigenvalue of B. Also assume that A is an $n \times n$ nonnegative matrix in following form*

$$A = \begin{pmatrix} A_1 & a \\ b^T & \mu_1 \end{pmatrix},$$

where A_1 is an $(n-1) \times (n-1)$ matrix, a and b are arbitrary vectors in $\mathbb{R}^{n-1}$ and $M_2 = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ is the set of eigenvalues of A. Then there exists an $(m+n-1) \times (m+n-1)$ nonnegative matrix such that $M = \{\mu_2, \ldots, \mu_m, \lambda_1, \lambda_2, \ldots, \lambda_m\}$ is its eigenvalues.

The above theorem presents a recursive method for solving NIEP. In this paper we use this theorem and some of the results presented in the paper [7] in the interval form and try to construct a similar method for solving the nonnegative interval inverse eigenvalue problem matrices which is briefly denoted by NIIEP. Also we study some conditions for realization of the problem.

Some necessary conditions for NIIEP on the list of complex interval number $\sigma = \left([\underline{\lambda_1}, \overline{\lambda_1}], [\underline{\lambda_2}, \overline{\lambda_2}], \ldots, [\underline{\lambda_n}, \overline{\lambda_n}]\right)$ to be the spectrum of a nonnegative interval matrix are listed below.

1. The Perron eigenvalue $\max\{|[\underline{\lambda_i}, \overline{\lambda_i}]|; [\underline{\lambda_i}, \overline{\lambda_i}] \in \sigma\}$ belongs to σ (Perron-Frobenius theorem in interval case);
2. The list σ is closed under complex conjugation;
3. $s_k = \sum_{i=1}^{n} \left|[\underline{\lambda_i}, \overline{\lambda_i}]\right|^k \geq 0$.

2 The case $n = 2$

In this section for a given interval set of eigenvalues with two real elements (both elements must necessarily be real), we find a nonnegative interval matrix, such that the given set is its spectrum.

Theorem 2. *Assume that $[\underline{\lambda_1}, \overline{\lambda_1}]$ and $[\underline{\lambda_2}, \overline{\lambda_2}]$ are two interval real numbers and $[\underline{\lambda_1}, \overline{\lambda_1}] \geq 0$ (this means both $\underline{\lambda_1}$ and $\overline{\lambda_1}$ are nonnegative). Also assume that $\sigma = \{[\underline{\lambda_1}, \overline{\lambda_1}], |[\underline{\lambda_2}, \overline{\lambda_2}]\}$ and $[\underline{\lambda_1}, \overline{\lambda_1}] \geq |[\underline{\lambda_2}, \overline{\lambda_2}]|$. Then there exists the nonnegative interval matrix C^I, such that for every piont eigenvalue σ_1 of σ there exists a nonnegative point matrix of C^I such that σ_1 is its spectrum.*

Proof. If $[\underline{\lambda_2}, \overline{\lambda_2}] \geq 0$, then the nonnegative interval matrix

$$C^I = \mathrm{diag}\left([\underline{\lambda_1}, \overline{\lambda_1}], [\underline{\lambda_2}, \overline{\lambda_2}]\right),$$

is the solution of problem. Otherwise, the following nonngative real interval matrix

$$C^I = \begin{pmatrix} 0 & -[\underline{\lambda_1}, \overline{\lambda_1}][\underline{\lambda_2}, \overline{\lambda_2}] \\ 1 & [\underline{\lambda_1}, \overline{\lambda_1}] + [\underline{\lambda_2}, \overline{\lambda_2}] \end{pmatrix}, \quad (1)$$

has interval eigenvalues $[\underline{\lambda_1}, \overline{\lambda_1}]$ and $[\underline{\lambda_2}, \overline{\lambda_2}]$, where the number 0 or 1 are point interval numbers as $0 = [0, 0]$ and $1 = [1, 1]$. □

3 The case $n = 3$

In this section for a given interval set of eigenvalues with three real or complex elements that satisfies necessary conditions of NIIEP, we find a 3×3 nonnegative interval matrix, such that for every given set of eigenvalues of this interval set we can find a nonnegative matrix from nonnegative interval matrix in which this set is its spectrum.

Theorem 3. *Let* $\sigma = \big([\underline{\lambda_1}, \overline{\lambda_1}], [\underline{\lambda_2}, \overline{\lambda_2}], [\underline{\lambda_3}, \overline{\lambda_3}]\big)$ *be a set of real and complex interval numbers and*

$$\begin{aligned} p &= [\underline{\lambda_1}, \overline{\lambda_1}][\underline{\lambda_2}, \overline{\lambda_2}][\underline{\lambda_3}, \overline{\lambda_3}], \\ \alpha_1 &= [\underline{\lambda_1}, \overline{\lambda_1}][\underline{\lambda_2}, \overline{\lambda_2}] + [\underline{\lambda_1}, \overline{\lambda_1}][\underline{\lambda_3}, \overline{\lambda_3}] + [\underline{\lambda_2}, \overline{\lambda_2}][\underline{\lambda_3}, \overline{\lambda_3}], \\ \alpha_2 &= [\underline{\lambda_1}, \overline{\lambda_1}] - [\underline{\lambda_2}, \overline{\lambda_2}] - [\underline{\lambda_3}, \overline{\lambda_3}] - \left|[\underline{\lambda_2}, \overline{\lambda_2}]\right|^2 - 1, \\ \alpha_3 &= [\underline{\lambda_1}, \overline{\lambda_1}] - \left|[\underline{\lambda_2}, \overline{\lambda_2}]\right|^2, \\ \alpha_4 &= [\underline{\lambda_2}, \overline{\lambda_2}] + [\underline{\lambda_3}, \overline{\lambda_3}] + \left|[\underline{\lambda_2}, \overline{\lambda_2}]\right|^2. \end{aligned}$$

Furthermore if σ *satisfies the following conditions:*
(1) $[\underline{\lambda_1}, \overline{\lambda_1}] + [\underline{\lambda_2}, \overline{\lambda_2}] + [\underline{\lambda_3}, \overline{\lambda_3}] \geq 0$,
(2) $\sigma = \overline{\sigma}$,
(3) $\underline{\lambda_1}, \overline{\lambda_1} \in \mathbb{R}, [\underline{\lambda_1}, \overline{\lambda_1}] \geq \left|[\underline{\lambda_i}, \overline{\lambda_i}]\right|, \quad i = 2, 3$,
(4) *if* $([\underline{\lambda_2}, \overline{\lambda_2}], [\underline{\lambda_3}, \overline{\lambda_3}] \notin \mathbb{IR}, \alpha_1 > 0) \Rightarrow \alpha_2, \alpha_3, \alpha_4 \geq 0$,
then there exists a nonnegative matrix C^I*, such that for every* σ_1 *of* σ *there exists a nonnegative matrix of* C^I *such that* σ_1 *is its spectrum.*

Proof. At first, we assume that all elements of σ are real interval numbers. In accordance with the above conditions we consider the following cases:

(a) If $[\underline{\lambda_2}, \overline{\lambda_2}], [\underline{\lambda_3}, \overline{\lambda_3}] \geq 0$, then the nonnegative interval matrix $C^I = \operatorname{diag}\big([\underline{\lambda_1}, \overline{\lambda_1}], [\underline{\lambda_2}, \overline{\lambda_2}], [\underline{\lambda_3}, \overline{\lambda_3}]\big)$ is a desired matrix.

(b) If $[\underline{\lambda_2}, \overline{\lambda_2}], [\underline{\lambda_3}, \overline{\lambda_3}] < 0$, since two nonnegative interval matrices A^I and B^I with spectrum $\sigma_1 = \{[\underline{\lambda_1}, \overline{\lambda_1}], [\underline{\lambda_2}, \overline{\lambda_2}]\}$, $\sigma_2 = \{[\underline{\lambda_1}, \overline{\lambda_1}] + [\underline{\lambda_2}, \overline{\lambda_2}], [\underline{\lambda_3}, \overline{\lambda_3}]\}$ in the following form:

$$A^I = \begin{pmatrix} 0 & -[\underline{\lambda_1}, \overline{\lambda_1}][\underline{\lambda_2}, \overline{\lambda_2}] \\ 1 & [\underline{\lambda_1}, \overline{\lambda_1}] + [\underline{\lambda_2}, \overline{\lambda_2}] \end{pmatrix},$$

$$B^I = \begin{pmatrix} 0 & -\big([\underline{\lambda_1}, \overline{\lambda_1}] + [\underline{\lambda_2}, \overline{\lambda_2}]\big)[\underline{\lambda_3}, \overline{\lambda_3}] \\ 1 & [\underline{\lambda_1}, \overline{\lambda_1}] + [\underline{\lambda_2}, \overline{\lambda_2}] + [\underline{\lambda_3}, \overline{\lambda_3}] \end{pmatrix},$$

respectively, satisfy the conditions of Theorem 1, in its interval form, then it is easy to verify that the vector

$$s = \left(-\frac{[\underline{\lambda_3}, \overline{\lambda_3}]}{\sqrt{1+\left|[\underline{\lambda_3}, \overline{\lambda_3}]\right|^2}}, \quad \frac{1}{\sqrt{1+\left|[\underline{\lambda_3}, \overline{\lambda_3}]\right|^2}} \right)^T,$$

is the normalized eigenvector associated to Perron interval eigenvalue $[\underline{\lambda_1}, \overline{\lambda_1}] + [\underline{\lambda_2}, \overline{\lambda_2}]$, of nonnegative interval matrix B^I, where

$\sqrt{[\underline{\lambda_1}, \overline{\lambda_1}]} = [\sqrt{\underline{\lambda_1}}, \sqrt{\overline{\lambda_1}}]$. Then by Theorem 1 the nonnegative interval matrix

$$C^I = \begin{pmatrix} 0 & \frac{[\underline{\lambda_1},\overline{\lambda_1}][\underline{\lambda_2},\overline{\lambda_2}][\underline{\lambda_3},\overline{\lambda_3}]}{\sqrt{1+|[\underline{\lambda_3},\overline{\lambda_3}]|^2}} & -\frac{[\underline{\lambda_1},\overline{\lambda_1}][\underline{\lambda_2},\overline{\lambda_2}]}{\sqrt{1+|[\underline{\lambda_3},\overline{\lambda_3}]|^2}} \\ -\frac{[\underline{\lambda_3},\overline{\lambda_3}]}{\sqrt{1+|[\underline{\lambda_3},\overline{\lambda_3}]|^2}} & 0 & -\left([\underline{\lambda_1},\overline{\lambda_1}]+[\underline{\lambda_2},\overline{\lambda_2}]\right)[\underline{\lambda_3},\overline{\lambda_3}] \\ \frac{1}{\sqrt{1+|[\underline{\lambda_3},\overline{\lambda_3}]|^2}} & 1 & [\underline{\lambda_1},\overline{\lambda_1}]+[\underline{\lambda_2},\overline{\lambda_2}]+[\underline{\lambda_3},\overline{\lambda_3}] \end{pmatrix},$$

is a solution of the problem.

(c) If $[\underline{\lambda_2}, \overline{\lambda_2}] < 0$ and $[\underline{\lambda_3}, \overline{\lambda_3}] \geq 0$, then the following nonnegative interval matrix

$$C^I = \begin{pmatrix} A^I & a \\ b^T & [\underline{\lambda_3}, \overline{\lambda_3}] \end{pmatrix}$$

are the the solution for the problem, where A^I is the matrix (1) and a and b^T is interval zero vector of dimension 2×1.

Now let $[\underline{\lambda_2}, \overline{\lambda_2}]$ and $[\underline{\lambda_3}, \overline{\lambda_3}]$ be a conjugate complex pair. One of the following cases will be happened.

(d) If $\alpha_1 \leq 0$, then the nonnegative interval matrix

$$C^I = \begin{pmatrix} 0 & p & 0 \\ 0 & 0 & 1 \\ 1 & -\alpha_1 & [\underline{\lambda_1},\overline{\lambda_1}]+[\underline{\lambda_2},\overline{\lambda_2}]+[\underline{\lambda_3},\overline{\lambda_3}] \end{pmatrix}.$$

is a solution of our problem.

(e) If $\alpha_1 > 0$, then by condition (4), we must have $\alpha_2, \alpha_3, \alpha_4 \geq 0$. So the nonnegative interval matrix

$$C^I = \begin{pmatrix} \alpha_3 & \alpha_2|[\underline{\lambda_2},\overline{\lambda_2}]|^2 & 1 \\ 1 & \alpha_4 & 0 \\ 0 & p & 0 \end{pmatrix},$$

is a solution of our problem.

□

Remark 1. *We can continue this method for $n = 4, 5$ and state some theorems similar to the theorem of [7].*

Example 1. Let $\sigma = \{[4, 6], [-4, -4]\}$. Then the following interval matrix has spectrum σ,

$$C^I = \begin{pmatrix} 0 & -[\underline{\lambda_1},\overline{\lambda_1}][\underline{\lambda_2},\overline{\lambda_2}] \\ 1 & [\underline{\lambda_1},\overline{\lambda_1}]+[\underline{\lambda_2},\overline{\lambda_2}] \end{pmatrix} = \begin{pmatrix} 0 & [16, 24] \\ 1 & [0, 2] \end{pmatrix}.$$

Example 2. Consider $\sigma = \{[10,11],[-5,-3] \pm [2,4]i\}$. Whereas

$$\begin{aligned} p &= [\underline{\lambda_1},\overline{\lambda_1}][\underline{\lambda_2},\overline{\lambda_2}][\underline{\lambda_3},\overline{\lambda_3}] = [130,451] > 0 \\ \alpha_1 &= [\underline{\lambda_1},\overline{\lambda_1}][\underline{\lambda_2},\overline{\lambda_2}] + [\underline{\lambda_1},\overline{\lambda_1}][\underline{\lambda_3},\overline{\lambda_3}] + [\underline{\lambda_2},\overline{\lambda_2}][\underline{\lambda_3},\overline{\lambda_3}] \\ &= 2 \times [-55,-30] + [13,41] = [-97,-19] < 0, \end{aligned}$$

then by the case (d) of Theorem 3 the following interval matrix is solution of problem and has spectrum σ:

$$\begin{aligned} C^I &= \begin{pmatrix} 0 & p & 0 \\ 0 & 0 & 1 \\ 1 & -\alpha_1 & [\underline{\lambda_1},\overline{\lambda_1}] + [\underline{\lambda_2},\overline{\lambda_2}] + [\underline{\lambda_3},\overline{\lambda_3}] \end{pmatrix} \\ &= \begin{pmatrix} 0 & [130,451] & 0 \\ 0 & 0 & 1 \\ 1 & [19,97] & [0,5] \end{pmatrix}. \end{aligned}$$

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