Journal of Mathematical Modeling
Vol. 3, No. 1, 2015, pp. 91-101

JMM

Determining the order of minimal realization of descriptor systems without use of the Weierstrass canonical form

Kamele Nassiri Pirbazari* and Mehdi Azari

Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran
Emails: k-nasiri@guilan.ac.ir, mr.mehdiazari@yahoo.com

Abstract. A common method to determine the order of minimal realization of a continuous linear time invariant descriptor system is to decompose it into slow and fast subsystems using the Weierstrass canonical form. The Weierstrass decomposition should be avoided because it is generally an ill-conditioned problem that requires many complex calculations especially for high-dimensional systems. The present study finds the order of minimal realization of a continuous linear time invariant descriptor system without use of the Weierstrass canonical form.

Keywords: Descriptor system, minimal realization, Weierstrass canonical form.
AMS Subject Classification: 93B10, 93B15, 93B20, 93B17.

1 Introduction

Consider the continuous linear time invariant descriptor system

$$\begin{cases} E\dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \tag{1}$$

where constant matrices $A,E\in R^{n\times n}, B \in R^{n\times m}, C \in R^{p\times n}, D \in R^{p\times m}$ are the coefficient matrices and $x(t) \in R^n$, $u(t) \in R^m$ and $y(t) \in R^p$ are the state, input and output vectors, respectively.

*Corresponding author.
Received: 15 October 2014 / Revised: 1 February 2015 / Accepted: 5 February 2015.

Matrix $\lambda E - A$ is called a matrix pencil. A matrix pencil can also be denoted by the matrix pair (E, A). It is assumed that system (1) is regular, i.e., there exists a $\lambda \in \mathbb{C}$ such that $\det(\lambda E - A) \neq 0$.

Matrix $G(s) = C(sE - A)^{-1}B + D$ is referred to as the transfer matrix of system (1). The dimension of state vector $x(t)$ is called the order of system. System (1) is C-controllable if rank$[\lambda E - A \ B] = n$, $\forall \lambda \in$ C and rank$[E \ B] = n$. System (1) is C-observable if rank $\begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = n$, $\forall \lambda \in$ C and rank $\begin{bmatrix} E \\ C \end{bmatrix} = n$. For the sake of brevity denote the sequel of system (1) as $[E,\ A,\ B,\ C,\ D]$.

Systems $[E, A, B, C, D]$ and $[\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}]$ are called equivalent if their orders and number of inputs and outputs are equal and there exist two nonsingular matrices P and Q such that

$$\tilde{E} = PEQ, \quad \tilde{A} = PAQ, \quad \tilde{B} = PB, \quad \tilde{C} = CQ, \quad \tilde{D} = D.$$

Further details about descriptor systems can be found in [2] and [4].

2 Determining the order of minimal realization of descriptor systems using Weierstrass canonical form

Since discernment of the proposed method for finding the minimal realization order requires knowledge about the common method to calculate the realization using Weierstrass canonical form, this section briefly explains the necessary related concepts.

Theorem 1. *If system $[E, A, B, C, D]$ is regular with an order of n, it is equivalent to system $[\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}]$, where*

$$\tilde{E} = \begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix}, \qquad \tilde{A} = \begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix},$$

$n_f + n_\infty = n$, $J \in R^{n_f \times n_f}$, $N \in R^{n_\infty \times n_\infty}$. Submatrices J and N are in Jordan canonical form and matrix N is a nilpotent matrix with the nilpotence index $\upsilon \geq 1$.

Proof. Since matrix pencil $(E,\ A)$ is regular, there exists a scalar $\lambda \in$ C such that $\det(\lambda E - A) \neq 0$. Define

$$\overline{E} = (\lambda E - A)^{-1}E, \qquad \overline{A} = (\lambda E - A)^{-1}A.$$

It is easy to obtain

$$\overline{A} = (\lambda E - A)^{-1}(\lambda E + A - \lambda E) = \lambda(\lambda E - A)^{-1}E - I_n = \lambda\bar{E} - I_n.$$

System (1) can be written as follows:

$$\begin{cases} (\lambda E - A)^{-1}E\dot{x}(t) = (\lambda E - A)^{-1}Ax(t) + (\lambda E - A)^{-1}Bu(t), \\ y(t) = Cx(t) + Du(t). \end{cases}$$

Assuming $(\lambda E - A)^{-1}B = \bar{B}$, it follows that

$$\begin{cases} \bar{E}\dot{x}(t) = \bar{A}x(t) + \bar{B}u(t), \\ y(t) = Cx(t) + Du(t). \end{cases} \tag{2}$$

There is a nonsingular matrix, such as M, which transforms $\bar{E}$ into Jordan canonical form

$$M^{-1}\bar{E}M = \begin{bmatrix} \bar{E}_1 & 0 \\ 0 & \bar{E}_2 \end{bmatrix}, \tag{3}$$

where $\bar{E}_1 \in R^{n_f \times n_f}$ is nonsingular and $\bar{E}_2 \in R^{n_\infty \times n_\infty}$ is nilpotent and therefore $\lambda\bar{E}_2 - I_{n_\infty}$ is nonsingular. In addition

$$\begin{aligned} M^{-1}\bar{A}M &= M^{-1}(\lambda\bar{E} - I_n)M \\ &= \lambda M^{-1}\bar{E}M - I_n = \begin{bmatrix} \lambda\bar{E}_1 - I_{n_f} & 0 \\ 0 & \lambda\bar{E}_2 - I_{n_\infty} \end{bmatrix}. \end{aligned} \tag{4}$$

Assuming $M^{-1}x(t) = z(t)$, system (2) can be written as follows

$$\begin{cases} M^{-1}\bar{E}M\dot{z}(t) = M^{-1}\bar{A}Mz(t) + M^{-1}\bar{B}u(t), \\ y(t) = CMz(t) + Du(t). \end{cases}$$

Inserting equations (3) and (4) into the previous system, produces

$$\begin{cases} \begin{bmatrix} \bar{E}_1 & 0 \\ 0 & \bar{E}_2 \end{bmatrix}\dot{z}(t) = \begin{bmatrix} \lambda\bar{E}_1 - I_{n_f} & 0 \\ 0 & \lambda\bar{E}_2 - I_{n_\infty} \end{bmatrix} z(t) + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} u(t), \\ y(t) = [\bar{C}_1 \ \ \bar{C}_2]z(t) + Du(t), \end{cases} \tag{5}$$

where $M^{-1}\bar{B} = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix}$ and $CM = [\bar{C}_1 \ \ \bar{C}_2]$. System (5) can be written as follows

$$\begin{cases} \begin{bmatrix} I_{n_f} & 0 \\ 0 & N_1 \end{bmatrix}\dot{z}(t) = \begin{bmatrix} J_1 & 0 \\ 0 & I_{n_\infty} \end{bmatrix} z(t) + \begin{bmatrix} (\bar{E}_1)^{-1}\bar{B}_1 \\ (\lambda\bar{E}_2 - I_{n_\infty})^{-1}\bar{B}_2 \end{bmatrix} u(t), \\ y(t) = [\bar{C}_1 \ \ \bar{C}_2]\, z(t) + Du(t), \end{cases} \tag{6}$$

where

$$\begin{aligned} J_1 &= \left(\bar{E}_1\right)^{-1}\left(\lambda\bar{E}_1 - I_{n_f}\right) = \lambda I_{n_f} - \left(\bar{E}_1\right)^{-1}, \\ N_1 &= \left(\lambda\bar{E}_2 - I_{n_\infty}\right)^{-1}\bar{E}_2 \quad . \end{aligned}$$

If the index of nilpotence of $\bar{E}_2$ is $\upsilon \geq 1$, it follows from

$$\left(\lambda\bar{E}_2 - I_{n_\infty}\right)^{-1}\bar{E}_2 = \bar{E}_2\left(\lambda\bar{E}_2 - I_{n_\infty}\right)^{-1},$$

that N_1 is a nilpotent with nilpotence index $\upsilon \geq 1$. Matrices N_1 and J_1 in system (6) are not in Jordan canonical form, however, there exists two nonsingular matrices $G \in R^{n_f \times n_f}$ and $R \in R^{n_\infty \times n_\infty}$ such that

$$G^{-1}J_1G = J, \qquad R^{-1}N_1R = N,$$

and matrices J and N are in Jordan canonical form. In addition matrix N is nilpotent.

Define $S=\begin{bmatrix} G & 0 \\ 0 & R \end{bmatrix}$ and assume that $S^{-1}z(t) = w(t)$. System (6) can be written as:

$$\left\{ \begin{array}{l} \begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix} \dot{w}\left(t\right) = \begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix} w\left(t\right) + \begin{bmatrix} G^{-1}\left(\bar{E}_1\right)^{-1}\bar{B}_1 \\ R^{-1}\left(\lambda\bar{E}_2 - I_{n_\infty}\right)^{-1}\bar{B}_2 \end{bmatrix} u\left(t\right), \\ y\left(t\right) = \left[\bar{C}_1G \quad \bar{C}_2R\right] w\left(t\right) + Du\left(t\right). \end{array} \right. \tag{7}$$

Suppose that

$$\begin{aligned} G^{-1}\left(\bar{E}_1\right)^{-1}\bar{B}_1 &= B_1, \quad R^{-1}\left(\lambda\bar{E}_2 - I_{n_\infty}\right)^{-1}\bar{B}_2 = B_2, \\ \bar{C}_1G &= C_1, \qquad\qquad\qquad\qquad \bar{C}_2R = C_2. \end{aligned}$$

So

$$\left\{ \begin{array}{l} \begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix} \dot{w}\left(t\right) = \begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix} w\left(t\right) + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u\left(t\right), \\ y\left(t\right) = \left[C_1 \quad C_2\right] w\left(t\right) + Du\left(t\right). \end{array} \right.$$

By defining

$$\begin{aligned} P &= \mathrm{diag}[G^{-1}\left(\bar{E}_1\right)^{-1}, R^{-1}\left(\lambda\bar{E}_2 - I_{n_\infty}\right)^{-1}]M^{-1}(\lambda E - A)^{-1}, \\ Q &= MS. \end{aligned}$$

Then

$$\tilde{E} = PEQ= \begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix}, \quad \tilde{A} = PAQ = \begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix},$$

$$\tilde{B} = PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad \tilde{C} = CQ = [\mathrm{C}_1 \quad \mathrm{C}_2] \ , \quad \tilde{D}{=}D \ .$$

As a result, the systems are equivalent and the conditions of the theorem are established. □

The basis of the proof for Theorem 1 is based on Duan [4].

Note 1. Representation

$$\lambda PEQ - PAQ = \lambda \begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix},$$

of pencil λE–A is referred to as the Weierstrass canonical form of λE–A.

Note 2. Using Theorem 1, system (1) can be decomposed into subsystems:

$$\begin{cases} \dot{x}_1(t) = Jx_1(t) + B_1u(t), \\ \mathrm{y}\,(\mathrm{t}) = C_1x_1(t)\,, \end{cases} \quad \text{and} \quad \begin{cases} N\dot{x}_2(t) = x_2(t) + B_2u(t)\,, \\ \mathrm{y}\,(\mathrm{t}) = C_2x_2(t) + Du(t), \end{cases}$$

which are called slow and fast subsystems, respectively. The transfer matrices of the slow and fast subsystems are $G_1(s) = C_1(sI - J)^{-1}B_1$ and $G_2(s) = C_2(sN - I)^{-1}B_2 + D$, respectively.

Notation 1. The slow system is shown with triple $[J,\ B_1,\ C_1]$.

Definition 1. $[E,\ A,\ B,\ C,\ D]$ is the realization of transfer matrix *G(s)* if $G(s) = C(sE - A)^{-1}B + D$. This realization is minimal, when it has the smallest possible order.

Definition 2. The realization $[E,\ A,\ B,\ C,\ 0]$ of transfer matrix *G(s)* is conditionally minimal if it has the smallest possible order among all realizations of form $G(s) = C(sE - A)^{-1}B$.

Theorem 2. *The realization [E, A, B, C, 0] of transfer matrix G(s) is conditionally minimal if and only if [E, A, B, C, 0] is C-controllable and C-observable.*

Proof. See [2] and [4]. □

Definition 3. The realization $[E,\ A,\ B,\ C,\ D]$ of transfer matrix *G(s)* is called a deflated minimal realization if

1. System $[E,\ A,\ B,\ C,\ D]$ is C-controllable and C-observable.
2. Nilpotent matrix N in the Weierstrass canonical form of pencil $sE - A$ does not contain nilpotent Jordan blocks of index one.

Theorem 3. *A minimal realization of given rational matrix G(s) has the same order as a deflated minimal realization of G(s).*

Proof. See [6]. □

Note 3. Observe that, for every rational matrix *G(s)*, there is representation $G(s) = G_1(s) + G_2(s)$ where $G_1(s)$ is a strictly proper rational matrix and $G_2(s)$ is a polynomial matrix. Suppose that: $G_1(s) = C_1(sI - J)^{-1}B_1$ and $G_2(s) = C_2(sN - I)^{-1}B_2 + D$ are minimal realizations of $G_1(s)$ and $G_2(s)$, respectively. Then

$$G(s) = [C_1 \quad C_2] \left[s \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right]^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + D,$$

is a minimal realization of *G(s)*.

Theorem 4. *If $G_2(s)$ is a polynomial matrix, the order of its deflated minimal realization will be equal to the order of minimal realization of $G_2(s)$.*

Proof. See [6]. □

Theorem 5. *If $G_2(s) = C_2(sN - I)^{-1}B_2 + D$ is the deflated minimal realization of polynomial matrix $G_2(s)$, its order will equal:*

$$n_1^* = \sum_{i=2}^{m} i d_i \quad ,$$

where d_i equals the number of Jordan blocks N of size of i and m is the largest order of Jordan block in N.

Proof. See [6]. □

3 Determining order of minimal realization of polynomial matrix $\mathbf{G_2(s) = C_2(s}N\mathbf{-I)^{-1}B_2 + D}$ without use of Weierstrass canonical form

Lemma 1. *The matrix R defined in Theorem 1 can be chosen such that $N = \bar{E}_2$.*

Proof. Assume that $\bar{E}_2 = \mathrm{diag}[J_{\alpha_1}, J_{\alpha_2}, \ldots, J_{\alpha_k}]$ is the nilepotent matrix of size $n_\infty \times n_\infty$, in which submatrices $J_{\alpha_i} (i = 1, 2, \ldots, k)$ are the Jordan block, nilpotent matrices of dimension α_i. Moreover $\sum_{i=1}^{k} \alpha_i = n_\infty$ and

$$J_{\alpha_i} = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}, \quad i = 1, 2, \ldots, k.$$

Using the block diagonal form of matrix $\bar{E}_2$, produces

$$\begin{aligned} N_1 &= (\lambda \bar{E}_2 - I_{n_\infty})^{-1} \bar{E}_2 \\ &= \mathrm{diag}[(\lambda J_{\alpha_1} - I_{\alpha_1})^{-1} J_{\alpha_1}, (\lambda J_{\alpha_2} - I_{\alpha_2})^{-1} J_{\alpha_2}, \ldots, (\lambda J_{\alpha_k} - I_{\alpha_k})^{-1} J_{\alpha_k}]. \end{aligned}$$

Since matrix J_{α_i} is upper triangular, for each $i = 1, 2, \ldots, k$, we have

$$\begin{aligned} (\lambda J_{\alpha_i} - I_{\alpha_i}) &= \begin{bmatrix} -1 & \lambda & & \\ & -1 & \ddots & \\ & & \ddots & \lambda \\ & & & -1 \end{bmatrix} \\ \Longrightarrow (\lambda J_{\alpha_i} - I_{\alpha_i})^{-1} &= \begin{bmatrix} -1 & -\lambda & -\lambda^2 & \cdots & -\lambda^{\alpha_i - 1} \\ & -1 & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & -\lambda^2 \\ & & & \ddots & -\lambda \\ & & & & -1 \end{bmatrix} \\ \Longrightarrow (\lambda J_{\alpha_i} - I_{\alpha_i})^{-1} J_{\alpha_i} &= \begin{bmatrix} 0 & -1 & -\lambda & -\lambda^2 & \cdots & -\lambda^{\alpha_i - 2} \\ & 0 & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \ddots & -\lambda^2 \\ & & & \ddots & \ddots & -\lambda \\ & & & & \ddots & -1 \\ & & & & & 0 \end{bmatrix}. \end{aligned}$$

Now, by choosing R appropriately, the above matrix can be transformed into the Jordan canonical form

$$J_{\alpha_i} = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix},$$

using elementary row operations. This means that the Jordan canonical form of each block of matrix N_1 is equal to its corresponding block in matrix $\bar{E}_2$. Matrix N_1 is, thus, transformed into Jordan canonical form which is represented by N. In fact, each block of matrix N is equal to its corresponding block in $\bar{E}_2$; Therefore, $N = \bar{E}_2$. □

Lemma 2. *If nilpotent matrix $\bar{E}_2$ does not contain nilpotent Jordan blocks of index one, then the order of minimal realization $G_2(s) = C_2(sN - I)^{-1}B_2 + D$ equals:*

$$n_1^* = \sum_{i=2}^{m} ie_i \quad ,$$

where e_i equals the number of Jordan blocks $\bar{E}_2$ of size of i and m is the largest order of Jordan blocks in $\bar{E}_2$.

Proof. This immediately follows from Theorem 5 and Lemma 1. □

Therefore, to calculate the order of minimal realization of a fast subsystem, write the Jordan form $\bar{E}_2$ and calculate n_1^*.

4 Determining order of minimal realization of strictly proper transfer matrix $\mathbf{G_1(s) = C_1(s}I\mathbf{-J)^{-1}B_1}$ without use of Weierstrass canonical form

Since subsystem $[J_1, (\bar{E}_1)^{-1}\bar{B}_1, \bar{C}_1]$ of system (6) is equivalent to subsystem $[J, B_1, C_1]$ of system (7), the order of their minimal realizations is the same. Next consider subsystem $[J_1, (\bar{E}_1)^{-1}\bar{B}_1, \bar{C}_1]$. Matrix J_1 can be directly obtained using $\bar{E}_1$ without using the Weierstrass canonical form or by calculating matrices P and Q. In fact, $J_1 = \lambda I_{n_f} - (\bar{E}_1)^{-1}$. Also $(\bar{E}_1)^{-1}\bar{B}_1$ can be calculated with Jordan decomposition $\tilde{E}$ and $\bar{B}_1$; therefore, the order of minimal realization of $[J_1, (\bar{E}_1)^{-1}\bar{B}_1, \bar{C}_1]$ can be obtained using common methods ([1], [3] and [5]). If the order of minimal realization of this subsystem is n_2^*, the order of minimal realization of system (1) is equal to $n = n_1^* + n_2^*$.

Note 4. Taking into account both the controllability and observability of a control system leads to the concept of minimal realization. This important characteristic forms a system with the lowest possible order and it is important to study and understand the control system. This is more important when high dimensional systems are involved. Such systems allow the study of a minimal system with lower dimensions than a main system. Knowing and obtaining the minimal order for the system is important for various reasons, one of them is that the main step to obtain minimal realization of a system in existing algorithms is to know the order of the minimal system ([2] and [4]). This knowledge makes each realization of minimal order, controllable and observable and eliminates the need for determining controllability and observability of the system.

5 Algorithm to determine order of minimal realization of descriptor systems without use of Weierstrass canonical form

For a given C-controllable and C-observable system $[E, A, B, C, D]$:

Step 1. *Select* $\lambda \in \mathrm{C}$ *such that* $\det(\lambda E - A) \neq 0$.

Step 2. *Compute matrices* $\bar{E} = (\lambda E - A)^{-1}E$ *and* $\bar{B} = (\lambda E - A)^{-1}B$.

Step 3. *Find matrices* $M \in R^{n\times n}$, $\bar{E}_1$ *and* $\bar{E}_2$, *such that equation (3) is satisfied, where* $\bar{E}_1 \in R^{n_f\times n_f}$ *is nonsingular and* $\bar{E}_2 \in R^{n_\infty\times n_\infty}$ *is nilpotent.*

Step 4. *Compute matrices* $M^{-1}\bar{B} = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix}$ *and* $CM = [\bar{C}_1 \quad \bar{C}_2]$, *in which* $\bar{B}_1 \in R^{n_f\times m}$, $\bar{B}_2 \in R^{n_\infty\times m}$, $\bar{C}_1 \in R^{p\times n_f}$ *and* $\bar{C}_2 \in R^{p\times n_\infty}$.

Step 5. *Compute the order of minimal realization for slow subsystem* $[J_1, (\bar{E}_1)^{-1}\bar{B}_1, \bar{C}_1]$ *using common methods.*

Step 6. *Compute the order of minimal realization of the fast subsystem by removing blocks with nilpotence indices equal to 1 in nilpotent matrix* $\bar{E}_2$.

Step 7. *Add orders of minimal realization from the slow and fast systems, from steps 5 and 6 to be the order of minimal realization of descriptor system* $[E, A, B, C, D]$.

Example 1. Consider system $[E, A, B, C, 0]$ such that

$$E = \begin{bmatrix} E_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = [C_1 \ \ C_2],$$

where

$$E_1 = \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1&0&0&0&-1&0&0&0 \\ 0&1&0&0&0&-1&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&-1&0&0&0 \\ 0&0&0&0&0&-1&0&0 \\ 0&0&0&0&3&0&1&0 \\ 0&0&0&0&0&2&0&1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & -2 \\ -2 & 3 \\ 0 & 0 \\ 3 & -3 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

The order of minimal of this system is 11. A program written in MATLAB was used to obtain the order of minimal of this system. The program implements both the Weierstrass canonical form and Jordan canonical form for $\lambda = 0$. The first step used the common Weierstrass method, to execute all stages mentioned in the proof of Theorem 1 to find transformation matrices, P and Q. Using these transformation matrices, the main system was decomposed into descriptor and standard subsystems. The blocks of dimension 1 in nilpotent matrix N, resulting from decomposition of the main system were removed to obtain the order of the minimal descriptor subsystem. Next, the order of the standard subsystem was obtained using matrix J from the decomposition. The order of the main system equals the summation of these two orders. The time consumed by the common Weierstrass canonical form was 2.720401 sec. The Jordan canonical method, does not need to perform all stages of the proof of Theorem 1; it is only necessary to continue up to equation (3). Matrices $\bar{E}_1$ and $\bar{E}_2$ were used to compute the orders of the descriptor subsystem and the standard subsystem respectively. The order of the main system was obtained as the summation of these two values. The time consumed by this method (without using the Weierstrass canonical form) was 1.676645 sec. The lack of a need to employ transformation matrices which eliminates the need to compute their inverses, and the decrease in the amount of calculation are advantages of the proposed method.

6 Conclusion

This study found the order of minimal realization of transfer matrix $\mathrm{G}(s) = \mathrm{C}(sE - A)^{-1}B + D$ without the use of the Weierstrass canonical form and transform matrices P and Q. The use of the Jordan canonical form, eliminated the use of the Weierstrass canonical form, which decreased the size of the calculations and the time required to perform the calculations.

Acknowledgements

We are grateful to the anonymous referee for valuable recommendations and comments which have improved the quality of the paper.

References

[1] C. T. Chen, *Linear systems theory and design*, 3rd Edition, Oxford University Press, 1999.

[2] L. Dai, *Singular control systems*, volume 118 of Lecture notes in control and information sciences. Springer-Verlag, Berlin, Heidelberg, 1989.

[3] B. N. Datta, *Numerical methods for linear control systems*, Elsevier Science & Technology Books, 2003.

[4] G. R. Duan, *Analysis and design of descriptor linear systems*, Springer, 2010.

[5] B. D. Schutter, *Minimal state-space realization in linear system theory: an overview*, J. Comput. Appl. Math. **121** (2000) 331-354.

[6] V. I. Sokolov, *Contributions to the minimal realization problem for descriptor systems*, Ph.D. thesis, Chemnitz, 2006.