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An $\mathcal{O}(h^8)$ optimal B-spline collocation for solving higher order boundary value problems

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Abstract. As we know the approximation solution of seventh order two points boundary value problems based on B-spline of degree eight has only $\mathcal{O}(h^2)$ accuracy and this approximation is non-optimal. In this work, we obtain an optimal spline collocation method for solving the general nonlinear seventh order two points boundary value problems. The $\mathcal{O}(h^8)$ convergence analysis, mainly based on the Green's function approach, has been proved. Numerical illustration demonstrate the applicability of the purposed method. Three test problems have been solved and the computed results have been compared with the results obtained by recent existing methods to verify the accurate nature of our method.

Keywords: Nonlinear boundary value problems, eighth degree B-spline, collocation method, convergence analysis, Green's function.
AMS Subject Classification: 65L10, 65L12, 65L20, 65L 70.

1 Introduction

We consider the general nonlinear seventh order two point boundary value problems (BVPs) of the following form:

$$Ly \equiv y^{(7)}(x) - f(x, y(x), y'(x), \ldots, y^{(6)}(x)) = 0, \quad a \le x \le b, \tag{1}$$

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with the boundary conditions,

$$By \equiv \sum_{j=0}^{6}(\alpha_{ij}y^{(j)}(a) + \beta_{ij}y^{(j)}(b)) = \eta_i, \quad 0 \le i \le 6, \tag{2}$$

where α_{ij}, β_{ij} and η_i are given real constants, f is a continuous function, $y(x)$ is an unknown function, and L and B are differential operators.

The formulation of many mathematical models in engineering and other branches of sciences are in the form of differential equations with initial or boundary conditions and boundary value problems generally. Obtaining the analytic solution for these problems is impossible, because of this, many authors attempt to use different numerical methods such as finite difference, Galerkin and Sinc collocation methods [12, 14, 15].

The literature on the numerical solution of seventh order two point boundary value problems is seldom. These problems are generally arise in modelling induction motors with two rator circuits. Behaviour of such models have been studied by Richards and Sarma [18]. The solution of seventh order BVPs based on variational iteration and differential transformation method are given by Siddiqi et al. [21, 22]. In [23] the authors used the homotopy analysis method for solving higher order BVPs. Reproducting kernel method for the solution of seventh order BVPs has been studied in [3].

Many researchers applied the collocation methods for solution of BVPs [2, 4, 19]. The spline functions has been applied to solve BVPs in [5], with order $\mathcal{O}(h^2)$. After that many authors [1, 7, 8, 20] examined the collocation method based on cubic spline for BVPs.

An optimal cubic spline collocation method at grid points was developed by Danial and Swatrz in [6], which gives $\mathcal{O}(h^4)$ accuracy. In [10] the authors used optimal collocation method on midpoints based on quadratic spline for approximate the solution of second order BVPs. Irodotou-Ellina and Houstis applied the optimal quintic spline collocation method for solving linear fourth order two point BVPs which lead to an $\mathcal{O}(h^6)$ approximation[11].

In [16] Rashidinia et al. developed an optimal method based on sextic spline at the grid points for solving of nonlinear fifth order two point BVPs. Also, Rashidinia and Ghasemi [17] applied an optimal sextic spline at the midpoints for the numerical solution of sixth order nonlinear two point BVPs.

In this paper we applied optimal collocation method based on B-spline of degree eight at the nodal points of the interval $[a, b]$ and obtained $\mathcal{O}(h^8)$ approximation for the numerical solution of boundary value problems (1) – (2). The approximation is assumed to satisfy a high order approximation of the problem. In Section 2, we obtain the consistency relations for spline of degree eight at the nodal points of the partition. In Section 3, the description of the method based on spline for the solution of (1) − (2) is explained. The convergence analysis of the presented method is given in detail, in Section 4. In Section 5, numerical experiments are conducted to demonstrate the applicability of the proposed method computationally. Conclusion is presented in Section 6.

2 Spline interpolation

We define the spline of degree eight as basis functions to construct an interpolant $S(x)$, satisfying certain end conditions and then derive several relations that are useful in the formulation of the optimal spline collocation method.

Now let $\Delta \equiv \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a uniform partition of the interval $[a, b]$ with the step size $h = \frac{b-a}{n}$. We consider smooth spline of degree eight $S(x)$, that is an element of $Sp_8(\Delta) \equiv \{q(x)|q(x) \in C^7[a, b]\}$ and $q(x)$ is a polynomial of degree at most 8 on the partition Δ. The set of B-splines $\{B_k(x)\}_{k=-3}^{k=n+4}$, form a basis for $Sp_8(\Delta)$, so we can define our spline of degree eight in the following form:

$$S(x) = \sum_{k=-3}^{n+4} c_k B_k(x), \quad x \in [x_i, x_{i+1}],$$

that satisfies the following interpolatory conditions:

$$S(x_i) = y(x_i), \quad 0 \le i \le n, \tag{3}$$

associated with the end conditions:

$$S^{(7)}(x_i) = y^{(7)}(x_i) - \frac{h^2}{12} y^{(9)}(x_i) + \frac{h^4}{240} y^{(11)}(x_i) - \frac{h^6}{6048} y^{(13)}(x_i), \tag{4}$$

for $i = 0, 1, 2, 3, n-2, n-1, n$. By using linear dependence relations, we have the following consistency relations for spline of degree eight and its first seventh derivatives for $4 \le i \le n-3$ at the grid points: [9, 24]

$$\begin{aligned}
(a)\quad \Gamma S_i^{(7)} &= \frac{40320}{h^7}(\mp S_{i-4,i+3} \pm 7S_{i-3,i+2} \mp 21S_{i-2,i+1} \pm 35S_{i-1,i}),\\
(b)\quad \Gamma S_i^{(6)} &= \frac{20160}{h^6}(S_{i-4,i+3} - 5S_{i-3,i+2} + 9S_{i-2,i+1} - 5S_{i-1,i}),\\
(c)\quad \Gamma S_i^{(5)} &= \frac{6720}{h^5}(\mp S_{i-4,i+3} \pm S_{i-3,i+2} \pm 9S_{i-2,i+1} \mp 25S_{i-1,i}),\\
(d)\quad \Gamma S_i^{(4)} &= \frac{1680}{h^4}(S_{i-4,i+3} + 7S_{i-3,i+2} - 27S_{i-2,i+1} + 19S_{i-1,i}),
\end{aligned}$$

$$\begin{aligned}
(e)\quad \Gamma S_i^{(3)} &= \frac{336}{h^3}(\mp S_{i-4,i+3} \mp 23S_{i-3,i+2} \pm 9S_{i-2,i+1} \pm 95S_{i-1,i}),\\
(f)\quad \Gamma S_i^{(2)} &= \frac{56}{h^2}(S_{i-4,i+3} + 55S_{i-3,i+2} + 189S_{i-2,i+1} - 245S_{i-1,i}),\\
(g)\quad \Gamma S_i^{(1)} &= \frac{8}{h}(\mp S_{i-4,i+3} \mp 119S_{i-3,i+2} \mp 1071S_{i-2,i+1} \mp 1225S_{i-1,i}),\\
(h)\quad \Gamma g_i &= (g_{i-4,i+3} + 247g_{i-3,i+2} + 4293g_{i-2,i+1} + 15619g_{i-1,i}), \qquad (5)
\end{aligned}$$

where the discrete operator Γ is defined for any function g on the interval $[a, b]$. For sake of convenience we set $S_i \equiv S(x_i)$, $S_i^j \equiv S^j(x_i)$, $i = 0, 1, \dots, n$, $j = 1, 2, \dots, 7$ where $g^{(j)} \equiv D^{(j)}g$. In order to obtain the error bounds for spline of degree eight S and its derivatives $S', \dots, S^{(7)}$, we present the next theorem.

Theorem 1. *Let $S(x)$ be the spline of degree eight, satisfying (3) − (4) and interpolating the function $y \in C^{14}[a, b]$, then for $i = 0, 1, \dots, n$ the following relations hold,*

$$\begin{aligned}
(a)\quad S_i^{(1)} &= y_i^{(1)} + \mathcal{O}(h^8),\\
(b)\quad S_i^{(2)} &= y_i^{(2)} + \mathcal{O}(h^8),\\
(c)\quad S_i^{(3)} &= y_i^{(3)} - \frac{h^6}{30240}y_i^{(9)} + \mathcal{O}(h^8),\\
(d)\quad S_i^{(4)} &= y_i^{(4)} + \frac{h^6}{6048}y_i^{(10)} + \mathcal{O}(h^8),\\
(e)\quad S_i^{(5)} &= y_i^{(5)} + \frac{h^4}{720}y_i^{(9)} - \frac{h^6}{3024}y_i^{(11)} + \mathcal{O}(h^8),\\
(f)\quad S_i^{(6)} &= y_i^{(6)} - \frac{h^4}{240}y_i^{(10)} + \frac{h^6}{3024}y_i^{(12)} + \mathcal{O}(h^8),\\
(g)\quad S_i^{(7)} &= y_i^{(7)} - \frac{h^2}{12}y_i^{(9)} + \frac{h^4}{240}y_i^{(11)} - \frac{h^6}{6048}y_i^{(13)} + \mathcal{O}(h^8), \qquad (6)
\end{aligned}$$

and we have the following error bounds,

$$\| (S - y)^{(k)} \| = \mathcal{O}(h^{9-k}), \quad k = 1, 2, \dots, 7. \qquad (7)$$

Proof. First we need to prove relation $(6g)$. Using Taylor's series expansion and taking into account the interpolatory condition $S_i = y_i$, $i = 0, 1, \dots, n$, in the relation $(5a)$ we have

$$\begin{aligned}
\Gamma S_i^{(7)} &= 40320y_i^{(7)} - 20160hy_i^{(8)} + 16800h^2y_i^{(9)} - 6720h^3y_i^{(10)} + 3192h^4y_i^{(11)}\\
&\quad -1064h^5y_i^{(12)} + \frac{1120}{3}h^6y_i^{(13)} - \frac{320}{3}h^7y_i^{(14)} + \mathcal{O}(h^8), \qquad (8)
\end{aligned}$$

for $4 \le i \le n-3$. Further, using Taylor's series expansion, for any function $g \in C^8[a,b]$ we obtain

$$\begin{aligned}\Gamma g_i &= 40320 g_i - 20160 h g_i' + 20160 h^2 g_i^{(2)} - 8400 h^3 g_i^{(3)} + 4704 h^4 g_i^{(4)} \\ &\quad -1680 h^5 g_i^{(5)} + 688 h^6 g_i^{(6)} - 215 h^7 g_i^{(7)} + \mathcal{O}(h^8), \quad 4 \le i \le n-3. \quad (9)\end{aligned}$$

Setting $g(x) = y_i^{(7)} - \frac{h^2}{12} y_i^{(9)} + \frac{h^4}{240} y_i^{(11)} - \frac{h^6}{6048} y_i^{(13)}$, we have the following relation

$$\begin{aligned}\Gamma g_i &= \Gamma\left(y_i^{(7)} - \frac{h^2}{12} y_i^{(9)} + \frac{h^4}{240} y_i^{(11)} - \frac{h^6}{6048} y_i^{(13)}\right) \\ &= 40320 y_i^{(7)} - 20160 h y_i^{(8)} + 16800 h^2 y_i^{(9)} - 6720 h^3 y_i^{(10)} + 3192 h^4 y_i^{(11)} \\ &= -1064 h^5 y_i^{(12)} + \frac{1120}{3} h^6 y_i^{(13)} - \frac{320}{3} h^7 y_i^{(14)} + \mathcal{O}(h^8). \quad (10)\end{aligned}$$

By subtracting Eq. (8) from (10), we obtain

$$\Gamma\left(S_i^{(7)} - y_i^{(7)} + \frac{h^2}{12} y_i^{(9)} - \frac{h^4}{240} y_i^{(11)} + \frac{h^6}{6048} y_i^{(13)}\right) = \mathcal{O}(h^8), \ \ 4 \le i \le n-3. \quad (11)$$

Denoting $R_i \equiv S_i^{(7)} - y_i^{(7)} + \frac{h^2}{12} y_i^{(9)} - \frac{h^4}{240} y_i^{(11)} + \frac{h^6}{6048} y_i^{(13)}$, then by associating the Eq. (4) and consistency equation (11), we get the following system of equations,

$$\Gamma R_i = \mathcal{O}(h^8) \parallel y^{(14)} \parallel, \ 4 \le i \le n-3,$$

$$R_0 = R_1 = R_2 = R_3 = R_{n-2} = R_{n-1} = R_n = 0. \quad (12)$$

Since the coefficient matrix of the above system is positive definite, it is nonsingular and its inverse has a finite norm. Thus we have $R_i = \mathcal{O}(h^8)$, $i = 0, 1, \ldots, n$, this concludes the proof of relation (6g).

To prove relation (6b) consider the following relations, which can be easily obtained via long straightforward calculations for any spline of degree eight at the interior grid points x_i,

$$\begin{aligned}S_i^{(6)} &= \frac{-1}{40320 h^6}[-40320 S_{i,i+6} + 241920 S_{i+1,i+5} - 604800 S_{i+2,i+4} \\ &\quad +806400 S_{i+3} - h^7(20159 S_i^{(7)} + 40072 S_{i+1}^{(7)} + 35779 S_{i+2}^{(7)} + 20160 S_{i+3}^{(7)} \\ &\quad +4541 S_{i+4}^{(7)} + 248 S_{i+5}^{(7)} + S_{i+6}^{(7)})], \quad 0 \le i \le n-6,\end{aligned}$$

and

$$\begin{aligned} S_i^{(6)} &= \frac{-1}{40320h^6}[\pm 116081280 S_{i-7,i} \mp 812609280 S_{i-6,i-1} \pm 2437948800 S_{i-5,i-2} \\ &\mp 4063449600 S_{i-4,i-3} + h^7(2879 S_{i-7}^{(7)} + 711112 S_{i-6}^{(7)} + 12359299 S_{i-5}^{(7)} \\ &+ 44962560 S_{i-4}^{(7)} + 44946941 S_{i-3}^{(7)} + 12323768 S_{i-2}^{(7)} + 671041 S_{i-1}^{(7)} \\ &-17280 S_i^{(7)})], \quad 7 \le i \le n. \end{aligned}$$

Using relation part (g) of Eq. (6) in the above relations and applying Taylor's series expansion of $y_{i\pm l}^{(k)}$ for $k = 0, 7, 9, 11$ we get

$$S_i^{(6)} = y_i^{(6)} - \frac{h^4}{240} y_i^{(10)} + \frac{h^6}{3024} y_i^{(12)} + \mathcal{O}(h^8), \quad 0 \le i \le n.$$

In a similar manner applying some appropriate consistency relations we can prove the other relations in this Theorem. □

To improve the order of numerical solution of the system of equations (1)−(2) we need to donate the following discrete operators for convenience:

$$\begin{aligned} \lambda_0 g_i &= g_{i-3,i+3} - 6g_{i-2,i+2} + 15g_{i-1,i+2} - 20g_i, \quad 3 \le i \le n-3 \\ \lambda_1 g_i &= -\frac{1}{6}[g_{i-3,i+3} - 12g_{i-2,i+2} + 39g_{i-1,i+1} - 56g_i], \quad 3 \le i \le n-3 \\ \lambda_2 g_i &= -\frac{1}{12}[g_{i-3,i+3} - 18g_{i-2,i+2} + 63g_{i-1,i+1} - 92g_i], \quad 3 \le i \le n-3 \\ \lambda_3 g_i &= g_{i-1} - 2g_i + g_{i+1}, \quad 3 \le i \le n-3, \end{aligned}$$

These operators define the relations of eight degree spline S with respect to the higher derivatives $y^{(9)}, \dots, y^{(13)}$.

Lemma 1. *If $y \in C^{14}[a, b]$, then using the above operators we have*

$$\begin{aligned} y_i^{(r)} &= \frac{\lambda_0 S_i^{(r-6)}}{h^6} + \mathcal{O}(h^2), \qquad 9 \le r \le 13,\ 3 \le i \le n-3, \\ y_i^{(r)} &= \frac{\lambda_1 S_i^{(r-4)}}{h^4} + \mathcal{O}(h^4), \qquad 9 \le r \le 10,\ 3 \le i \le n-3, \\ y_i^{(11)} &= \frac{\lambda_2 S_i^{(7)}}{h^4} + \mathcal{O}(h^4), \qquad 3 \le i \le n-3, \\ y_i^{(9)} &= \frac{\lambda_3 S_i^{(7)}}{h^2} + \mathcal{O}(h^6), \qquad 1 \le i \le n-1. \end{aligned}$$

Proof. The proof is state forward by using Lemma 2.1 and Theorem 2.1. □

Corollary 1. *Let S be the spline of degree eight which used to interpolate $y \in C^{14}[a,b]$ then for $i = 3(1)n-3$, the following relations hold*

$$\begin{aligned}
y_i^{(7)} &= S_i^{(7)} + \frac{1}{12}\lambda_3 S_i^{(7)} - \frac{1}{240}\lambda_2 S_i^{(7)} + \frac{1}{6048}\lambda_0 S_i^{(7)} + \mathcal{O}(h^8),\\
y_i^{(6)} &= S_i^{(6)} + \frac{1}{240}\lambda_1 S_i^{(6)} - \frac{1}{3024}\lambda_0 S_i^{(6)} + \mathcal{O}(h^8),\\
y_i^{(5)} &= S_i^{(5)} - \frac{1}{720}\lambda_1 S_i^{(5)} + \frac{1}{3024}\lambda_0 S_i^{(5)} + \mathcal{O}(h^8),\\
y_i^{(4)} &= S_i^{(4)} - \frac{1}{6048}\lambda_0 S_i^{(4)} + \mathcal{O}(h^8),
\end{aligned}$$

$$
\begin{aligned}
y_i^{(3)} &= S_i^{(3)} + \frac{1}{30240}\lambda_0 S_i^{(3)} + \mathcal{O}(h^8),\\
y_i^{(2)} &= S_i^{(2)} + \mathcal{O}(h^8),\\
y_i^{(1)} &= S_i^{(1)} + \mathcal{O}(h^8).
\end{aligned}
$$

Now we need to obtain the similar relations at the boundary and its neighbour points, so that we conclude the following Corollary 2.

Corollary 2. *Let $y \in C^{14}[a,b]$, denoting the index $\sigma_j = j$, $j = 0,1,2$ for the grid points, near the left end point and $\sigma_j = n-j$, $j = n-2, n-1, n$ for the grid points, near the right end point, then the following approximations to the higher order derivatives of y hold at the boundary and its neighbour points,*

$$
\begin{aligned}
y_{\sigma_0}^{(r)} &= \lambda_1\Big(\frac{20S_{\sigma_3}^{(r-4)} - 45S_{\sigma_4}^{(r-4)} + 36S_{\sigma_5}^{(r-4)} - 10S_{\sigma_6}^{(r-4)}}{h^4}\Big) + \mathcal{O}(h^4), \quad r = 9,10,\\
y_{\sigma_1}^{(r)} &= \lambda_1\Big(\frac{10S_{\sigma_3}^{(r-4)} - 20S_{\sigma_4}^{(r-4)} + 15S_{\sigma_5}^{(r-4)} - 4S_{\sigma_6}^{(r-4)}}{h^4}\Big) + \mathcal{O}(h^4), \quad r = 9,10,\\
y_{\sigma_2}^{(r)} &= \lambda_1\Big(\frac{4S_{\sigma_3}^{(r-4)} - 6S_{\sigma_4}^{(r-4)} + 4S_{\sigma_5}^{(r-4)} - S_{\sigma_6}^{(r-4)}}{h^4}\Big) + \mathcal{O}(h^4), \quad r = 9,10,\\
y_{\sigma_0}^{(9)} &= \lambda_3\Big(\frac{6S_{\sigma_1}^{(7)} - 15S_{\sigma_2}^{(7)} + 20S_{\sigma_3}^{(7)} - 15S_{\sigma_4}^{(7)} + 6S_{\sigma_5}^{(7)} - S_{\sigma_6}^{(7)}}{h^2}\Big) + \mathcal{O}(h^6),\\
y_{\sigma_0}^{(11)} &= \lambda_2\Big(\frac{20S_{\sigma_3}^{(7)} - 45S_{\sigma_4}^{(7)} + 36S_{\sigma_5}^{(7)} - 10S_{\sigma_6}^{(7)}}{h^4}\Big) + \mathcal{O}(h^4),\\
y_{\sigma_1}^{(11)} &= \lambda_2\Big(\frac{10S_{\sigma_3}^{(7)} - 20S_{\sigma_4}^{(7)} + 15S_{\sigma_5}^{(7)} - 4S_{\sigma_6}^{(7)}}{h^4}\Big) + \mathcal{O}(h^4),\\
y_{\sigma_2}^{(11)} &= \lambda_2\Big(\frac{4S_{\sigma_3}^{(7)} - 6S_{\sigma_4}^{(7)} + 4S_{\sigma_5}^{(7)} - S_{\sigma_6}^{(7)}}{h^4}\Big) + \mathcal{O}(h^4),\\
y_{\sigma_0}^{(r)} &= \lambda_0\Big(\frac{4S_{\sigma_3}^{(r-6)} - 3S_{\sigma_4}^{(r-6)}}{h^6}\Big) + \mathcal{O}(h^2), \quad r = 9,10,11,12,13,\\
y_{\sigma_1}^{(r)} &= \lambda_0\Big(\frac{3S_{\sigma_3}^{(r-6)} - 2S_{\sigma_4}^{(r-6)}}{h^6}\Big) + \mathcal{O}(h^2), \quad r = 9,10,11,12,13,\\
y_{\sigma_2}^{(r)} &= \lambda_0\Big(\frac{2S_{\sigma_3}^{(r-6)} - S_{\sigma_4}^{(r-6)}}{h^6}\Big) + \mathcal{O}(h^2), \quad r = 9,10,11,12,13.
\end{aligned}
$$

3 Description of the method

For solution of system of boundary value problems (1)–(2) by using $S(x) \in Sp_8(\Delta)$ and to achieve an $\mathcal{O}(h^8)$ optimal order method. We approximate

$y', \ldots, y^{(7)}$ by their spline relations, which prescribe in Theorem 1, Lemma 1 and Corollaries 1 and 2. Finally this approach lead to the following nonlinear system:

$$
\begin{aligned}
&S^{(7)}_{\sigma_0} + \frac{\lambda_3}{12}(6S^{(7)}_{\sigma_1} - 15S^{(7)}_{\sigma_2} + 20S^{(7)}_{\sigma_3} - 15S^{(7)}_{\sigma_4} + 6S^{(7)}_{\sigma_5} - S^{(7)}_{\sigma_6}) \\
&\quad -\frac{\lambda_2}{240}(20S^{(7)}_{\sigma_3} - 45S^{(7)}_{\sigma_4} + 36S^{(7)}_{\sigma_5} - 10S^{(7)}_{\sigma_6}) + \frac{\lambda_0}{6048}(4S^{(7)}_{\sigma_3} - 3S^{(7)}_{\sigma_4}) \\
&= f(x_{\sigma_0}, S_{\sigma_0}, S'_{\sigma_0}, S''_{\sigma_0}, S^{(3)}_{\sigma_0} + \frac{\lambda_0}{30240}(4S^{(3)}_{\sigma_3} - 3S^{(3)}_{\sigma_4}), \\
&\quad S^{(4)}_{\sigma_0} - \frac{\lambda_0}{6048}(4S^{(4)}_{\sigma_3} - 3S^{(4)}_{\sigma_4}), \\
&\quad S^{(5)}_{\sigma_0} - \frac{\lambda_1}{720}(20S^{(5)}_{\sigma_3} - 45S^{(5)}_{\sigma_4} + 36S^{(5)}_{\sigma_5} - 10S^{(5)}_{\sigma_6}) + \frac{\lambda_0}{3024}(4S^{(5)}_{\sigma_3} - 3S^{(5)}_{\sigma_4}), \\
&\quad S^{(6)}_{\sigma_0} + \frac{\lambda_1}{240}(20S^{(6)}_{\sigma_3} - 45S^{(6)}_{\sigma_4} + 36S^{(6)}_{\sigma_5} - 10S^{(6)}_{\sigma_6}) - \frac{\lambda_0}{3024}(4S^{(6)}_{\sigma_3} - 3S^{(6)}_{\sigma_4})) \\
&\quad +\mathcal{O}(h^8), \quad i = 0, n,
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
&S^{(7)}_{\sigma_1} + \frac{\lambda_3}{12}S^{(7)}_{\sigma_1} - \frac{\lambda_2}{240}(10S^{(7)}_{\sigma_3} - 20S^{(7)}_{\sigma_4} + 15S^{(7)}_{\sigma_5} - 4S^{(7)}_{\sigma_6}) \\
&\quad +\frac{\lambda_0}{6048}(3S^{(7)}_{\sigma_3} - 2S^{(7)}_{\sigma_4}) = f(x_{\sigma_1}, S_{\sigma_1}, S'_{\sigma_1}, S''_{\sigma_1}, \\
&\quad S^{(3)}_{\sigma_1} + \frac{\lambda_0}{30240}(3S^{(3)}_{\sigma_3} - 2S^{(3)}_{\sigma_4}), S^{(4)}_{\sigma_1} - \frac{\lambda_0}{6048}(3S^{(4)}_{\sigma_3} - 2S^{(4)}_{\sigma_4}), \\
&\quad S^{(5)}_{\sigma_1} - \frac{\lambda_1}{720}(10S^{(5)}_{\sigma_3} - 20S^{(5)}_{\sigma_4} + 15S^{(5)}_{\sigma_5} - 4S^{(5)}_{\sigma_6}) + \frac{\lambda_0}{3024}(3S^{(5)}_{\sigma_3} - 2S^{(5)}_{\sigma_4}), \\
&\quad S^{(6)}_{\sigma_1} + \frac{\lambda_1}{240}(10S^{(6)}_{\sigma_3} - 20S^{(6)}_{\sigma_4} + 15S^{(6)}_{\sigma_5} - 4S^{(6)}_{\sigma_6}) - \frac{\lambda_0}{3024}(3S^{(6)}_{\sigma_3} - 2S^{(6)}_{\sigma_4})) \\
&\quad +\mathcal{O}(h^8), \quad i = 1, n-1,
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
&S^{(7)}_{\sigma_2} + \frac{\lambda_3}{12}S^{(7)}_{\sigma_2} - \frac{\lambda_2}{240}(4S^{(7)}_{\sigma_3} - 6S^{(7)}_{\sigma_4} + 4S^{(7)}_{\sigma_5} - S^{(7)}_{\sigma_6}) + \frac{\lambda_0}{6048}(2S^{(7)}_{\sigma_3} - S^{(7)}_{\sigma_4}) \\
&= f(x_{\sigma_2}, S_{\sigma_2}, S'_{\sigma_2}, S''_{\sigma_2}, S^{(3)}_{\sigma_2} + \frac{\lambda_0}{30240}(2S^{(3)}_{\sigma_3} - S^{(3)}_{\sigma_4}), \\
&\quad S^{(4)}_{\sigma_2} - \frac{\lambda_0}{6048}(2S^{(4)}_{\sigma_3} - S^{(4)}_{\sigma_4}), \\
&\quad S^{(5)}_{\sigma_2} - \frac{\lambda_1}{720}(4S^{(5)}_{\sigma_3} - 6S^{(5)}_{\sigma_4} + 4S^{(5)}_{\sigma_5} - S^{(5)}_{\sigma_6}) + \frac{\lambda_0}{3024}(2S^{(5)}_{\sigma_3} - S^{(5)}_{\sigma_4}), \\
&\quad S^{(6)}_{\sigma_2} + \frac{\lambda_1}{240}(4S^{(6)}_{\sigma_3} - 6S^{(6)}_{\sigma_4} + 4S^{(6)}_{\sigma_5} - S^{(6)}_{\sigma_6}) - \frac{\lambda_0}{3024}(2S^{(6)}_{\sigma_3} - S^{(6)}_{\sigma_4})) \\
&\quad +\mathcal{O}(h^8), \quad i = 2, n-2,
\end{aligned} \tag{15}
$$

$$S_i^{(7)} + \frac{\lambda_3}{12} S_i^{(7)} - \frac{\lambda_2}{240} S_i^{(7)} + \frac{\lambda_0}{6048} S_i^{(7)} = f(x_i, S_i, S_i', S_i'', S_i^{(3)} + \frac{\lambda_0}{30240} S_i^{(3)},$$
$$S_i^{(4)} - \frac{\lambda_0}{6048} S_i^{(4)}, S_i^{(5)} - \frac{\lambda_1}{720} S_i^{(5)} + \frac{\lambda_0}{3024} S_i^{(5)},$$
$$S_i^{(6)} + \frac{\lambda_1}{240} S_i^{(6)} - \frac{\lambda_0}{3024} S_i^{(6)}) + \mathcal{O}(h^8), \quad 3 \le i \le n-3, \tag{16}$$

associated with the boundary formulas,

$$\begin{aligned}
BS \equiv\ & \alpha_{i,0} S_0 + \alpha_{i,1} S_0' + \alpha_{i,2} S_0'' + \alpha_{i,3}(S_0^{(3)} + \frac{\lambda_0}{30240}(4S_3^{(3)} - 3S_4^{(3)})) \\
& + \alpha_{i,4}(S_0^{(4)} - \frac{\lambda_0}{6048}(4S_3^{(4)} - 3S_4^{(4)})) \\
& + \alpha_{i,5}(S_0^{(5)} - \frac{\lambda_1}{720}(20S_{\sigma_3}^{(5)} - 45S_{\sigma_4}^{(5)} + 36S_{\sigma_5}^{(5)} - 10S_{\sigma_6}^{(5)}) \\
& + \frac{\lambda_0}{3024}(4S_3^{(5)} - 3S_4^{(5)})) \\
& + \alpha_{i,6}(S_0^{(6)} + \frac{\lambda_1}{240}(20S_{\sigma_3}^{(6)} - 45S_{\sigma_4}^{(6)} + 36S_{\sigma_5}^{(6)} - 10S_{\sigma_6}^{(6)}) \\
& - \frac{\lambda_0}{3024}(4S_3^{(6)} - 3S_4^{(6)})) + \beta_{i,0} S_n + \beta_{i,1} S_n' + \beta_{i,2} S_n'' \\
& + \beta_{i,3}(S_n^{(3)} + \frac{\lambda_0}{30240}(4S_{n-3}^{(3)} - 3S_{n-4}^{(3)})) \\
& + \beta_{i,4}(S_n^{(4)} - \frac{\lambda_0}{6048}(4S_{n-3}^{(4)} - 3S_{n-4}^{(4)})) \\
& + \beta_{i,5}(S_n^{(5)} - \frac{\lambda_1}{720}(20S_{n-3}^{(5)} - 45S_{n-4}^{(5)} + 36S_{n-5}^{(5)} - 10S_{n-6}^{(5)}) \\
& + \frac{\lambda_0}{3024}(4S_{n-3}^{(5)} - 3S_{n-4}^{(5)})) \\
& + \beta_{i,6}(S_n^{(6)} + \frac{\lambda_1}{240}(20S_{n-3}^{(6)} - 45S_{n-4}^{(6)} + 36S_{n-5}^{(6)} - 10S_{n-6}^{(6)}) \\
& - \frac{\lambda_0}{3024}(4S_{n-3}^{(6)} - 3S_{n-4}^{(6)})) = \eta_i, \quad i = 0, 1, \ldots, 6.
\end{aligned} \tag{17}$$

Let L' be the approximation of L defined as follows,

$$\begin{aligned}
L' g_i \equiv\ & g_i^{(7)} + \frac{1}{12}\lambda_3 g_i^{(7)} \\
& - \frac{1}{240}\lambda_2 g_i^{(7)} + \frac{1}{6048}\lambda_0 g_i^{(7)} - f(x_i, g_i, g_i', g_i'', g_i^{(3)} + \frac{1}{30240}\lambda_0 g_i^{(3)}, \\
& g_i^{(4)} - \frac{1}{6048}\lambda_0 g_i^{(4)}, g_i^{(5)} - \frac{1}{720}\lambda_1 g_i^{(5)} + \frac{1}{3024}\lambda_0 g_i^{(5)}, \\
& g_i^{(6)} + \frac{1}{240}\lambda_1 g_i^{(6)} - \frac{1}{3024}\lambda_0 g_i^{(6)}),
\end{aligned}$$

and let B' be the approximation of B defined in (17), and $S(x)$ be the spline of degree eight which is the solution of the system (1) – (2), then the following relations hold,

$$\begin{cases} L'S_i = \mathcal{O}(h^8), & 0 \le i \le n, \\ B'S = \mathcal{O}(h^8). \end{cases} \tag{18}$$

For the convergence analysis first we need to recall and prove the following Lemmas.

Lemma 2. *If $p = \{p_{ij}\}$ is an $m \times m$ matrix and $p_{ii} \ge \sum_{j=1, i\neq j}^{m} |p_{ij}| + \epsilon$, for $i = 1, 2, \ldots, m$, where $\epsilon > 0$, then we have $\|p^{-1}\|_\infty \le \epsilon^{-1}$.*

Proof. See Lemma 4 in [13]. □

Lemma 3. *If the coefficients matrix of $S_i^{(7)}$ in the equation $L'S_i = \mathcal{O}(h^8)$, $i = 0, 1, \ldots, n$, is denoted by Q_7, then Q_7 is nonsingular and $\|Q_7^{-1}\|_\infty$ is bounded.*

Proof. Using relations (13) – (16) we can obtain the $(n+1) \times (n+1)$ coefficients matrix Q_7 as

$$Q_7 = \frac{1}{60480} \times$$

$$\begin{pmatrix}
91180 & -144855 & 327246 & -466043 & \cdots & 4482248 & -291687 & 124918 & -32853 \\
5280 & 46000 & 26715 & -52434 & \cdots & 74077 & -64992 & 35493 & -11402 \\
104 & 3272 & 58404 & -12709 & \cdots & 23054 & -18771 & 9656 & -2971 \\
31 & -438 & 6513 & 48268 & \cdots & 6513 & -438 & 31 & 0 \\
0 & 31 & -438 & 6513 & \cdots & 48268 & 6513 & -438 & 31 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 31 & -438 & \cdots & 6513 & 48268 & 6513 & -438 \\
0 & 0 & 0 & 31 & \cdots & -438 & 6513 & 48268 & 6513 \\
-21 & 4625 & -2971 & 9656 & \cdots & -18771 & 23054 & -12709 & 58404 \\
-84 & 1827 & -11402 & 35493 & \cdots & -64992 & 74077 & -52434 & 26715 \\
-210 & 4536 & -32853 & 124918 & \cdots & -291687 & 4482248 & -466043 & 327246
\end{pmatrix},$$

Let E_i, be the i-th row of Q_7. Then by using the following elementary row operations, this matrix can be converted to a strictly diagonally dominant:

$$\begin{aligned}
&E_2 + \frac{1}{2}E_3 - \frac{1}{4}E_4 + \frac{1}{4}E_5 \to E_2, \\
&E_1 - \frac{1}{2}E_2 + E_3 - 1.4E_4 + 1.35E_5 - 0.833E_6 + 0.333E_7 \to E_1, \\
&0.1E_0 + 0.314E_1 - 0.847E_2 + 1.61E_3 - 1.56E_4 + 1.172E_5 \\
&\qquad -0.4859E_6 + 0.1666E_7 + 0.0407E_8 \to E_0, \\
&E_{n-2} + \frac{1}{2}E_{n-3} - \frac{1}{4}E_{n-4} + \frac{1}{4}E_{n-5} \to E_{n-2},
\end{aligned}$$

$$\begin{aligned} &E_{n-1} - \frac{1}{2}E_{n-2} + E_{n-3} - 1.4E_{n-4} + 1.35E_{n-5} - 0.833E_{n-6} \\ &\quad +0.333E_{n-7} \to E_{n-1}, \\ &0.1E_n + 0.314E_{n-1} - 0.847E_{n-2} + 1.61E_{n-3} - 1.56E_{n-4} + 1.172E_{n-5} \\ &\quad -0.4859E_{n-6} + 0.1666E_{n-7} + 0.0407E_{n-8} \to E_n, \end{aligned}$$

Hence, the matrix Q_7 is strictly diagonally dominant and positive definite. Therefore, using Lemma 2 we can conclude that $\|Q_7^{-1}\|_\infty$ is finite. $\square$

4 Convergence analysis

We prove the convergence of the presented method via Green's function scheme. If we assume that the boundary value condition $y^{(7)} = 0$ subjected to homogeneous boundary conditions $By = 0$, has a unique solution, it implies that there is a Green's function $G(x,t)$ for this problem [19]. Let $y^{(7)} = \phi$ and $\hat{S}^{(7)} = \psi$, be the exact and the spline solutions of the problem (1) which satisfy the boundary conditions (2). Then $y(x)$ and $\hat{S}(x)$ and its first sixth derivatives can be obtained as follows:

$$y^{(i)}(x) = \int_a^b \frac{\partial^i G(x,t)}{\partial x^i}\phi(t)dt, \quad \hat{S}^{(i)}(x) = \int_a^b \frac{\partial^i G(x,t)}{\partial x^i}\psi(t)dt,$$

for $i = 0, 1, \ldots, 6$. We define the operators $F_n, \mathbf{M}_n, \mathbf{k}$ and $\mathbf{R}$ as:

$$F_n : C[a,b] \to R^{n+1}, \quad F_n g = [g(x_0), \ldots, g(x_n)]^T,$$

$$\mathbf{M}_n : R^{n+1} \to C[a,b], \quad \text{via piecewise linear interpolation at } \{x_i\}_0^n,$$

$$\mathbf{k} : C[a,b] \to C[a,b], \quad \mathbf{k}g = f(x, G_{p,0}(x), G_{p,1}(x), \ldots, G_{p,6}(x)),$$

$$\mathbf{R} : C[a,b] \to C[a,b], \quad \mathbf{R}g = f(x, Q_0F_nG_{p,0}(x), \ldots, Q_6F_nG_{p,6}(x)),$$

where $g \in C[a,b]$, $G_{p,i}(x) = \displaystyle\int_a^b \frac{\partial^i G_p(x,t)}{\partial x^i}g(t)dt,\ i = 0, 1, \ldots, 6$ and

$$Q_i = \begin{cases} I_{(n+1)\times(n+1)}, & 0 \le i \le 2, \\ \text{The coefficients matrix of } S^{(i)} \text{ in Eq. (18)}, & 3 \le i \le 7. \end{cases}$$

With the introduced notations, we can rewrite Eqs. (1) and (13)-(16) respectively as:

$$y^{(7)} - f(x, y(x), y'(x), \ldots, y^{(6)}(x)) = \phi - \mathbf{k}\phi = (I - \mathbf{k})\phi = 0, \tag{21}$$

$$Q_7F_n\hat{S}^{(7)} - f(x, Q_0F_n\hat{S}, Q_1F_n\hat{S}', \ldots, Q_6F_n\hat{S}^{(6)}) = Q_7F_n\hat{S}^{(7)} - F_n\mathbf{R}\psi = 0.$$

Since Q_7 is nonsingular and $\hat{S}^{(7)}(x)$ is a linear polynomial, therefore we have the following relations:

$$F_n\hat{S}^{(7)} - Q_7^{-1}F_n\mathbf{R}\psi = 0 \Rightarrow \mathbf{M}_nF_n\hat{S}^{(7)} - \mathbf{M}_nQ_7^{-1}F_n\mathbf{R}\psi = 0,$$

$$\hat{S}^{(7)} - \mathbf{M}_nQ_7^{-1}F_n\mathbf{R}\psi = (I - \mathbf{M}_nQ_7^{-1}F_n\mathbf{R})\psi = (I - p_n\mathbf{R})\psi = 0, \tag{22}$$

where $p_n = \mathbf{M}_nQ_7^{-1}F_n$. Notice that p_n is an operator from $C[a, b]$ into the continuous piecewise linear functions with grid points x_i.

Lemma 4. *Let $\{\Delta\}$ be a sequence of partitions of the interval $[a, b]$. Then the sequence of operators $p_n = \boldsymbol{M}_nQ_7^{-1}F_n$ converges to the identity operator as h approaches zero.*

Proof. We want to show that $|p_ng - g| \to 0$ for each $g \in C[a, b]$. To do so, we have

$$\begin{aligned}
\|p_ng - g\| &\leq \|\mathbf{M}_nQ_7^{-1}F_ng - \mathbf{M}_nF_ng\| \\
&\leq \|\mathbf{M}_n\|\|Q_7^{-1}\|\|F_ng - Q_7F_ng\| \\
&\leq C^*\|F_ng - Q_7F_ng\| \\
&\leq C^*\omega(g, 10h),
\end{aligned}$$

where C^* is a finite constant and $\omega(g, \epsilon) = \sup\{|g(x+\epsilon') - g(x)| : x, x+\epsilon' \in [a, b], |\epsilon'| \leq \epsilon\}$. When $h \to 0$ we have, $\omega(g, 10h) \to 0$. □

Lemma 5. *Let $g \in C[a, b]$, then $p_n\boldsymbol{R}$ converges to $\boldsymbol{k}$.*

Proof. By using the definitions of $\mathbf{k}$ and $\mathbf{R}$ we obtain

$$\begin{aligned}
\|p_n\mathbf{R}g - \mathbf{k}g\| &= \|\mathbf{M}_nQ_7^{-1}F_n\mathbf{R}g - \mathbf{k}g\| \\
&\leq \|\mathbf{M}_nQ_7^{-1}F_n\mathbf{R}g - \mathbf{M}_nF_n\mathbf{k}g\| + \|\mathbf{M}_nF_n\mathbf{k}g - \mathbf{k}g\| \\
&\leq \|\mathbf{M}_nQ_7^{-1}\|\ \|F_n\mathbf{R}g - Q_7F_n\mathbf{K}g\| + \mathcal{O}(h^2),
\end{aligned}$$

and since $\|\mathbf{M}_n\|$ and $\|Q_7^{-1}\|$ are bounded, we have

$$\|p_n\mathbf{R}g - \mathbf{k}g\| \leq \hat{C}\|F_n\mathbf{R}g - Q_7F_n\mathbf{K}g\| \leq \hat{C}\omega(g, 10\delta),$$

with

$$\delta = \max\{10h, \omega(G_{p,0}(x), 17h), \omega(G_{p,1}(x), 17h), \ldots, \omega(G_{p,6}(x), 17h)\}. \tag{23}$$

$\omega(G_{p,j}(x), 17h)$, $0 \leq j \leq 6$ convergence to zero as h approaches zero for continuous functions $G_{p,j}(x)$, $0 \leq j \leq 6$, so that by using Eq (23), $\delta \to 0$ and $\omega(g, 10\delta)$ convergence to zero. □

Now we present the main convergence theorem.

Theorem 2. *The error bounds for collocation approximation $\hat{S}(x) \in Sp_8(\Delta)$ satisfies,*

$$\begin{aligned} &\|(y-\hat{S})^{(j)}\| = \mathcal{O}(h^{8-j}), \quad j = 0,1,\ldots,7,\\ &|(y-\hat{S})_i^{(j)}| = \mathcal{O}(h^8), \quad j = 0,1,2,\\ &|(y-\hat{S})_i^{(j)}| = \mathcal{O}(h^6), \quad j = 3,4,\\ &|(y-\hat{S})_i^{(j)}| = \mathcal{O}(h^4), \quad j = 5,6,\\ &|(y-\hat{S})_i^{(j)}| = \mathcal{O}(h^2), \quad j = 7. \end{aligned}$$

Proof. We consider the problem: $S^{(7)} = \nu,\;\; BS = \mathcal{O}(h^8)$. Let $\{\Delta\}$ be a sequence of partitions of the $[a,b]$ and the problem $y^{(7)} = 0,\; By = 0$ has a unique solution. So there exists a polynomial $\xi(x)$ of order 6 as follows

$$B\xi = BS = \mathcal{O}(h^8), \quad \|\xi^{(k)}\| = \mathcal{O}(h^8), \quad k = 0,1,\ldots,6. \tag{24}$$

From solvability of $(S-\xi)^{(7)} = \nu,\; B(S-\xi) = 0$ we obtain

$$(I - \mathbf{M}_n Q_7^{-1} F_n \mathbf{R})(S^{(7)} - \xi^{(7)}) = \mathbf{M}_n Q_7^{-1}(Q_7 F_n - F_n \mathbf{R})(S-\xi)^{(7)}.$$

Using (18) and the boundedness of $\|\mathbf{M}_n\|$ and $\|Q_7^{-1}\|$, we have

$$(I - \mathbf{M}_n Q_7^{-1} F_n \mathbf{R})(S^{(7)} - \xi^{(7)}) = \mathbf{M}_n Q_7^{-1}(\mathcal{O}(h^8)) = \mathcal{O}(h^8). \tag{25}$$

Subtracting (22) and (25), we obtain,

$$(I - \mathbf{M}_n Q_7^{-1} F_n \mathbf{R})(S^{(7)} - \xi^{(7)} - \hat{S}^{(7)}) = \mathcal{O}(h^8),$$

and we have

$$(S^{(7)} - \xi^{(7)} - \hat{S}^{(7)}) = p_n \mathbf{R}(S^{(7)} - \xi^{(7)} - \hat{S}^{(7)}) + \mathcal{O}(h^8). \tag{26}$$

The operator $\mathbf{R}$ is continuously differentiable. So Eq. (26) has an integral equation form as following

$$\begin{aligned}(S^{(7)} - \xi^{(7)} - \hat{S}^{(7)}) = p_n &\left(\int_0^1 (\mathbf{R}'[\hat{S}^{(7)} + t(S^{(7)} - \xi^{(7)} - \hat{S}^{(7)})]dt \right) \\ &\times (S^{(7)} - \xi^{(7)} - \hat{S}^{(7)}) + \mathcal{O}(h^8), \end{aligned} \tag{27}$$

where $\{\tau_n\} = p_n \left(\int_0^1 (\mathbf{R}'[\psi + t(S^{(7)} - \xi^{(7)} - \hat{S}^{(7)})]dt \right)$, is a sequence of linear operators converging to $\mathbf{R}'(y^{(7)})$. So we have

$$(S^{(7)} - \xi^{(7)} - \hat{S}^{(7)}) = \tau_n (S^{(7)} - \xi^{(7)} - \hat{S}^{(7)}) + \mathcal{O}(h^8).$$

Since $(I-\tau_n)^{-1}$ exists and its norm is bounded, we obtain

$$\|(S-\xi-\hat{S})^{(7)}\|_\infty = \mathcal{O}(h^8). \tag{28}$$

According to the hypotheses of the problem, $(S-\xi-\hat{S})^{(7)} = r,\ B(S-\xi-\hat{S}) = 0$, has unique solution. So we can write $(S-\xi-\hat{S})^{(i)}$ in the following form

$$(S-\xi-\hat{S})^{(i)} = \int \frac{\partial^i G(x,t)}{\partial x^i}(S^{(7)}-\xi^{(7)}-\hat{S}^{(7)})(t)dt,\ i=0,1,\ldots,6, \tag{29}$$

which implies that

$$\|(S-\xi-\hat{S})^{(i)}\|_\infty = \mathcal{O}(h^8), \quad i=0,1,\ldots,6. \tag{30}$$

Using the triangular inequality we obtain

$$\|(y-S)^{(i)}\| \le \|(y-\hat{S})^{(i)}\| + \|(S-\hat{S})^{(i)}\| + \|\xi^{(i)}\|, \quad i=0,1,\ldots,6,$$

by using equations (18) and (24) and Theorem 1, we can obtain the results of Theorem 2. This completes of proof. □

5 Numerical experiments

We present the results from numerical experiments to demonstrate the performance of the presented method and verify the results of the analysis. The obtained results has been compared with the references [21, 23, 3] and the results tabulated in Tables 1–6, these results verify the accurate nature of our purposed method in applications. The numerical computations have done by the software Mathematica 10.

Example 1. The following linear seventh order boundary value problem is considered:

$$y^{(7)}(x) = xy(x) + e^x(x^2-2x-6), \quad 0 \le x \le 1,$$

subjected to the boundary conditions

$$\begin{aligned} &y(0) = y(1) = 1, \quad y'(0) = 0,\ y'(1) = -e, \\ &y''(0) = -1, \quad y''(1) = -2e, \quad y^{(3)}(0) = -2. \end{aligned}$$

The exact solution of the problem is $y(x) = (1-x)e^x$. This example has been solved by our method with $h = \frac{1}{10}$, the maximum absolute errors in the certain points are tabulated in Table 1 and compared with [21, 23], which shows that our method is accurate. Also the example has been solved with $h = \frac{1}{9}, \frac{1}{18}, \frac{1}{36}, \frac{1}{72}, \frac{1}{144}$ and the maximum absolute errors in the solutions are tabulated in Table 2. In this table $E_i = \|y^{(i)} - \hat{S})^{(i)}\|_\infty, \quad 0 \le i \le 6$ and O_i is the order of convergence of i-th derivatives of y. This table also verified that our approach are applicable and accurate.

Table 1: The maximum absolute errors in the solution of Example 1.

x	our method	method in [23]	method in [21]
0.1	1.03(−15)	3.42(−13)	4.66(−13)
0.2	2.08(−15)	6.25(−14)	5.71(−12)
0.3	5.71(−14)	1.42(−13)	2.13(−11)
0.4	9.52(−14)	8.84(−14)	4.69(−11)
0.5	8.82(−14)	6.43(−14)	7.43(−11)
0.6	5.05(−13)	1.52(−12)	8.92(−11)
0.7	1.91(−13)	1.48(−12)	7.98(−11)
0.8	1.82(−13)	4.94(−12)	4.67(−11)
0.9	1.05(−13)	5.38(−12)	1.09(−11)

Table 2: The maximum absolute errors in the solution of Example 1 with various values of h.

h	$\frac{1}{9}$	$\frac{1}{18}$	$\frac{1}{36}$	$\frac{1}{72}$	$\frac{1}{144}$
E_0, O_0	1.5(−13), −	5.3(−16), 8.1	1.9(−18), 8.1	7.6(−21), 7.9	3.0(−23), 7.9
E_1, O_1	5.6(−13), −	2.0(−15), 8.1	7.8(−18), 8	2.9(−20), 8.1	1.1(−22), 8
E_2, O_2	4.2(−12), −	1.6(−14), 8	5.9(−17), 8.1	2.3(−19), 8	8.9(−22), 8
E_3, O_3	8.9(−10), −	1.5(−11), 5.9	2.3(−13), 6	3.7(−15), 6	5.7(−17), 6
E_4, O_4	5.0(−9), −	8.1(−11), 5.9	1.3(−12), 5.9	2.0(−14), 6	3.2(−16), 6
E_5, O_5	3.1(−6), −	1.9(−7), 4	1.3(−8), 3.9	7.9(−10), 4	4.9(−11), 4
E_6, O_6	1.0(−5), −	6.6(−7), 3.9	4.2(−8), 4	2.7(−9), 4	1.8(−10), 4

Example 2. Consider the following nonlinear seventh order boundary value problem,

$$y^{(7)}(x) = y^2(x)e^x, \quad 0 \le x \le 1,$$

subjected to the boundary conditions

$$y(0) = y'(0) = y''(0) = y^{(3)}(0) = 1, \quad y(1) = y'(1) = y''(1) = e.$$

Table 3: The maximum absolute errors in the solution of Example 2.

h	our method	method in [3]	method in [21]
$\frac{1}{10}$	$6.22(-15)$	$6.48(-11)$	$3.02(-14)$
$\frac{1}{30}$	$1.42(-18)$	$3.31(-14)$	$--$
$\frac{1}{50}$	$1.01(-19)$	$2.78(-15)$	$--$

The exact solution of this problem is $y(x) = e^x$. First of all we solve this problem for various values of $h = \frac{1}{10}, \frac{1}{30}, \frac{1}{50}$ and compare with the results in [21, 3]. Our results are shown in Table 3. Then we obtain E_i and O_i for various values of h. The results are tabulated in Table 4. This table shows that the orders of convergence in applications agree with those we obtained theoretically.

Table 4: The maximum absolute errors in the solution of Example 2 with various values of h.

h	$\frac{1}{9}$	$\frac{1}{18}$	$\frac{1}{36}$	$\frac{1}{72}$	$\frac{1}{144}$
E_0, O_0	$1.5(-14), -$	$5.3(-17), 8.1$	$2.0(-19), 8$	$7.7(-22), 8$	$2.9(-24), 8.1$
E_1, O_1	$5.9(-14), -$	$2.2(-16), 8.1$	$8.3(-19), 8.1$	$3.2(-21), 8$	$1.2(-23), 8.1$
E_2, O_2	$4.3(-13), -$	$1.6(-15), 8.1$	$6.2(-18), 8$	$2.4(-20), 8$	$9.3(-23), 8$
E_3, O_3	$1.0(-10), -$	$1.7(-12), 5.9$	$2.7(-14), 6$	$4.2(-16), 6$	$6.6(-18), 6$
E_4, O_4	$5.2(-10), -$	$8.4(-12), 6$	$1.3(-13), 6$	$2.1(-15), 6$	$3.3(-17), 6$
E_5, O_5	$3.6(-7), -$	$2.3(-8), 4$	$1.5(-9), 3.9$	$9.2(-11), 4$	$5.8(-12), 4$
E_6, O_6	$1.1(-6), -$	$6.9(-8), 4$	$4.4(-9), 4$	$2.8(-10), 4$	$1.7(-11), 4$

Example 3. Consider the following nonlinear seventh order boundary value problem,

$$\begin{aligned} y^{(7)}(x) + y^{(4)}(x) - y(x)e^{y(x)} &= e^x((-4(-3+x) \\ &+e^{(-e^x(x-1)\cos x)}(x-1))\cos x - 8(5+x)\sin x), \quad 0 \le x \le 1, \end{aligned}$$

subjected to the boundary conditions

$$\begin{aligned} &y(0) = 1,\ y'(0) = y(1) = 0,\ y'(1) = -e\cos 1, \\ &y''(0) = y^{(3)}(0) = -2,\ y''(1) = -2e\cos 1 + 2e\sin 1. \end{aligned}$$

The exact solution of the problem is given by $y(x) = e^x(1-x)\cos x$. First we solve this problem with $h = \frac{1}{30}$ and compared the errors in those special points given in [3]. These results are tabulated in Table 5, the results in this table verified that our method is more accurate. Then we obtain E_i and O_i for various values of h. The results are tabulated in Table 6.

Table 5: The maximum absolute errors in the solution of Example 3.

x	our method	method in [3]
0.125	1.15(−12)	4.74(−10)
0.250	1.35(−10)	5.20(−9)
0.375	2.89(−9)	1.53(−8)
0.500	5.42(−9)	2.45(−8)
0.625	4.96(−9)	2.53(−8)
0.750	2.69(−9)	1.56(−8)
0.875	1.94(−10)	3.29(−9)

Table 6: The maximum absolute errors in the solution of Example 3 with various values of h.

h	$\frac{1}{9}$	$\frac{1}{18}$	$\frac{1}{36}$	$\frac{1}{72}$	$\frac{1}{144}$
E_0, O_0	3.8(−13), −	1.1(−15), 8.4	6.3(−18), 7.4	2.9(−20), 7.8	1.2(−22), 7.9
E_1, O_1	4.6(−12), −	1.6(−14), 8.1	6.3(−17), 8	2.5(−19), 8	9.8(−22), 8
E_2, O_2	1.7(−11), −	5.4(−14), 8.3	2.4(−16), 7.8	9.9(−19), 7.9	4.1(−21), 7.9
E_3, O_3	1.2(−8), −	1.9(−10), 6	2.9(−12), 6	4.6(−14), 6	7.2(−16), 6
E_4, O_4	3.9(−8), −	5.8(−10), 6.1	9.1(−12), 6	1.4(−13), 6	2.2(−15), 6
E_5, O_5	4.1(−5), −	2.6(−6), 4	1.6(−7), 4	1.1(−8), 3.9	6.3(−10), 4.1
E_6, O_6	7.7(−5), −	4.8(−6), 4	2.9(−7), 4	1.8(−8), 4	1.2(−9), 3.9

6 Conclusion

We developed a numerical method to solve the general nonlinear seventh order boundary value problems by using eighth degree B-spline approximation. The numerical illustration shown the proposed method has the $\mathcal{O}(h^8)$ order of accuracy, so we can conclude that our method has highly accurate and efficient in comparison with the other existing methods. Our results obtained by the optimal $\mathcal{O}(h^8)$ method are in good agreement with the proposed numerical algorithm.

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