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Existence and uniqueness of integrable solutions of fractional order initial value equations

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Abstract. This paper is devoted to investigate some existence and uniqueness results of integrable solutions for nonlinear fractional order initial value differential equations involving Caputo operator. We develop the existence of integral solution using Schauder's fixed point theorem. In addition, by applying the Banach contraction principle, we establish uniqueness result. To illustrate the applicability of main results, two examples are presented.

Keywords: Integrable solution; existence and uniqueness; Caputo operator; fixed point theorems.
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1 Introduction

In recent years, the theory of fractional calculus have been studied by many researchers to describe the hereditary properties and memory effect of numerous processes arise in mathematical modeling of diverse realistic situations. Nowadays the study of nonlinear fractional differential

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equations is well established area of analysis due to its applications in several practical problems for investigating the nonlocal response of many phenomena which includes viscoelasticity, thermodynamics, diffusion processes, control theory, electromagnetic and many more, see the monographs (see [4, 5, 16, 17, 19, 22, 24]). Numerous methods have been investigated to find analytical and approximate solutions of nonlinear fractional differential equations (see [8, 14, 15, 26–28]). However, to find analytical solutions the preliminary knowledge of existence of solutions of these equations is also utmost importance. Hence, developing existence and uniqueness results of different kinds of fractional differential equations has emerged as vital field of research in recent times (see [1–3, 6, 10, 18, 20, 21, 23, 25]).

To the best of our knowledge, there are few works deal with the existence of L^1- solutions of fractional differential equations. Recently, the existence of integral and continuous solutions for quadratic integral equations is studied by El-Sayed et al. [11]. Besides this, in [12] the authors investigated L^P solutions for differential equations of a weighted Cauchy problem. Further, integrable and continuous solutions of a nonlinear Riemann-Liouville fractional order Cauchy problem is studied by Gaafar in [13]. Very recently, in [7] Benchohra and Souid developed existence of L^1-solutions for implicit differential equations having Caputo fractional operator.

Inspired by these ideas, this paper is devoted to investigate existence criteria using the Schauder's fixed point theorem and uniqueness result using the Banach contraction principle for integrable solutions of nonlinear differential equations involving the Caputo fractional operator given as

$$^cD^{\alpha}u(t) = f(t, u(t), D^{\beta}u(t)), \quad t \in \Omega = [0, \tau], \tag{1}$$

$$u^i(0) = \eta_i, \qquad i = 0, 1, 2, \ldots, n-1. \tag{2}$$

Here $f : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is a given function, $\alpha \in (n-1, n)$, $\beta \in (m-1, m)$, $n, m \in \mathbb{N}$ and $\beta < \alpha$, $\tau > 0$ and $\eta_i \in \mathbb{R}$. The rest of the manuscript is organized as follows. In Section 2, some useful definitions and introductory results required to prove our main theorems are presented. In Section 3, we focus on the proof of main theorems. In Section 4, two examples are included to demonstrate the applicability of obtained criteria. Finally we give concluding remarks in Section 5.

2 Preliminary concepts

We begin by introducing some necessary definitions and basic results required for further developments in this paper.

Let us denote the class of integrable functions in the sense of Lebesgue by $L^1([0,\tau])$ on $[0,\tau]$ having norm $\|w\|_{L_1} = \int_0^\tau |w(t)|dt$.

Definition 1. ([17]) The Riemann-Liouville fractional integral operator of the function $f \in L^1([a,b],\mathbb{R}_+)$ with order $\alpha \in \mathbb{R}_+$ is defined as

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \quad t > a.$$

Definition 2. ([17]) The Riemann-Liouville fractional derivative operator for the function $f \in L^1([a,b],\mathbb{R}_+)$ with order $\alpha \in \mathbb{R}_+$, and $n-1 < \alpha \leq n$, and $n \in \mathbb{N}$ is defined as

$$D_t^\alpha f(t) = D^n I^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s)ds, \quad t > a.$$

Definition 3. ([17]) The Caputo fractional derivative operator for the function $f \in L^1([a,b],\mathbb{R}_+)$ with order $\alpha \geq 0$ is defined as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-x)^{n-\alpha-1} \frac{d^n}{ds^n} f(s)ds, \quad t > a,$$

where $\alpha \in (n-1,n)$, and $n \in \mathbb{N}$.

Proposition 1. *([17]) Let $\alpha,\ \beta > 0$. Then we have*

(*i*) $I^\alpha : L^1(\Omega,\mathbb{R}_+) \to L^1(\Omega,\mathbb{R}_+)$, *and if* $f \in L^1(\Omega,\mathbb{R}_+)$, *then*

$$I^\alpha I^\beta f(t) = I^\beta I^\alpha f(t) = I^{\alpha+\beta} f(t).$$

(*ii*) *If* $f \in L^P(\Omega,\ \mathbb{R}_+)$ *and* $1 \leq P \leq +\infty$, *then*

$$\|I^\alpha f\|_{L_p} \leq \frac{\tau^\alpha}{\Gamma(\alpha+1)} \|f\|_{L_P}.$$

(*iii*) $\lim_{\alpha \to n} I^\alpha f(t) = I^n f(t)$, $n = 1, 2, \cdots$ *uniformly.*

We recall the following theorems.

Theorem 1. *(Schauder's fixed point theorem [9]) Let E be a Banach space and S be a nonempty, closed, bounded and convex subset of E. If $F : U \to U$ is continuous and compact map then the operator $Fu = u$ has at least one fixed point in U.*

Theorem 2. *([9]) Let $\Lambda \subseteq L^P(\Omega,\ \mathbb{R})$ and $1 \le P < \infty$. if*

(i) *Λ is bounded in $L^P(\Omega,\ \mathbb{R})$, and*

(ii) *$u_j \to u$ as $j \to 0$ uniformly for $u \in \Lambda$, then Λ is relatively compact in $L^P(\Omega,\ \mathbb{R})$, where*

$$u_j(t) = \frac{1}{j}\int_t^{t+j} u(s)ds.$$

The above theorem is also known as Kolmogorov compactness criterion. Motivated from [7], for further analysis we assume that the following hypothesis are satisfied

(H1) $f : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is a function with continuous derivatives and satisfies $f(0,0,0) = 0$ and $f(t,0,0) \neq 0$.

(H2) $f : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is measurable in $t \in \Omega, \forall (x_1, x_2) \in \mathbb{R}^2$ and continuous for almost all $t \in \Omega$ in $(x_1, x_2) \in \mathbb{R}^2$.

(H3) There exists a positive function $p \in L^1(\Omega,\ \mathbb{R})$ such that

$$|f(t, x_1, x_2)| \le |p(t)| + q_1|x_1| + q_2|x_2|, \qquad \forall (t, x_1, x_2) \in \Omega \times \mathbb{R}^2,$$

where $q_i > 0;\ i = 1, 2$ are constants.

(H4) For all $x_1,\ x_2,\ y_1,\ y_2 \in \mathbb{R}$, and $t \in \Omega$, there exist constants $\kappa_1,\ \kappa_2\ > 0$ such that

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \le \kappa_1|x_1 - x_2| + \kappa_2|y_1 - y_2|.$$

3 Main results

This section is devoted to establish existence and uniqueness criteria of integrable solutions for the IVP (1) and (2).

Definition 4. A function $u \in L^1(\Omega,\ \mathbb{R})$ is a solution of IVP (1) and (2) if u satisfies (1) and (2).

Lemma 1. *([18]) Let (H1) hold and $m - 1 < \beta < m,\ n - 1 < \alpha < n,\ n, m \in \mathbb{N},\ \ m < n$, then a function $u \in L^1(\Omega,\ \mathbb{R})$ is a solution of IVP (1) and (2) if and only if*

$$u(t) = \sum_{i=0}^{m-1} \frac{t^i}{i!}\eta_i + \int_0^t \frac{(t-s)^{m-1}}{(m-1)!} w(s)ds, \quad t \in [0,1], \tag{3}$$

where $w(t) = u^m(t) \in L^1(\Omega,\ \mathbb{R})$ *is given by*

$$w(t) = \sum_{k=0}^{n-m-1} \frac{\eta_{m+k} t^k}{k!} + \frac{1}{\Gamma(\alpha - m)} \int_0^t (t-s)^{\alpha-m-1} f\Bigg(s, \sum_{i=0}^{m-1} \frac{s^i}{i!}\eta_i \\ + \int_0^s \frac{(s-\nu)^{m-1}}{(m-1)!} w(\nu)d\nu, \frac{1}{\Gamma(m-\beta)} \int_0^s (s-\nu)^{m-\beta-1} w(\nu)d\nu\Bigg) ds. \tag{4}$$

Theorem 3. *Suppose that (H1), (H2) and (H3) hold. If*

$$\frac{q_1 \tau^\alpha}{\Gamma(1+\alpha)} + \frac{q_2 \tau^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} < 1, \tag{5}$$

then there exists at least one solution for IVP (1) *and* (2) *for some constants* $q_i > 0;\ i = 1, 2$ *and for* $\alpha \in (n-1, n),\ \beta \in (m-1, m),\ n, m \in \mathbb{N}$ *and* $\beta < \alpha$.

Proof. Let U be a subset of $L^1([0,\tau],\ \mathbb{R})$ which is closed, bounded and convex given by

$$U = \{w(t) | w(t) \in L^1(\Omega,\ \mathbb{R}),\quad \|w\|_{L_1} \le R,\quad t \in \Omega\},$$

where $R = C/(1-M)$ and

$$C \ = \sum_{k=0}^{n-m-1} \frac{\eta_{m+k} \tau^{k+1}}{(k+1)!} + \frac{\tau^{\alpha-m}}{\Gamma(\alpha-m+1)} \|p\|_{L_1} + \sum_{i=0}^{m-1} \frac{q_1\ \tau^{\alpha-m+i+1}}{\Gamma(\alpha-m+i+2)} \eta_i < \infty,$$

$$M \ = \frac{q_1\ \tau^\alpha}{\Gamma(\alpha+1)} + \frac{q_2\ \tau^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} < 1.$$

Let also $F : L^1(\Omega,\ \mathbb{R}) \longrightarrow L^1(\Omega,\ \mathbb{R})$ be the mapping defined by

$$F(w)(t) \ = \sum_{k=0}^{n-m-1} \frac{\eta_{m+k} t^k}{k!} + \frac{1}{\Gamma(\alpha - m)} \int_0^t (t-s)^{\alpha-m-1} f\Bigg(s, \sum_{i=0}^{m-1} \frac{s^i}{i!}\eta_i \\ + \int_0^s \frac{(s-\nu)^{m-1}}{(m-1)!} w(\nu)d\nu, \frac{1}{\Gamma(m-\beta)} \int_0^s (s-\nu)^{m-\beta-1} w(\nu)d\nu\Bigg) ds. \tag{6}$$

Then the solution of IVP (1) agrees with fixed operator F. The operator F is well defined, indeed, for each $w \in L^1([0,\tau],\ \mathbb{R})$ and from assumptions (H1), (H2) and (H3) we get

$$\|F(w)(t)\|_{L_1} = \int_0^\tau |F(w)(t)| dt$$

$$\le \int_0^\tau \sum_{k=0}^{n-m-1} \frac{\eta_{m+k} t^k}{k!} + \int_0^\tau I^{\alpha-m} f\Bigg(s, \sum_{i=0}^{m-1} \frac{s^i}{i!}\eta_i + \int_0^s \frac{(s-\nu)^{m-1}}{(m-1)!} w(\nu)d\nu,$$

$$
\begin{aligned}
&\frac{1}{\Gamma(m-\beta)}\int_0^s (s-\nu)^{m-\beta-1}w(\nu)d\nu\Big)dt\\
\leq&\sum_{k=0}^{n-m-1}\frac{\eta_{m+k}\tau^{k+1}}{(k+1)!}+\int_0^\tau I^{\alpha-m}\left(|p(s)|+q_1\left|\sum_{i=0}^{m-1}\frac{s^i}{i!}\eta_i+I^m w(s)\right|+q_2\left|I^{m-\beta}w(s)\right|\right)dt\\
\leq&\sum_{k=0}^{n-m-1}\frac{\eta_{m+k}\tau^{k+1}}{(k+1)!}+\int_0^\tau (I^{\alpha-m}|p(s)|)dt+q_1\int_0^\tau\left(I^{\alpha-m}\sum_{i=0}^{m-1}\frac{s^i}{i!}\eta_i\right)dt\\
&+q_1\int_0^\tau (I^\alpha w(s))dt+q_2\int_0^\tau (I^{\alpha-\beta}w(s))dt\\
\leq&\sum_{k=0}^{n-m-1}\frac{\eta_{m+k}\tau^{k+1}}{(k+1)!}+\frac{\tau^{\alpha-m}}{\Gamma(\alpha-m+1)}\|p\|_{L_1}+\sum_{i=0}^{m-1}\frac{q_1\ \tau^{\alpha-m+i+1}}{\Gamma(\alpha-m+i+2)}\eta_i\\
&+\frac{q_1\ \tau^\alpha}{\Gamma(\alpha+1)}\|w\|_{L_1}+\frac{q_2\ \tau^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\|w\|_{L_1}\\
=&C+\left[\frac{q_1\ \tau^\alpha}{\Gamma(\alpha+1)}+\frac{q_2\ \tau^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right]\|w\|_{L_1}=C+M\|w\|_{L_1}\leq C+MR=R.
\end{aligned}
$$

Thus $FU\subset U$. Hence, F is continuous by assumption (H2). Further to show that F is compact, we prove that FU is relatively compact. Clearly FU is bounded in $L^1(\Omega,\ \mathbb{R})$, therefore the first condition of Kolmogorov compactness criterion is fulfilled.

Next we show that $(Fw)_j\to(Fw)$ in $L^1(\Omega,\ \mathbb{R})$, $\ \forall\ w\in U$. Let $w\in U$, then we get

$$
\begin{aligned}
\|(Fw)_j-(Fw)\|_{L_1}&=\int_0^\tau |(Fw)_j(t)-(Fw)(t)|dt\\
&=\int_0^\tau\left|\frac{1}{j}\int_t^{t+j}(Fw)(s)ds-(Fw)(t)\right|dt\\
&\leq\int_0^\tau\left(\frac{1}{j}\int_t^{t+j}|(Fw)(s)-(Fw)(t)|ds\right)dt\\
&\leq\int_0^\tau\frac{1}{j}\int_t^{t+j}\Bigg|I^{\alpha-m}f\left(s,\sum_{i=0}^{m-1}\frac{s^i}{i!}\eta_i+I^m w(s),I^{m-\beta}w(s)\right)\\
&\quad-I^{\alpha-m}f\left(t,\sum_{i=0}^{m-1}\frac{t^i}{i!}\eta_i+I^m w(t),I^{m-\beta}w(t)\right)\Bigg|dsdt.
\end{aligned}
$$

Since $w\in U\subset L^1(\Omega,\ \mathbb{R})$, hypothesis (H3) implies that $I^{\alpha-m}f\in L^1(\Omega,\ \mathbb{R})$. Therefore, we have

$$
\frac{1}{j}\int_t^{t+j}\left|I^{\alpha-m}f\left(s,\sum_{i=0}^{m-1}\frac{s^i}{i!}\eta_i+I^m w(s),I^{m-\beta}w(s)\right)-I^{\alpha-m}f\left(t,\sum_{i=0}^{m-1}\frac{t^i}{i!}\eta_i\right.\right.
$$

$$+ I^m w(t), I^{m-\beta} w(t)\Big)\Big| ds \longrightarrow 0 \quad as \quad j \to 0, \quad t \in [0,\tau].$$

Hence

$$(Fw)_j \to (Fw) \text{ uniformly as } j \to 0.$$

Thus all conditions of Theorem 2 are satisfied. Hence, FU is relatively compact which further gives at least one solution to the IVP (1) and (2) in U using Theorem 1. $\square$

Theorem 4. *Suppose that the condition (H4) holds. If*

$$\frac{\kappa_1 \tau^\alpha}{\Gamma(1+\alpha)} + \frac{\kappa_2 \tau^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} < 1, \tag{7}$$

then the IVP (1) and (2) has unique solution $w \in L^1(\Omega,\ \mathbb{R})$ for some constants $\kappa_i > 0;\ i = 1,2$ and for $\alpha \in (n-1,n),\ \beta \in (m-1,m),\ n,m \in \mathbb{N}$ and $\beta < \alpha$.

Proof. We shall prove that the operator F given by (6) has a fixed point. Let $v, w \in L^1(\Omega,\ \mathbb{R})$, and $t \in \Omega$. Then we have

$$\begin{aligned}
|F(w_1)(t) - F(w_2)(t)| &= \bigg| I^{\alpha-m}\Big[f\Big(t, \sum_{i=0}^{m-1} \frac{t^i}{i!}\eta_i + I^m w_1, I^{m-\beta} w_1\Big) \\
&\quad - f\Big(t, \sum_{i=0}^{m-1} \frac{t^i}{i!}\eta_i + I^m w_2, I^{m-\beta} w_2\Big)\Big]\bigg| \\
&\le I^{\alpha-m}\Big[\kappa_1 I^m |w_1(t) - w_2(t)| + \kappa_2 I^{m-\beta}|w_1(t) - w_2(t)|\Big] \\
&\le \kappa_1 I^\alpha |w_1(t) - w_2(t)| + \kappa_2 I^{\alpha-\beta}|w_1(t) - w_2(t)|.
\end{aligned}$$

Hence

$$\begin{aligned}
\|F(w_1)(t) - F(w_2)(t)\|_{L_1} &= \int_0^\tau [\kappa_1 I^\alpha |w_1(t) - w_2(t)| + \kappa_2 I^{\alpha-\beta}|w_1(t) - w_2(t)|]dt \\
&\le \frac{\kappa_1 \tau^\alpha}{\Gamma(1+\alpha)}\|w_1 - w_2\|_{L_1} + \frac{\kappa_2 \tau^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\|w_1 - w_2\|_{L_1} \\
&\le \left(\frac{\kappa_1 \tau^\alpha}{\Gamma(1+\alpha)} + \frac{\kappa_2 \tau^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)\|w_1 - w_2\|_{L_1}.
\end{aligned}$$

Thus, by the Banach contraction principle and using equation (7) the operator F is a contraction and consequently has a fixed point. This fixed point gives unique solution for the problem (1) and (2). $\square$

4 Illustration

To demonstrate the applicability of theorems, we present two examples.

Example 1. Consider the IVP involving Caputo fractional differential operator:

$$ {}^cD^{\frac{15}{4}}u(t) = \frac{t^2}{9} + \frac{e^{-t}}{e^t+4}u(t) + \frac{1}{7}\,{}^cD^{\frac{3}{2}}u(t), \qquad 0 < t \le 2, \tag{8}$$

$$u^{(i)}(0) = k, \quad \text{ for } i = 0,1,2,3 \text{ and } k = 1,2,3,4. \tag{9}$$

Let $f(t,u(t),{}^cD^{\frac{3}{2}}u(t)) = \frac{t^2}{9} + \frac{e^{-t}}{e^t+4}u(t) + \frac{1}{7}\,{}^cD^{\frac{3}{2}}u(t)$, then by hypothesis (H3) we have

$$|f(t,u(t),{}^cD^{\frac{3}{2}}u(t))| \le |\frac{t^2}{9}| + \frac{1}{5}|u(t)| + \frac{1}{7}\,|{}^cD^{\frac{3}{2}}u(t)|, \qquad t\in[0,2].$$

Therefore, we get $\tau = 2,\ n = 4,\ \ m = 2,\ \alpha = \frac{15}{4},\ \beta = \frac{3}{2},\ \eta_0 = 1,\ \eta_1 = 2,\ \eta_2 = 3,$ and $\eta_3 = 4$. Obviously the hypothesis (H1)-(H3) are satisfied with $p(t) = \frac{t^2}{9},\ q_1 = \frac{1}{5},\ q_2 = \frac{1}{7}$ and in this case

$$\begin{aligned} M &= \frac{q_1\ \tau^{\alpha}}{\Gamma(\alpha+1)} + \frac{q_2\ \tau^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} = \frac{2^{\frac{15}{4}}}{5\ \Gamma(\frac{19}{4})} + \frac{2^{\frac{9}{4}}}{7\ \Gamma(\frac{13}{4})} = 0.428802 < 1,\\ C &= \sum_{k=0}^{n-m-1}\frac{\eta_{m+k}\tau^{k+1}}{(k+1)!} + \frac{\tau^{\alpha-m}}{\Gamma(\alpha-m+1)}\|p\|_{L_1} + \sum_{i=0}^{m-1}\frac{q_1\ \tau^{\alpha-m+i+1}}{\Gamma(\alpha-m+i+2)}\eta_i < \infty,\end{aligned}$$

$$\sum_{k=0}^{1}\frac{\eta_{k+2}2^{k+1}}{(k+1)!} = 14, \quad \|p\|_{L_1} = \int_0^{\tau}\frac{t^2}{9}dt = \frac{8}{27}$$

$$\sum_{i=0}^{m-1}\frac{q_1\ \tau^{\alpha-m+i+1}}{\Gamma(\alpha-m+i+2)}\eta_i = \sum_{i=0}^{1}\frac{2^{\frac{11}{4}+i}}{5\ \Gamma(\frac{15}{4}+i)}\eta_i = 0.628662,$$

hence

$$C = 14 + \frac{8}{27}\frac{2^{\frac{7}{4}}}{\Gamma(\frac{11}{4})} + 0.628662 = 15.2483 < \infty.$$

By Theorem 3 there exists a solution of equation (8) and (9).

Example 2. Consider the IVP involving Caputo fractional differential operator:

$$ {}^cD^{\frac{9}{2}}u(t) = \frac{e^{-t}}{e^t+8}u(t) + \frac{(t-0.5)^2}{7}\,{}^cD^{\frac{5}{2}}u(t), \qquad 0 < t \le 2, \tag{10}$$

$$u^{(i)}(0) = k, \quad \text{ for } i = 0,1,2,3,4 \text{ and } k = 1,2,3,4,5. \tag{11}$$

Let

$$f(t, u(t), {}^c D^{\frac{5}{2}} u(t)) = \frac{e^{-t}}{e^t + 8} u(t) + \frac{(t-0.5)^2}{7} \, {}^c D^{\frac{5}{2}} u(t).$$

Then we get

$$\begin{aligned}
&|f(t, u(t), {}^c D^{\frac{5}{2}} u(t)) - f(t, v(t), {}^c D^{\frac{5}{2}} v(t))| \\
&\qquad \le \frac{e^{-t}}{e^t + 8} |u(t) - v(t)| + \frac{5(t-0.5)^2}{7} |{}^c D^{\frac{5}{2}} u(t) - {}^c D^{\frac{5}{2}} v(t)| \\
&\qquad \le \frac{1}{9} |u(t) - v(t)| + \frac{2.25}{7}{}^c D^{\frac{5}{2}} u(t) - {}^c D^{\frac{5}{2}} v(t)|.
\end{aligned}$$

Hence the condition (H3) holds with $\kappa_1 = \frac{1}{9}$ and $\kappa_2 = \frac{2.25}{7}$. Now we shall check that condition (7) is satisfied with $\tau = 2$:

$$\begin{aligned}
\frac{l_1 \tau^\alpha}{\Gamma(1+\alpha)} + \frac{l_2 \tau^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} &= \frac{2^\alpha}{9\,\Gamma(1+\alpha)} + \frac{2.25\ 2^{\alpha-\beta}}{7\,\Gamma(\alpha-\beta+1)} \\
&= \frac{2^{4.5}}{9\,\Gamma(5.5)} + \frac{2.25 \times 2^2}{7\,\Gamma(3)} = 0.69089 < 1.
\end{aligned}$$

Therefore by Theorem 4, there exists a unique integrable solution to the IVP (10) and (11) on [0, 2].

5 Conclusions

In this paper, the existence and uniqueness criteria of integrable solutions for the class of fractional differential equations (1) and (2) having Caputo fractional operator were obtained using the Schauder's fixed point theorem and the Banach contraction principle. Obtained results were illustrated using two initial value problems of fractional order.

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