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SDO relaxation approach to fractional quadratic minimization with one quadratic constraint

Maziar Salahi[*] and Arezo Zare

Faculty of Mathematical Sciences, University of Guilan,Rasht, Iran
Emails: salahim@guilan.ac.ir, ze.arezou@gmail.com

Abstract. In this paper, we study the problem of minimizing the ratio of two quadratic functions subject to a quadratic constraint. First we introduce a parametric equivalent of the problem. Then a bisection and a generalized Newton-based method algorithms are presented to solve it. In order to solve the quadratically constrained quadratic minimization problem within both algorithms, a semidefinite optimization relaxation approach is presented. Finally, two set of examples are presented to compare the performance of algorithms.

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1 Introduction

Quadratically constrained quadratic fractional optimization problems arise in wide-range applications including signal processing, communications, financial analysis, location theory, portfolio selection problem, stochastic decision making problems [11,13,17,18]. These class of problems are in general nonconvex and hard to solve. However several among them are successfully solved using semidefinite optimization (SDO) relaxation [1,3,15,16,19]. In

[*]Corresponding author.
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2006, Beck et al. [2] solved a class of fractional quadratic problem subject to a quadratic constraint with application to the regularized total least squares problem. Moreover in 2010, Beck and Teboulle [4], considered the nonconvex problem minimizing the ratio of two quadratic functions over finitely many nonconvex quadratic inequalities. Using the homogenization technique, they established a sufficient condition that warrantees the attainment of an optimal solution. In [22], Zhang and Hayashi focused on fractional optimization problems that minimize the ratio of two indefinite quadratic functions subject to two quadratic constraints. Using the relationship between fractional and parametric optimization problems, they transformed the original problem into a univariate nonlinear equation and proposed a bisection method and a generalized Newton algorithm. Within both algorithms, they need to solve a problem of minimizing a nonconvex quadratic function subject to two quadratic constraints, which is commonly called a Celis-Dennis-Tapia (CDT) subproblem.

In this paper, we consider the problem analogous to the one in [22] but with one constraint as follows:

$$\begin{aligned} \min \qquad & \frac{f_1(x)}{f_2(x)} \qquad (1) \\ \text{subject to} \qquad & x^T Bx - 2d^T x + e \leq 0, \end{aligned}$$

where

$$f_i = x^T A_i x - 2b_i^T x + c_i, \qquad i = 1, 2, \qquad (2)$$

$A_1, A_2, B \in \mathbb{R}^{n\times n}$ are symmetric matrices, $b_1, b_2, d \in \mathbb{R}^n$ and $c_1, c_2, e \in \mathbb{R}$. Furthermore, we require that the denominator of the objective function is away from zero in the feasible region. For simplicity we denote the feasible region of the problem by $\mathcal{F}$.

In Section 2, we represent the fractional problem as a parametric optimization problem. Then the bisection and generalized Newton methods are discussed to solve the parametric problem. In Section 3, we give an SDO relaxation approach to solve the quadratically constrained quadratic problem within both algorithms. Finally in Section 4, we give some numerical results to examine the efficiency of both algorithms.

2 Parametric approach

The following proposition gives the relationship between fractional and parametric problems [8, 22].

Proposition 1. *The following two statements are equivalent:*

1. $\min_{x\in\mathcal{F}} \dfrac{f_1(x)}{f_2(x)} = \alpha.$
2. $$F(\alpha) := \min_{x\in\mathcal{F}}\{f_1(x) - \alpha f_2(x)\} = 0. \tag{3}$$

Using this proposition, finding the minimum of (1) is equivalent to a root-finding problem. Now, we give some properties of the univariate function F.

Theorem 1. *The following statements hold.*
(a) *F is concave over $\mathbb{R}$.*
(b) *F is continuous at any $\alpha \in \mathbb{R}$.*
(c) *F is strictly decreasing.*
(d) *$F(\alpha) = 0$ has a unique solution.*

Proof. See [22] □

In what follows, we give the bisection-based method algorithm to find the root of F [22].

Algorithm 1: Bisection Algorithm

1: Choose l_0 and u_0 that $l_0 \leq \min_{x\in\mathcal{F}} \dfrac{f_1(x)}{f_2(x)} \leq u_0$ holds. Set $k := 1$.

2: Let

$$\alpha_k := \tfrac{l_{k-1}+u_{k-1}}{2},$$

then calculate $F(\alpha_k)$ by solving the following minimization problem:

$$\min_{x\in\mathcal{F}} \quad f_1(x) - \alpha_k f_2(x) \tag{4}$$

3: If $\mid F(\alpha_k) \mid \leq \epsilon$, then terminate. Otherwise, update l_k and u_k as follows:
If $F(\alpha_k) \leq 0$,

$$l_k := l_{k-1}, \qquad u_k := \alpha_k.$$

If $F(\alpha_k) > 0$,

$$l_k := \alpha_k, \qquad u_k := u_{k-1}.$$

Let $k = k + 1$. Return to Step 1.

As we see, the main step of the algorithm is Step 2 which requires solving an indefinite quadratic optimization problem. Beside this main difficulty, it is known that bisection method has slow convergence rate, thus in the sequel a generalized Newton-based method is presented to solve the problem.

The function F is is not differentiable, however, there exists an explicit expression of its subgradient [22].

Theorem 2. *For any $\alpha \in R$, let x_α be $x_\alpha \in \operatorname{argmin}_{x\in\mathcal{F}}\{-f_1(x) + \alpha f_2(x)\}$. Then, a subgradient of F at α is given by $f_2(x_\alpha)$, i.e.,*

$$f_2(x_\alpha) \in \partial F(\alpha),$$

where ∂F denotes the clarke subdifferential of F.

Proof. See [22]. □

Algorithm 2: Generalized Newton Algorithm

1: Choose $\alpha_1 \in \mathbb{R}$. Set $k := 1$.

2: Solve the following minimizing problem to obtain a global optimum x_k and its optimal value $F(\alpha_k)$:

$$\min_{x\in\mathcal{F}} \quad f_1(x) - \alpha_k f_2(x) \tag{5}$$

3: If $\mid F(\alpha_k) \mid \leq \epsilon$, then terminate. Otherwise, Let:

$$\alpha_k := \frac{f_1(x)}{f_2(x)}$$

Let $k = k + 1$ and return to Step 1.

As we see, in both algorithms one requires to solve the following nonconvex quadratically constrained quadratic minimization problem:

$$\min_{x\in\mathcal{F}} \quad x^T Ax - 2b^T x + c \tag{6}$$

where

$$A = A_1 - \alpha A_2,$$

$$b = b_1 - \alpha b_2,$$

$$c = c_1 - \alpha c_2,$$

In general this subproblem is nonconvex, thus classical algorithms may not lead to a global solution. In the next section, an SDO relaxation approach is used to solve (6) globally.

3 SDO relaxation approach

Problem (6) except the constant c in the homogenized form is given by

$$\min_{x\in\mathbb{R}^n,\ t\in\mathbb{R}} \quad x^T Ax - 2tb^T x \tag{7}$$

$$\text{subject to} \quad x^T Bx - 2td^T x + t^2 e \leq 0,$$

$$t^2 = 1.$$

Obviously if (x, t) solves this problem, then x/t is an optimal solution of (1). Furthermore, (7) is equivalent to

$$\begin{array}{rl} \min & M_0 \bullet \hat{X} \\ \text{subject to} & M_1 \bullet \hat{X} \leq 0, \\ & M_2 \bullet \hat{X} = 1, \end{array} \tag{8}$$

where $A \bullet B = Trace(A^T B)$ and

$$\hat{X} = \begin{pmatrix} t^2 & tx^T \\ tx & xx^T \end{pmatrix}, \qquad M_0 = \begin{pmatrix} 0 & -b^T \\ -b & A \end{pmatrix},$$

$$M_1 = \begin{pmatrix} e & -d^T \\ -d & B \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & 0_{n\times n} \end{pmatrix}.$$

The SDO relaxation of (8) is

$$\begin{array}{rl} \min\limits_{X\in S^n} & M_0 \bullet X \\ \text{subject to} & M_1 \bullet X \leq 0, \\ & M_2 \bullet X = 1, \\ & X \succeq 0_{(n+1)\times(n+1)} \end{array} \tag{9}$$

where

$$X = \begin{pmatrix} 1 & x_0^T \\ x_0 & \bar{X} \end{pmatrix}.$$

The dual of (9) is given by

$$\begin{aligned} \max \quad & y_2 \\ \text{subject to} \quad & y_1 M_1 + y_2 M_2 \preceq M_0 \\ & y_1 \leq 0, \quad y_2 \text{ free} \end{aligned} \tag{10}$$

or

$$\begin{aligned} \max \quad & y_2 \\ \text{subject to} \quad & Z = M_0 - y_1 M_1 - y_2 M_2 \\ & Z \succeq 0_{(n+1)\times(n+1)}, \quad y_1 \leq 0, \quad y_2 \text{ free.} \end{aligned} \tag{11}$$

Theorem 3. *Suppose (1) has a strictly feasible solution x_0 and $\lambda_1 A + \lambda_2 B \succ 0$ for some $\lambda_1, \lambda_2 \geq 0$. Then, both problems (9) and (11) also satisfy the Slater regularity conditions. Hence, both problems attain their optimal value and the duality gap is zero.*

Proof. Let X be as follows:

$$X = \begin{pmatrix} 1 & x_0^T \\ x_0 & \bar{X} \end{pmatrix},$$

where $\bar{X} = x_0 x_0^T + Q$ and $Q = \mathrm{diag}(q_1, ..., q_n)$ with all $q_i > 0$. Obviously by the Schur complement theorem, X is positive definite. Moreover,

$$\begin{aligned} M_1 \bullet X < 0 \Longleftrightarrow M_1 \bullet X &= \begin{pmatrix} e & -d^T \\ -d & B \end{pmatrix} \bullet \begin{pmatrix} 1 & x_0^T \\ x_0 & x_0 x_0^T + Q \end{pmatrix} \\ &= e - d^T x_0 - d^T x_0 + x_0^T B x_0 + \sum_{i=1}^{n} (B)_{ii} q_i < 0. \end{aligned}$$

Now since x_0 is strictly feasible for (1), then by choosing q_i's sufficiently small, the inequality holds. Also

$$M_2 \bullet X = \begin{pmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & 0_{n\times n} \end{pmatrix} \bullet \begin{pmatrix} 1 & x_0^T \\ x_0 & x_0 x_0^T + Q \end{pmatrix} = 1.$$

Therefore, X is a strictly feasible solution for the primal problem. Now for the dual problem we have

$$\begin{aligned} Z &= M_0 - y_1 M_1 - y_2 M_2 \\ &= \begin{pmatrix} 0 & -b^T \\ -b & A \end{pmatrix} - y_1 \begin{pmatrix} e & -d^T \\ -d & B \end{pmatrix} - y_2 \begin{pmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & 0_{n\times n} \end{pmatrix} \\ &= \begin{pmatrix} -(ey_1+y_2) & -b^T + y_1 d^T \\ -b + y_1 d & A - y_1 B \end{pmatrix}. \end{aligned}$$

By the Schur complement theorem

$$Z \succ 0 \Longleftrightarrow A - y_1 B + \frac{1}{ey_1+y_2}(-b+y_1 d)(-b+y_1 d)^T \succ 0. \tag{12}$$

Since $(-b+y_1d)(-b+y_1d)^T$ is positive semidefinite, then by choosing y_2 sufficiently large, the term $\frac{1}{ey_1+y_2}(-b+y_1d)(-b+y_1d)^T$ will be always positive semidefinite. Moreover, since $\lambda_1 A + \lambda_2 B \succ 0$, then $A - y_1 B \succ 0$ holds with appropriately chosen y_1, which implies the Slater regularity of (11). □

Lemma 1. *Let X be a positive semidefinite matrix with rank r and G be an arbitrary symmetric matrix such that $G \bullet X \le 0$. Then there exists a rank−one decomposition for X such that $X = \sum_{i=1}^r x_i x_i^T$ and $x_i^T G x_i \le 0$ for all $i = 1, 2, \dots, r$. If, in particular, if $G \bullet X = 0$, then $x_i^T G x_i = 0$ for all $i = 1, 2, \dots, r$.*

Proof. See [9]. □

Now let us assume that X^* with rank r and (y_1^*, y_2^*, Z^*) are optimal solutions for (9) and (11), respectively. Moreover let $x_i^* = \begin{bmatrix} t_i^* \\ \bar{x_i^*} \end{bmatrix}$, then by the second constraint of (9), for at least one $k = 1, \dots, r$, we have $t_k^* \neq 0$. Now let us assume $y_k^* = [1, (\bar{y_k^*})^T]^T$ with $\bar{y_k^*} = \frac{\bar{x_k^*}}{t_k^*}$ and $Y^* = y_k^*(y_k^*)^T$. Then we have

$$M_1 \bullet Y^* \le 0.$$

Similarly one has

$$M_2 \bullet Y^* = 1, \; Z^* \bullet Y^* = 0.$$

Therefore Y^*, (y^*, Z^*) are optimal solution for (9) and (11), respectively. Now since (9) is equivalent to (7), then Y^* is optimal for (7) as well. This further implies $\bar{y_k^*}$ is optimal for (1).

4 Numerical Results

In this section, we consider two sets of examples to compare the two previous algorithms for dimensions 50 to 550 for different densities. In order to solve the subproblem within both algorithms, we use "fmincon" command of Matlab and the SDO relaxation approach of the previous section. For both algorithms, ϵ is set to be 10^{-6} . Results are summarized in Tables 1 to 3. For each dimension, we generate 5 test problems and report the average CPU time and roots. All computations are performed on MATLAB 8.1 using a PC computer with Intel(R) Core Duo CPU 2.40 GHz and 8.00 GB of RAM. In all tables "———" means the algorithm is not able to solve the problem.

Example 1. Consider the following problem

$$\begin{array}{ll} \displaystyle\min_{x\in\mathbb{R}^n} & \dfrac{x^TA_1x - 2b_1^Tx + c_1}{\| x \|^2 + 1}, \\ \text{subject to} & x^TA_2x - 2b_2^Tx + c_2 \leq 0, \end{array}$$

where A_1, A_2 are two symmetric matrices, A_2 is positive semidefinite, $b_2, b_1 \in \mathbb{R}^n$, $c_2, c_1 \in \mathbb{R}$. As wee see in Table 1, both algorithms solve all problems and the one using "fmincon" command is faster than the SDO relaxation approach on most problems. However, from Table 2 we observe that when algorithm is using "fmincon", it fails to solve some problems and for densities 1 to 0.1, among the problems which can be solved by both algorithms, SDO-based algorithm is faster on most problems, while the "fmincon"-based algorithm is better for sparse problems.

Example 2. Consider the following problem

$$\begin{array}{ll} \displaystyle\min_{x\in\mathbb{R}^n} & \dfrac{x^TA_1x - 2b_1^Tx + c_1}{x^TA_2x - 2b_2^Tx + c_2}, \\ \text{subject to} & x^TA_3x - 2b_3^Tx + c_3 \leq 0, \end{array}$$

where A_1, A_2, A_3 are symmetric and A_3 is positive semidefinite, $b_3, b_1 \in \mathbb{R}^n$, $b_2 \in \mathbb{R}^n$ $c_3, c_2, c_1 \in \mathbb{R}$. For this class of examples, we do not report the results of Algorithm 1 due to its slow convergence rate. In Table 3 we have reported the results for Algorithm 2. As wee see, when the algorithm is using "fmincon" command, it fails to solve some problems, while SDO-based algorithm solves all problems. Moreover, SDO is faster than "fmincon" command for most of those problems which "fmincon" can solve. Overally, from tables we can observe that the SDO-based algorithms are robust although they might be slow one some problems.

Table 1: Numerical results for Example 1 using Algorithm 1

		SDO		**fmincon**	
n	density	α	time	α	time
50	1	-1.203041e+00	2.244538e+01	-1.203041e+00	2.433823e+01
100	1	-2.272280e+00	8.279166e+01	-2.272280e+00	3.879564e+01
150	1	-4.507551e+00	2.475396e+02	-4.507551e+00	2.053157e+02
50	0.5	-1.532171e+00	2.406932e+01	-1.532171e+00	3.432474e+01
100	0.5	-2.761989e+00	6.920756e+01	-2.761989e+00	3.961989e+01
150	0.5	-3.886832e+00	2.148970e+02	-3.886832e+00	4.304446e+01
200	0.5	-4.253636e+00	5.382952e+02	-4.253636e+00	3.546856e+02
50	0.25	-1.477334e+00	2.233867e+01	-1.477334e+00	2.803656e+01
100	0.25	-4.311943e+00	4.770934e+01	-4.311943e+00	4.112678e+01
150	0.25	-4.874942e+00	1.963069e+02	-4.874942e+00	4.976305e+01
200	0.25	-5.595645e+00	4.529062e+02	-5.595645e+00	5.356899e+01
50	1e-1	-3.259227e+00	2.344456e+01	-3.259227e+00	1.943978e+01
100	1e-1	-4.540142e+00	7.320052e+01	-4.540142e+00	6.825349e+01
150	1e-1	-6.206322e+00	2.010590e+02	-6.206322e+00	1.182349e+02
200	1e-1	-7.045596e+00	4.717287e+02	-7.045596e+00	2.123551e+02
250	1e-1	-7.983914e+00	7.373832e+02	-7.983914e+00	2.600914e+02
300	1e-1	-8.456371e+00	1.236395e+03	-8.456371e+00	3.085462e+02
350	1e-1	-9.817050e+00	1.799365e+03	-9.817050e+00	2.755520e+02
400	1e-1	-9.954717e+00	3.242619e+03	-9.954717e+00	8.302876e+02
50	1e-2	-2.976855e+00	1.101675e+01	-2.976855e+00	4.646819e+01
100	1e-2	-5.387870e+00	5.903953e+00	-5.387870e+00	3.217152e+01
150	1e-2	-6.953154e+00	1.091339e+02	-6.953154e+00	4.768078e+01
200	1e-2	-7.916123e+00	2.917711e+02	-7.916123e+00	1.246565e+02
250	1e-2	-8.843726e+00	5.393416e+02	-8.843726e+00	1.864699e+02
300	1e-2	-9.959038e+00	8.960638e+02	-9.959038e+00	3.126383e+02
350	1e-2	-1.048172e+00	1.375094e+03	-1.048172e+00	3.671712e+02
400	1e-2	-1.173159e+01	2.084403e+03	-1.173159e+01	4.431602e+02
450	1e-2	-1.240355e+01	2.880718e+03	-1.240355e+01	2.629137e+02
50	1e-3	-3.302979e+00	2.348483e+01	-3.302979e+00	4.225397e+01
100	1e-3	-5.323466e+00	5.761741e+01	-5.323466e+00	1.630380e+01
150	1e-3	-6.879773e+00	1.404970e+02	-6.879773e+00	3.890080e+01
200	1e-3	-7.851325e+00	2.649818e+02	-7.851325e+00	6.413956e+01
250	1e-3	-8.823625e+00	4.857750e+02	-8.823625e+00	7.708184e+01
300	1e-3	-9.801415e+00	8.593537e+02	-9.801415e+00	1.191771e+02
350	1e-3	-1.040392e+01	1.313989e+03	-1.040392e+01	1.862124e+02
400	1e-3	-1.147393e+01	1.992004e+03	-1.147393e+01	2.406859e+02
450	1e-3	-1.216152e+01	2.902077e+03	-1.216152e+01	2.774110e+02
500	1e-3	-1.233325e+01	6.111908e+03	-1.233325e+01	2.983806e+02
550	1e-3	-1.342142e+01	5.405368e+03	-1.342142e+01	2.962538e+02

5 Conclusions

In this paper, we have studied minimizing the ratio of two quadratic functions subject to a quadratic constraint. Two existing algorithms from the literature are presented and a SDO relaxation approach is introduced to solve the underlying subproblems within both algorithms. Our computational experiments on several randomly generated test problems with various dimensions and densities show that the SDO relaxation is better than the case when we use "fmincon" command of MATLAB to solve the

Table 2: Numerical results for Example 1 using Algorithm 2

		SDO		fmincon	
n	density	α	time	α	time
50	1	-1.203041e+00	1.776531e+00	-1.203041e+00	4.524196e+00
100	1	-2.272280e+00	7.345768e+00	——	——
150	1	-4.507551e+00	2.564052e+01	——	——
50	0.5	-1.532171e+00	2.031175e+00	-1.532171e+00	1.747989e+01
100	0.5	-2.761989e+00	9.091808e+00	——	——
150	0.5	-3.886832e+00	2.339637e+01	——	——
200	0.5	-4.253636e+00	5.528322e+01	——	——
50	0.25	-1.477334e+00	1.815958e+00	-1.477334e+00	5.562837e+00
100	0.25	-4.311943e+00	7.726511e+00	——	——
150	0.25	-4.874942e+00	2.110719e+01	——	——
200	0.25	-5.595645e+00	4.611992e+01	——	——
50	1e-1	-3.259227e+00	1.852506e+00	-3.259227e+00	2.215671e+00
100	1e-1	-4.540142e+00	7.854478e+00	-4.540142e+00	5.663952e+01
150	1e-1	-6.206322e+00	1.994895e+01	-6.206322e+00	2.830753e+01
200	1e-1	-7.045596e+00	4.784027e+01	-7.045596e+00	6.111840e+00
250	1e-1	-7.983914e+00	9.147257e+01	-7.983914e+00	1.593589e+01
300	1e-1	-8.456371e+00	1.544323e+02	——	——
350	1e-1	-9.817050e+00	2.456413e+02	——	——
400	1e-1	-9.954717e+00	4.316310e+02	——	——
50	1e-2	-2.976855e+00	2.228447e+00	-2.976855e+00	4.435214e+00
100	1e-2	-5.387870e+00	6.048665e+00	-5.387870e+00	1.300060e+01
150	1e-2	-6.953154e+00	1.972976e+01	-6.953154e+00	2.368148e+01
200	1e-2	-7.916123e+00	3.390616e+01	-7.916123e+00	2.582431e+01
250	1e-2	-8.843726e+00	7.312239e+01	-8.843726e+00	6.351422e+01
300	1e-2	-9.959038e+00	1.226811e+02	-9.959038e+00	8.001024e+01
350	1e-2	-1.048172e+00	1.836862e+02	-1.048172e+00	8.253071e+01
400	1e-2	-1.173159e+01	3.411169e+02	-1.173159e+01	2.567219e+02
450	1e-2	-1.240355e+01	4.867198e+02	-1.240355e+01	9.986375e+01
50	1e-3	-3.302979e+00	2.269103e+00	-3.302979e+00	9.012562e+00
100	1e-3	-5.323466e+00	5.523672e+00	-5.323466e+00	2.257842e+01
150	1e-3	-6.879773e+00	1.899749e+01	-6.879773e+00	6.615520e+01
200	1e-3	-7.851325e+00	3.446472e+01	-7.851325e+00	1.019773e+01
250	1e-3	-8.823625e+00	6.622839e+01	-8.823625e+00	1.177376e+01
300	1e-3	-9.801415e+00	1.392504e+02	-9.801415e+00	1.585505e+01
350	1e-3	-1.040392e+01	2.155496e+02	-1.040392e+01	2.725475e+01
400	1e-3	-1.147393e+01	3.275338e+02	-1.147393e+01	4.230004e+01
450	1e-3	-1.216152e+01	4.756332e+02	-1.216152e+01	4.613759e+01
500	1e-3	-1.233325e+01	6.439025e+02	-1.233325e+01	5.409772e+01
550	1e-3	-1.342142e+01	8.917905e+02	-1.342142e+01	5.987055e+01

subproblems. Extending our approach to the other classes of fractional problems are left for interested readers.

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Table 3: Numerical results for Example 2 using Algorithm 2

		SDO		**fmincon**	
n	density	α	time	α	time
50	1	-1.258963e+00	2.958644e+00	———	———
100	1	-2.125057e+00	1.631298e+01	———	———
50	0.5	-1.576781e+00	3.450615e+00	———	———
100	0.5	-1.167992e+00	1.378850e+01	———	———
150	0.5	-1.354655e+00	4.054407e+01	———	———
200	0.5	-1.846205e+00	9.175490e+01	———	———
50	0.25	-9.593844e+00	2.693359e+00	———	———
100	0.25	-1.671113e+00	1.756213e+01	———	———
150	0.25	-1.510739e+00	4.062985e+01	———	———
200	0.25	-2.464268e+00	9.385546e+01	———	———
50	1e-1	-1.660130e+00	1.425864e+00	-1.660130e+00	1.81655e+01
100	1e-1	-1.975422e+00	1.558024e+01	———	———
150	1e-1	-1.769742e+00	4.223061e+01	———	———
200	1e-1	-2.049265e+00	9.533966e+01	———	———
250	1e-1	-1.543671e+00	9.906579e+01	———	———
300	1e-1	-2.854732e+00	2.922332e+02	———	———
350	1e-1	-3.953304e+00	5.148218e+02	———	———
400	1e-1	-3.973742e+00	4.755629e+02	———	———
50	1e-2	-1.157849e+00	3.288964e+00	-1.157849e+00	9.785501e+00
100	1e-2	-1.356978e+00	1.142670e+01	-1.356978e+00	1.296221e+01
150	1e-2	-1.624859e+00	2.880941e+01	———	———
200	1e-2	-1.938739e+00	6.325097e+01	———	———
250	1e-2	-2.828952e+00	1.182690e+02	———	———
300	1e-2	3.801268e+00	2.356515e+02	———	———
350	1e-2	-2.884995e+00	3.463546e+02	———	———
400	1e-2	-3.973742e+00	4.755629e+02	———	———
450	1e-2	-4.494638e+00	6.343319e+02	———	———
50	1e-3	-1.489773e+00	4.015604e+00	-1.489773e+00	1.315874e+00
100	1e-3	-1.612217e+00	1.641235e+01	-1.612217e+00	1.200856e+01
150	1e-3	-2.159040e+00	2.797469e+01	-2.159040e+00	1.973654e+01
200	1e-3	-2.646711e+00	3.724827e+01	-2.646711e+00	5.874495e+01
250	1e-3	-3.007925e+00	1.152657e+01	———	———
300	1e-3	-4.139799e+00	2.046470e+02	———	———
350	1e-3	-2.602452e+00	3.088694e+02	———	———
400	1e-3	-4.211007e+00	7.803894e+02	———	———
450	1e-3	-4.505398e+00	9.766945e+02	———	———
500	1e-3	-6.072257e+00	1.371127e+03	———	———
550	1e-3	-3.349576e+00	1.320868e+03	———	———

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