Journal of Mathematical Modeling
Vol. 3, No. 1, 2015, pp. 77-89

Bernoulli matrix approach for matrix differential models of first-order

Ahmad Golbabai[*] **and Samaneh Panjeh Ali Beik**

School of Mathematics, Iran University of Science and Technology, Tehran, Iran
Emails: golbabai@iust.ac.ir, panjehali@iust.ac.ir

Abstract. The current paper contributes a novel framework for solving a class of linear matrix differential equations. To do so, the operational matrix of the derivative based on the shifted Bernoulli polynomials together with the collocation method are exploited to reduce the main problem to system of linear matrix equations. An error estimation of presented method is provided. Numerical experiments are reported to demonstrate the applicably and efficiency of the propounded technique.

Keywords: Linear matrix differential equation, Bernoulli polynomials, operational matrix of derivative, error estimation .
AMS Subject Classification: 34A30, 41A10, 65L05.

1 Introduction

In the present work, we focus on the following first-order linear matrix differential problem:

$$\begin{cases} Y'(t) = A(t)Y(t) + B(t), & t_0 \leq t \leq t_f, \\ Y(t_0) = Y_0, \end{cases} \tag{1}$$

where $Y \in \mathbb{R}^{p\times q}$ is an unknown matrix, the matrices $Y_0 \in \mathbb{R}^{p\times q}$, $A : [t_0, t_f] \to \mathbb{R}^{p\times p}$ and $B : [t_0, t_f] \to \mathbb{R}^{p\times q}$ are given. Let us assume that in

[*]Corresponding author.
Received: 12 November 2014 / Revised: 2 January 2015 / Accepted: 14 January 2015.

(1), $A, B \in C^s([t_0, t_f])$ for $s \geq 1$ which guarantees that (1) has a unique and continuously differentiable solution [13].

Considering the fundamental role of matrix differential models in the numerous areas of applied sciences and engineering, development and implementation of the accurate methods for solving these equations are the subject of interest and have been examined intensively in the literature; see [1, 2, 9, 10, 14, 20] and the references therein.

It is well-known that the collection methods are the specific type of spectral methods. These methods, have been utilized for various of differential and integral equations. Two main advantages of collocation methods are the ease of applying the method and High accuracy; for further details see [7, 8]. The main purpose of the current work is to implement the collocation method for evaluating a rough solution of (1). To this end, first, each entry of the approximate solution $Y(t)$ is expanded in terms of the Bernoulli polynomials up to degree m with unknown coefficient. Afterwards with the suitable collocation knots and properties of Bernoulli polynomials, we reach to a system of linear matrix equations. So the computations can be handled in a simple way and the unknown coefficients will obtained by solving the matrix equations.

The outline of this paper is organized as follows. Section 2 presents a brief survey on some preparatory definitions and concepts of the Bernoulli polynomials which are required for our subsequent development. In Section 3, first, it reveals that how the Bernoulli polynomials can be implemented to reduce solving (1) into resolving matrix equations. In addition, an upper error bound for the approximated solution is established. Section 4 is devoted to reporting some numerical experiments which turn out the accuracy of the proposed numerical scheme for solving (1). Eventually, the paper is ended with a brief conclusion in Section 5.

2 An overview on Bernoulli polynomials

The Bernoulli polynomials of order i are specified as follows (see [16])

$$B_i(\tau) = \sum_{k=0}^{i} \begin{pmatrix} i \\ k \end{pmatrix} \alpha_k \tau^{i-k},$$

where

$$\begin{pmatrix} i \\ k \end{pmatrix} = \frac{i!}{k!(i-k)!},$$

and α_k's are called the Bernoulli numbers and determined by

$$\frac{\tau}{e^{\tau}-1}=\sum_{k=0}^{\infty}\alpha_k\frac{\tau^k}{k!}.$$

It is known that the Bernoulli polynomials form a complete basis over $[0,1]$; for more details see [15]. Bernoulli polynomials satisfy the following relations

$$\frac{dB_i(\tau)}{d\tau}=iB_{i-1}(\tau),\quad i\geq 1. \tag{2}$$

The advantages of Bernoulli polynomials in comparison with classical orthogonal polynomials such as Legendre or Chebyshev polynomials are due to less terms and more sparse operational matrices which provide less CPU time and computational errors [16, 19]; In order to utilize these polynomials on an arbitrary interval $[t_0,t_f]$, we define the so-called shifted Bernoulli polynomials by applying the change of variable

$$\tau=\frac{(t-t_0)}{h},\qquad\qquad t_0\leq t\leq t_f,$$

where $h=t_0-t_f$. Consequently,

$$\beta_i(t)=\sum_{k=0}^{i}\left(\begin{array}{c} i\\ k\end{array}\right)\alpha_k\frac{(t-t_0)^{i-k}}{h^{i-k}}.$$

Presume that $H:=L^2[t_0,t_f]$ and

$$Y=\mathrm{span}\{\beta_0,\beta_1,\ldots,\beta_m\},$$

where $m\in\mathbb{N}\cup\{0\}$ and β_i's are the Bernoulli polynomials. Since $Y\subset H$ is a finite dimensional vector space, for every $f\in H$, there exists a unique $y_0\in Y$ such that

$$\|f-y_0\|_2\leq\|f-y\|_2\qquad\forall y\in Y,$$

in which $\|f\|_2=\sqrt{\langle f,f\rangle}$. Here, the function y_0 is called the best approximation to f out of Y. As $y_0\in Y$, we may conclude that

$$f(t)\approx y_0(t)=\sum_{j=0}^{m}c_j\beta_j(t)=C^T\Psi(t),$$

where

$$\Psi^T(t)=(\beta_0(t),\beta_1(t),\ldots,\beta_m(t)), \tag{3}$$

and $C^T = (c_0, c_1, \ldots, c_m)$ such that C^T uniquely calculated by

$$C^T Q = \int_{t_0}^{t_f} f(t)\Psi(t)dt, \tag{4}$$

where $Q \in \mathbb{R}^{(m+1)\times(m+1)}$ is called the dual matrix of $\Psi(t)$ and given by

$$Q = \int_{t_0}^{t_f} \Psi(t)\Psi^T(t)dt.$$

For more details about best approximation see [15]. The subsequent proposition is useful for the next sections. As in [8], the weighted $L^2_\omega[t_0, t_f]$ norm is defined as

$$\|f\|^2_{L^2_\omega[t_0,t_f]} = \int_{t_0}^{t_f} |f(t)|^2\omega(t)dx.$$

Definition 1. A function, $f : [t_0, t_f] \to \mathbb{R}$, belongs to Sobolev space $H^{k,2}_\omega$, if its jth weak derivative, lies in $L^2_\omega[t_0, t_f]$ for $0 \le j \le k$ with the norm

$$\|f\|^2_{H^{k,2}_\omega(t_0,t_f)} = \sum_{j=0}^{k} \left\| f^{(j)} \right\|^2_{L^2_\omega}.$$

Proposition 1. *([8]) Assume that $f \in H^{k,2}_\omega(t_0, t_f)$, $\mathcal{I}_m f$ is the interpolation of f at Chebyshev-Gauss points and the weight function defined as $\omega(t) = (1 - t^2)^{-1/2}$. Then*

$$\|f - \mathcal{I}_m f\|_{L^2_\omega(t_0,t_f)} \le Ch^{\min(k,m)} m^{-k} |f|_{H^{k,2}_\omega(t_0,t_f)}, \tag{5}$$

where

$$|f|^2_{H^{k,2}_\omega(t_0,t_f)} = \sum_{j=\min(k,m+1)}^{k} \left\| f^{(j)} \right\|^2_{L^2_\omega},$$

and C is a constant independent of m and f

Definition 2. The derivative of the vector $\Psi(t)$ in (3) can be determined by

$$\frac{d\Psi(t)}{dt} = D\Psi(t), \quad t_0 \le t \le t_f, \tag{6}$$

where $D \in \mathbb{R}^{(m+1)\times(m+1)}$ is called the operational matrix of derivative. From equation (2) it is easy to see that [19] this matrix is given by

$$D = \frac{1}{h} \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & m-1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & m & 0 \end{pmatrix}.$$

3 Proposing the main approach

Let us approximate each of the entries of $Y(t) = [y_{ij}(t)]_{p\times q}$ in (1), on $[t_0, t_f]$ by the Bernoulli polynomials. That is,

$$Y(t) \approx \overline{\mathcal{A}}(I_q \otimes \Psi(t)). \tag{7}$$

where the notation $\otimes$ stands for the well-known Kronecker product, I_q is the identity matrix of order q, $\Psi(t)$ is given by (3) and $\overline{\mathcal{A}} \in \mathbb{R}^{p\times q(m+1)}$ is unknown constant matrix to be determined. The definition of the operational matrix of derivative implies that

$$Y'(t) \approx \overline{\mathcal{A}}(I_q \otimes D\Psi(t)), \tag{8}$$

By Substituting Eqs. (7) and (8) in (1), we derive

$$\overline{\mathcal{A}}(I_q \otimes D\Psi(t)) = A(t)\overline{\mathcal{A}}(I_q \otimes \Psi(t)) + B(t) + R_m(t). \tag{9}$$

In order to specify the unknown coefficients in (9), we collocate this equation at m collocation points, which is named ς_i and $R_m(\varsigma_i) = 0$ for $i = 1, \ldots, m$. We chose the the Chebyshev-Gauss nodes in $[t_0, t_f]$ as suitable collocation points. By replacing the above knots in (9), we reach the following coupled linear matrix equations

$$\overline{\mathcal{A}}\mathcal{C}_i = \mathcal{D}_i\overline{\mathcal{A}}\mathcal{E}_i + \mathcal{F}_i, \qquad i = 1, 2, \ldots, m,$$

where $\mathcal{C}_i = I_q \otimes D\Psi(\varsigma_i)$, $\mathcal{D}_i = A(\varsigma_i)$, $\mathcal{E}_i = I_q \otimes \Psi(\varsigma_i)$ and $\mathcal{F}_i = B(\varsigma_i)$. In addition for $i = m+1$ from initial condition, we set $\mathcal{C}_{m+1} = [0]_{q(m+1)\times q}$, $\mathcal{D}_{m+1} = I_p$, $\mathcal{E}_{m+1} = I_q \otimes \Psi(t_0)$ and $\mathcal{F}_{m+1} = -Y(t_0)$. Therefore, in order to numerically solve the problem (1), we may solve the following coupled linear matrix equation

$$X\mathcal{C}_i - \mathcal{D}_i X\mathcal{E}_i = \mathcal{F}_i, \quad i = 1, 2, \ldots, m+1, \tag{10}$$

where $\mathcal{C}_i$, $\mathcal{D}_i$, $\mathcal{E}_i$ and $\mathcal{F}_i$ are constant matrices and the unknown matrix $X := \overline{\mathcal{A}}$ is to be determined. Using the following relation (see [6]),

$$\mathrm{vec}(AXB) = (B^T \otimes A)\mathrm{vec}(X),$$

it can be found that the coupled matrix equations (10) are equivalent to the linear system $Ax = b$, with subsequent parameters

$$\left\{ \begin{array}{l} A = \begin{pmatrix} \mathcal{C}_1^T \otimes I_p - \mathcal{E}_1^T \otimes \mathcal{D}_1 \\ \vdots \\ \mathcal{C}_{m+1}^T \otimes I_p - \mathcal{E}_{m+1}^T \otimes \mathcal{D}_{m+1} \end{pmatrix}, \\ \\ x = \mathrm{vec}(X), \qquad b = \begin{pmatrix} \mathrm{vec}(\mathcal{F}_1) \\ \vdots \\ \mathrm{vec}(\mathcal{F}_{m+1}) \end{pmatrix}, \end{array} \right. \tag{11}$$

The classical methods such as the GMRES or conjugate gradient method [17] can be applied for solving the above linear system. However, the size of the coefficient matrix of the linear system (11) is $pq(m + 1)$ and it may become too large even for moderate values of p, q and m. This motivates us to apply an iterative method for solving the coupled linear matrix equations(10) rather than the linear system. In the literature, a large number of papers are devoted to applying different kinds of iterative algorithm for solving various linear coupled matrix equations, for more details see [3–5, 11, 12, 18] and the references therein.

3.1 Implementing the method

For solving (1), we use a step-by-step strategy. To do so, we first choose a step length $h \neq 0$ and consider the points $x_i = x_0 + ih$, $i = 1, 2, 3, \ldots$. Then, starting with the given initial values $x_0 := t_0$, $\mathcal{P}_0 := Y(x_0)$. Now by using the approach described in the previous section we solve the following matrix differential equation

$$\left\{ \begin{array}{ll} \mathcal{P}'(t) = A(t)\mathcal{P}(t) + B(t), & x_i \leq t < x_{i+1}, \\ \mathcal{P}(x_i) = \mathcal{P}_i, & \end{array} \right.$$

and successively compute the rough solution $\mathcal{P}(t)$ to $Y(t)$ on $[x_i, x_{i+1})$ for $i = 0, 1, 2, \ldots, [\frac{1}{h}] - 1$. Afterward we set $\mathcal{P}_{i+1} = \mathcal{P}(x_{i+1})$, to compute the approximate solution $\mathcal{P}(t)$ of $Y(t)$ on the next subinterval.

3.2 Estimation of an upper error bound

In this subsection, we present an upper error bound analytically that reveals the spectral rate of convergence. In what follows, the (i, j)th entry of the matrix $Y(t)$ is denoted by $y_{ij}(t)$.

Definition 3. Let $F(t) = [f_{ij}(t)]$ be an arbitrary $p \times q$ matrix defined on $[t_0, t_f]$ such that $f_{ij}(t) \in L^2[t_0, t_f]$. Then, we define

$$\parallel F \parallel_\infty = \max_{i,j} \parallel f_{ij} \parallel_{L^2_\omega}, \qquad 1 \le i \le p, \quad 1 \le j \le q.$$

Theorem 1. *Consider the problem* (1) *where* $y_{ij} \in H^{k,2}_\omega(x_l, x_{l+1})$*,* $A(t) = [a_{ij}(t)]_{p\times p}$ *and* $B(t) = [b_{ij}(t)]_{p\times q}$ *are given such that* $a_{ij}(t)$ *and* $b_{ij}(t)$ *are sufficiently smooth. In addition, assume that* $Y_m = \overline{\mathcal{A}}(I_q \otimes \Psi)$ *stands for the Bernoulli collocation approximation of* Y*. Furthermore, suppose that* $M_1 = \max_{ij} \max_{t\in(x_l,x_{l+1})} |a_{ij}(t)|$*. Then* $\lim_{m\to\infty} Y_m(t) = Y(t)$*. Besides the following statement holds:*

$$\begin{aligned}\|Y - Y_m\|_\infty \le \tilde{C}_1 M_1 h^{\min(k,m)+1} m^{-1-k} \max_j \sum_{\nu=1}^{p} |y_{\nu j}|_{H^{k,2}_\omega(x_l,x_{l+1})} \\ + \tilde{C}_2 h^{\min(k,m)} m^{-k} \max_{ij} |y_{ij}|_{H^{k,2}_\omega(x_l,x_{l+1})}.\end{aligned}$$

Proof. Integrating (1) in $[x_l, t]$ results in

$$Y(t) = \int_{x_l}^{t} (A(x)Y(x) + B(x))dx + Y(x_l). \tag{12}$$

Since we assume that $Y_m(x_l) = Y(x_l)$, we can rewrite (10) as follows:

$$Y_m(\xi_n) = \int_{x_l}^{\xi_n} (A(x)Y_m(x) + B(x))dx + Y(x_l), \quad n = 1, \dots, m+1, \tag{13}$$

where $\xi_n, n = 1, \dots, m$, are Chebyshev-Gauss knots on $[x_l, x_{l+1}]$ and $\xi_{m+1} = x_l$.

It follows form (13) that

$$Y_m(\xi_n) = \int_{x_l}^{\xi_n} A(x)E(x)dx + \int_{x_l}^{\xi_n} (A(x)Y(x) + B(x))dx + Y(x_l), \tag{14}$$

such that $E = [e_{ij}]_{p\times q} = Y_m - Y$. Multiplying both sides of the $n-$th equation of (14) by Lagrange interpolating polynomial, L_n and summing up over n form 1 to $m+1$ results in

$$\begin{aligned}\sum_{n=1}^{m+1} L_n(t) Y_m(\xi_n) = \sum_{n=1}^{m+1} L_n(t) \int_{x_l}^{\xi_n} A(x)E(x)dx \\ + \sum_{n=1}^{m+1} L_n(t)(\int_{x_l}^{\xi_n} (A(x)P(x) + B(x))dx + Y(x_l)).\end{aligned}$$

Subtracting form (12) yields

$$\sum_{n=1}^{m+1} L_n(t)Y_m(\xi_n) - Y(t) = \int_{x_l}^{t} A(x)E(x)dx + S_2(t) + S_1(t), \tag{15}$$

where

$$S_1(t) = \sum_{n=1}^{m+1} L_n(t)\int_{x_l}^{\xi_n} A(x)E(x)dx - \int_{x_l}^{t} A(x)E(x)dx,$$

and

$$S_2(t) = \sum_{n=1}^{m+1} L_n(t)(\int_{x_l}^{\xi_n} (A(x)P(x) + B(x))dx + Y(x_l))$$

$$- \int_{x_l}^{t} (A(x)P(x) + B(x))dx - Y(x_l)dx.$$

We may rewrite (15) in the following form

$$E(t) = \int_{x_l}^{t} A(x)E(x)dx + H(t), \tag{16}$$

By implying Gronwall inequality on (16) we have

$$\|E\|_\infty \leq C\|H\|_\infty. \tag{17}$$

Since we assume that the A and B are sufficiently smooth, for $S_1(t)$ and $S_2(t)$ we obtain the following results. From Definition 3

$$\|S_1\|_\infty = \max_{ij} \|\mathcal{I}_m f - f\|_{L^2_\omega},$$

in which $f(t) = \displaystyle\int_{x_l}^{t} \sum_{\upsilon=1}^{p} a_{i\upsilon}(x)e_{\upsilon j}(x)dx$. Using (5) for $k = 1$, it can be deduced that

$$\begin{aligned} \|S_1\|_\infty &= \tilde{C}hm^{-1} \max_{ij} \left\| \sum_{\upsilon=1}^{p} a_{i\upsilon}e_{\upsilon j} \right\|_{L^2_\omega} \\ &\leq \tilde{C}M_1 h^{\min(k,m)+1} m^{-1-k} \max_{j} \sum_{\nu=1}^{p} |y_{\nu j}|_{H^{k,2}_\omega(x_l,x_{l+1})} \end{aligned} \tag{18}$$

Also, for $S_2(t)$, we derive that

$$\begin{aligned} \|S_2\|_\infty &= \max_{ij} \|\mathcal{I}_m y_{ij} - y_{ij}\|_{L^2_\omega} \\ &\leq C_2 h^{min(k,m)} m^{-k} \max_{ij} |y_{ij}|_{H^{k,2}_\omega(x_l,x_{l+1})}. \end{aligned} \tag{19}$$

Now the assertion can be concluded form (17), (18) and (19). □

Table 1: Relative errors for Example 1 ($m = 4, 5$ and different values of h).

t	Method of [20] $h = 1/160$	*present method* $(m, h) = (4, 1/80)$	*present method* $(m, h) = (5, 1/10)$	*present method* $(m, h) = (5, 1/100)$
1	6.126×10^{-12}	4.293×10^{-12}	3.795×10^{-12}	1.834×10^{-15}
5	3.049×10^{-9}	2.533×10^{-10}	1.438×10^{-9}	2.065×10^{-13}
10	1.145×10^{-7}	6.165×10^{-9}	1.097×10^{-7}	3.233×10^{-13}

4 Numerical experiments

In this section, two numerical examples are examined to illustrate the efficiency of proposed method and the presented theoretical results. All of the numerical experiments are performed using Mathematica 7 with a machine unit round off precision of around 10^{-16}.

Example 1. Consider the following first-order linear matrix differential equation [20]

$$\begin{cases} Y'(t) = \begin{pmatrix} -\frac{19}{2}t - 12 & -14t - \frac{35}{2} \\ \frac{20}{3}t + \frac{25}{3} & \frac{59}{6}t + \frac{73}{6} \end{pmatrix} Y(t), \qquad 0 \le t \le 10, \\ Y(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{cases} \tag{20}$$

It can be verified that the exact solution of (20) is

$$Y(t) = \begin{pmatrix} 15y_1(t) - 14y_2(t) & 21y_1(t) - 21y_2(t) \\ 10y_2(t) - 10y_1(t) & 15y_2(t) - 14y_1(t) \end{pmatrix},$$

where

$$y_1(t) = e^{-\frac{t^2}{2} - \frac{t}{3}}, \qquad y_2(t) = e^{\frac{t^2}{4} + \frac{t}{2}}.$$

The comparison results between the method of [20] and present method are tabulated in Table 1. The numerical results for the first examples are also illustrated in Figure 1.

Example 2. Consider the following problem [10]

$$\begin{cases} Y'(t) = A(t)Y(t) + B(t), \\ Y(0) = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}, \qquad t \in [0, 1], \end{cases}$$

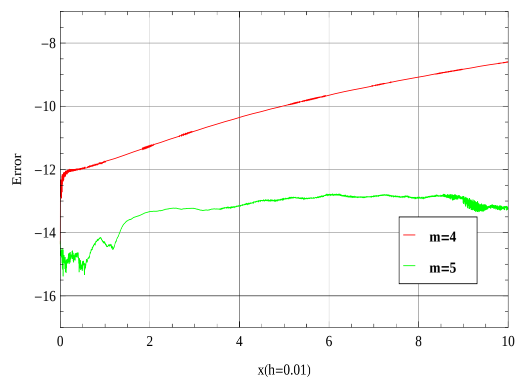

Figure 1: `Example` 1 `(`$m = 4, 5$ `and` $h = 0.01$`).`

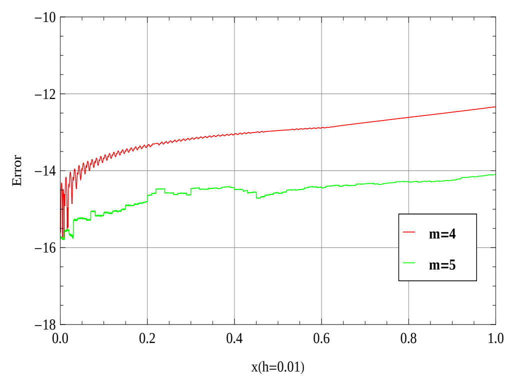

Figure 2: `Example` 2 `(` $m = 4, 5$ `and` $h = 0.01$`).`

where

$$A(t) = \begin{pmatrix} 1 & -1 \\ 1 & e^t \end{pmatrix}, \qquad B(t) = \begin{pmatrix} -3e^{-t} - 1 & 2 - 2e^{-t} \\ -3e^{-t} - 2 & 1 - 2\cosh(t) \end{pmatrix},$$

which has the exact solution

$$Y(x) = \begin{pmatrix} 2e^{-t} + 1 & e^{-t} - 1 \\ e^{-t} & 1 \end{pmatrix}.$$

The corresponding numerical results for the second examples with both the method proposed in [10] and present method are reported in Table 2. For more details, the numerical results for the second example are depicted in Figure 2.

Table 2: Absolute errors for Example 2 ($m = 4, 5$ and $h = 0.1$).

Interval	[0, 0.1]	[0.1,0.2]	[0.2,0.3]	[0.3,0.4]	[0.4,0.5]
present method ($m = 4$)	9.84×10^{-10}	1.31×10^{-9}	1.63×10^{-9}	1.92×10^{-9}	2.17×10^{-9}
spline Algorithm [10] ($m = 4$)	5.06×10^{-8}	1.02×10^{-7}	1.54×10^{-7}	2.10×10^{-7}	2.70×10^{-7}
present method ($m = 5$)	3.26×10^{-12}	2.99×10^{-12}	2.78×10^{-12}	2.53×10^{-12}	1.21×10^{-12}
spline Algorithm [10] ($m = 5$)	6.75×10^{-10}	1.36×10^{-9}	2.06×10^{-9}	2.80×10^{-9}	3.60×10^{-9}
Interval	[0.5,0.6]	[0.6,0.7]	[0.7,0.8]	[0.8,0.9]	[0.9,1]
present method ($m = 4$)	2.14×10^{-9}	3.53×10^{-9}	4.72×10^{-9}	5.61×10^{-9}	7.35×10^{-9}
spline Algorithm [10] ($m = 4$)	3.38×10^{-7}	4.19×10^{-7}	5.21×10^{-7}	6.59×10^{-7}	8.51×10^{-7}
present method ($m = 5$)	9.98×10^{-13}	2.36×10^{-12}	2.71×10^{-12}	2.83×10^{-12}	2.50×10^{-12}
spline Algorithm [10] ($m = 5$)	4.50×10^{-9}	5.57×10^{-9}	6.93×10^{-9}	8.75×10^{-9}	1.13×10^{-8}

5 Conclusion

The properties of the Bernoulli polynomials and their operational matrix of derivative have been utilized to numerically solve a class of the first-order matrix differential problems. The propounded approach reduces the main problem to a linear coupled matrix equations. An error estimation of offered method was presented. Numerical experiments have illustrated to demonstrate the efficiency and applicably of our offered approach. Finally, we point out that the proposed novel strategy can be examined for more complicated types of matrix differential models.

Acknowledgements

The authors would like to thank the anonymous referee for useful comments and suggestions.

References

[1] Z. A. A. Al-Zhour, *New techniques for solving some matrix and matrix differential equations*, Ain Shams Eng. J. **6** (2015) 347-354.

[2] S. U. Altinbasak and M. Demiralp, *Solutions to linear matrix ordinary differential equations via minimal, regular, and excessive space extension based universalization*, J. Math. Chem. **48** (2010) 266–286.

[3] F. P. A. Beik and D. K. Salkuyeh, *On the global Krylov subspace methods for solving general coupled matrix equation*, Comput. Math. Appl. **62** (2011) 4605–4613.

[4] F. P. A. Beik and D. K. Salkuyeh, *The coupled Sylvester-transpose matrix equations over generalized centrosymmetric matrices*, Int. J. Comput. Math. **90** (2013) 1546-1566.

[5] F. P. A. Beik, D. K. Salkuyeh and M. M. Moghadam, *Gradient-based iterative algorithm for solving the generalized coupled Sylvester-transpose and conjugate matrix equations over reflexive (anti-reflexive) matrices*, Trans. Inst. Measurement. Control. **36** (2014) 99-110.

[6] D. S. Bernstein, *Matrix Mathematics: theory, facts, and formulas*, Second edition, Princeton University Press, New Jersey, 2009.

[7] J. P. Boyd, *Chebyshev and Fourier Spectral Methods*, Second edition, Dover, New York, 2000.

[8] C. Canuto, M. Y. Hussaini, A. Quarteroni and T. A. Zang, *Spectral Methods in Single Domisns*, Springer-Verlag, 2006.

[9] J. P. Chehab, *Matrix differential equations and inverse preconditioners*, Comp. Appl. Math. **26** (2007) 95–128.

[10] E. Defez, A. Hervás, J. Ibáñez and M.M. Tung, *Numerical solutions of matrix differential models using higher-order matrix splines*, Mediterr. J. Math. **9** (2012) 865–882.

[11] F. Ding and T. Chen, *Iterative least-squares solutions of coupled Sylvester matrix equations*, Systems Control Lett. **54** (2005) 95-107.

[12] F. Ding and T. Chen, *On iterative solutions of general coupled matrix equations*, SIAM J. Control Optim. **44** (2006) 2269-2284.

[13] T. M. Flett, *Differential Analysis*, Cambridge University Press, 1980.

[14] A. Golbabai and S. Panjeh Ali Beik, *An efficient method based on operational matrices of Bernoulli polynomials for solving matrix differential equations*, Comp. Appl. Math. **34** (2015) 159–175.

[15] E. Kreyszig, *Introductory Functional Analysis with Applications*, John Wiley and Sons, Inc., 1978.

[16] S. Mashayekhi, Y. Ordokhani and M. Razzaghi, *Hybrid functions approach for optimal control of systems described by integro-differential equations*, Appl. Math. Model. **37** (2013) 3355-3368.

[17] Y. Saad, *Iterative Methods for Sparse linear Systems*, PWS press, New York, 1995.

[18] D. K. Salkuyeh and F. P. A. Beik, *On the gradient based algorithm for solving the general coupled matrix equations*, Trans. Inst. Measurement. Control. **36** (2014) 375–381.

[19] E. Tohidi, A.H. Bhrawy and K. Erfani, *A collocation method based on Bernoulli operational matrix for numerical solution of generalized pantograph equation*, Appl. Math. Model. **37** (2013) 4283-4294.

[20] H. Zheng and W. Han, *On some discretization methods for solving a linear matrix ordinary differential equation*, J. Math. Chem. **49** (2011) 1026–1041.