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A comparative study of fuzzy norms of linear operators on a fuzzy normed linear spaces

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Abstract. In the present paper, we first modify the concepts of weakly fuzzy boundedness, strongly fuzzy boundedness, fuzzy continuity, strongly fuzzy continuity and weakly fuzzy continuity. Then, we try to find some relations by making a comparative study of the fuzzy norms of linear operators.

Keywords: Fuzzy norm, Fuzzy normed linear space, Fuzzy bounded linear operator.
AMS Subject Classification: 47s40.

1 Introduction

In 1992, Felbin [5] has offered an alternative definition of a fuzzy norm on a linear space with an associated metric of the Kaleva and Seikkala type [10]. He has shown that every finite dimensional normed linear space has a completion. Then Xiao and Zhu [12] have modified the definition of this fuzzy norm and studied the topological properties of fuzzy normed linear spaces. Another fuzzy norm is defined by Bag and Samanta [2].

Bag and Samanta [4] have defined concepts of weakly fuzzy boundedness, strongly fuzzy boundedness, fuzzy continuity, strongly fuzzy continuity, weakly fuzzy continuity, sequentially fuzzy continuity and fuzzy norm

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of linear operators with an associated fuzzy norm defined in [2]. In [7] the authors have defined a norm of operator with an associated fuzzy norm defined by Felbin [5], and studied some of their properties. Then, in [9] norm of operator defined in [3] by Bag and Samanta is considered and bounded inverse theorem and compact operators on fuzzy normed linear spaces are studied.

In the present paper, at first we modify the concepts of weakly fuzzy boundedness, strongly fuzzy boundedness, fuzzy continuity, strongly fuzzy continuity, weakly fuzzy continuity defined by Bag and Samanta [4]. Finally, an attempt is made to find such relation by making a comparative study of the fuzzy norms of linear operators defined in [7] and [4].

2 Preliminaries

Definition 1. (Xiao and Zhu [12]) A mapping $\eta : \mathbb{R} \longrightarrow [0, 1]$ is called a fuzzy real number with α-level set $[\eta]_\alpha = \{t : \eta(t) \geq \alpha\}$, if it satisfies the following conditions:
(N1) there exists $t_0 \in \mathbb{R}$ such that $\eta(t_0) = 1$.
(N2) for each $\alpha \in (0, 1]$, there exist real numbers $\eta^-_\alpha \leq \eta^+_\alpha$ such that the $\alpha-$level set $[\eta]_\alpha$ is equal to the closed interval $[\eta^-_\alpha, \eta^+_\alpha]$.
The set of all fuzzy real numbers is denoted by $F(\mathbb{R})$. Since each $r \in \mathbb{R}$ can be considered as the fuzzy real number $\tilde{r} \in F(\mathbb{R})$ defined by

$$\tilde{r}(t) = \begin{cases} 1 & t = r \\ 0 & t \neq r, \end{cases}$$

it follows that $\mathbb{R}$ can be embedded in $F(\mathbb{R})$.

Definition 2. (Kaleva and Seikkala [10]) The arithmetic operations $+$, $-$, $\times$ and $/$ on $F(\mathbb{R}) \times F(\mathbb{R})$ are defined by

$$\begin{aligned} (\eta + \gamma)(t) &= \sup_{t=x+y} (\min(\eta(x), \gamma(y))), \\ (\eta - \gamma)(t) &= \sup_{t=x-y} (\min(\eta(x), \gamma(y))), \\ (\eta \times \gamma)(t) &= \sup_{t=xy} (\min(\eta(x), \gamma(y))), \\ (\eta / \gamma)(t) &= \sup_{t=x/y} (\min(\eta(x), \gamma(y))), \end{aligned}$$

which are special cases of Zadeh's extension principle.

Definition 3. (Kaleva and Seikkala [10]) The absolute value $|\eta|$ of $\eta \in F(\mathbb{R})$ is defined by

$$|\eta|(t) = \begin{cases} \max(\eta(t), \eta(-t)), & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Definition 4. (Kaleva and Seikkala [10]) Let $\eta \in F(\mathbb{R})$. If $\eta(t) = 0$ for all $t < 0$, then η is called a positive fuzzy real number. The set of all positive fuzzy real numbers is denoted by $F^+(\mathbb{R})$.

Lemma 1. *(Kaleva and Seikkala [10]) Let $\eta, \gamma \in F(\mathbb{R})$ and $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$, $[\gamma]_\alpha = [\gamma_\alpha^-, \gamma_\alpha^+]$. Then*
i) $[\eta + \gamma]_\alpha = [\eta_\alpha^- + \gamma_\alpha^-, \eta_\alpha^+ + \gamma_\alpha^+]$,
ii) $[\eta - \gamma]_\alpha = [\eta_\alpha^- - \gamma_\alpha^+, \eta_\alpha^+ - \gamma_\alpha^-]$,
iii) $[\eta \times \gamma]_\alpha = [\eta_\alpha^- \gamma_\alpha^-, \eta_\alpha^+ \gamma_\alpha^+]$ *for* $\eta, \gamma \in F^+(\mathbb{R})$,
iv) $[1/\eta]_\alpha = [\frac{1}{\eta_\alpha^+}, \frac{1}{\eta_\alpha^-}]$ *if* $\eta_\alpha^- > 0$,
v) $[|\eta|]_\alpha = [\max(0, \eta_\alpha^-, -\eta_\alpha^+), \max(|\eta_\alpha^-|, |\eta_\alpha^+|)]$.

Lemma 2. *(Kaleva and Seikkala [10]) Let $[a^\alpha, b^\alpha]$, $0 < \alpha \leq 1$, be a given family of non-empty intervals. Assume*
a)$[a^{\alpha_1}, b^{\alpha_1}] \supset [a^{\alpha_2}, b^{\alpha_2}]$ *for all* $0 < \alpha_1 \leq \alpha_2$,
b)$[\lim\limits_{k \to -\infty} a^{\alpha_k}, \lim\limits_{k \to \infty} b^{\alpha_k}] = [a^\alpha, b^\alpha]$ *whenever* $\{\alpha_k\}$ *is an increasing sequence in* $(0, 1]$ *converging to* α,
c)$-\infty < a^\alpha \leq b^\alpha < +\infty$, *for all* $\alpha \in (0, 1]$.
Then the family $[a^\alpha, b^\alpha]$ *represents the* α*-level sets of a fuzzy real number* $\eta \in F(\mathbb{R})$.

Conversely if $[a^\alpha, b^\alpha]$, $0 < \alpha \leq 1$, *are the* α*-level sets of a fuzzy number* $\eta \in F(\mathbb{R})$, *then the conditions (a),(b) and (c) are satisfied.*

Definition 5. (Kaleva and Seikkala [10]) Let $\eta, \gamma \in F(\mathbb{R})$ and $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$, $[\gamma]_\alpha = [\gamma_\alpha^-, \gamma_\alpha^+]$, for all $\alpha \in (0, 1]$. Define a partial ordering by $\eta \leq \gamma$ if and only if $\eta_\alpha^- \leq \gamma_\alpha^-$ and $\eta_\alpha^+ \leq \gamma_\alpha^+$, for all $\alpha \in (0, 1]$. Strict inequality in $F(\mathbb{R})$ is defined by $\eta < \gamma$ if and only if $\eta_\alpha^- < \gamma_\alpha^-$ and $\eta_\alpha^+ < \gamma_\alpha^+$, for all $\alpha \in (0, 1]$.

Lemma 3. *Let $\eta \in F(\mathbb{R})$. Then $\eta \in F^+(\mathbb{R})$ if and only if $0 \leq \eta$.*

Definition 6. (Felbin [5]) Let X be a vector space over $\mathbb{R}$. Assume the mappings $L, R : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ are symmetric and non-decreasing in both arguments, and that $L(0, 0) = 0$ and $R(1, 1) = 1$. Let $\|.\| : X \longrightarrow F^+(\mathbb{R})$. The quadruple $(X, \|.\|, L, R)$ is called a fuzzy normed linear space with the fuzzy norm $\|.\|$, if the following conditions are satisfied:

(F_1) if $x \neq 0$ then $\inf_{0<\alpha\leq 1} \|x\|_\alpha^- > 0$,
(F_2) $\|x\| = \tilde{0}$ if and only if $x = 0$,
(F_3) $\|rx\| = |\tilde{r}|\|x\|$ for $x \in X$ and $r \in \mathbb{R}$,
(F_4) for all $x, y \in X$,
$(F_4L)\|x+y\|(s+t) \geq L(\|x\|(s), \|y\|(t))$ whenever $s \leq \|x\|_1^-$, $t \leq \|y\|_1^-$ and $s+t \leq \|x+y\|_1^-$,
$(F_4R)\|x+y\|(s+t) \leq R(\|x\|(s), \|y\|(t))$ whenever $s \geq \|x\|_1^-$, $t \geq \|y\|_1^-$ and $s+t \geq \|x+y\|_1^-$.

We assume that
$(F5)$ for any sequence $\{\alpha_k\}$ in $(0,1]$ such that $\alpha_k \searrow \alpha \in (0,1]$ implies that $\|x\|_{\alpha_k}^+ \nearrow \|x\|_\alpha^+$ for all $x \in X$.

Lemma 4. *(Xiao and Zhu [12]) Let $(X, \|.\|, L, R)$ be a fuzzy normed linear space.*
(1)If $L \leq \min$, then (F_4L) holds whenever $\|x+y\|_\alpha^- \leq \|x\|_\alpha^- + \|y\|_\alpha^-$ for all $\alpha \in (0,1]$ and $x, y \in X$.
(2)If $L \geq \min$, then $\|x+y\|_\alpha^- \leq \|x\|_\alpha^- + \|y\|_\alpha^-$ for all $\alpha \in (0,1]$ and $x, y \in X$ whenever (F_4L) holds.
(3)If $R \geq \max$, then (F_4R) holds whenever $\|x+y\|_\alpha^+ \leq \|x\|_\alpha^+ + \|y\|_\alpha^+$ for all $\alpha \in (0,1]$ and $x, y \in X$.
(4)If $R \leq \max$, then $\|x+y\|_\alpha^+ \leq \|x\|_\alpha^+ + \|y\|_\alpha^+$ for all $\alpha \in (0,1]$ and $x, y \in X$ whenever (F_4R) holds.

In what follows $L(s,t) = \min(s,t)$ and $R(s,t) = \max(s,t)$ for all $s, t \in [0,1]$. We write $(X, \|.\|)$ or simply X when L and R are as above.

The following result is an analogue of the usual triangle inequality.

Theorem 1. *In a fuzzy normed linear space $(X, \|.\|)$, the condition (F_4) is equivalent to*

$$\|x+y\| \leq \|x\| + \|y\|.$$

Proof. The proof is a consequence of Lemmas 4. □

Definition 7. (Bag and Samanta [1]) Let X be a linear space over $\mathbb{R}$ (real number). Let N be a fuzzy subset of $X \times \mathbb{R}$ such that for all $x, u \in X$ and $c \in \mathbb{R}$
$(N1)$ $N(x,t) = 0$ for all $t \leq 0$,
$(N2)$ $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$,
$(N3)$ If $c \neq 0$ then $N(cx,t) = N(x, t/|c|)$ for all $t \in \mathbb{R}$,
$(N4)$ $N(x+u, s+t) \geq \min\{N(x,s), N(u,t)\}$ for all $s, t \in \mathbb{R}$,
$(N5)$ $N(x,.)$ is a nondecreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.
Then N is called a fuzzy norm on X.

Definition 8. (Bag and Samanta [1]) Let X be a linear space over $\mathbb{R}$ (real number). Let N^* be A fuzzy subset of $X \times \mathbb{R}$ such that for all $x, u \in X$ and $c \in \mathbb{R}$
(N^*1) $N^*(x,t) = 1$ for all $t \leq 0$,
(N^*2) $x = 0$ if and only if $N^*(x,t) = 0$ for all $t > 0$,
(N^*3) If $c \neq 0$ then $N^*(cx,t) = N^*(x, t/|c|)$ for all $t \in \mathbb{R}$,
(N^*4) $N^*(x+u, s+t) \leq max\{N^*(x,s), N^*(u,t)\}$ for all $s,t \in \mathbb{R}$,
(N^*5) $N^*(x,.)$ is a nonincreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N^*(x,t) = 0$.
Then N^* is called a fuzzy antinorm on X.

Definition 9. Let X be a linear space over $\mathbb{R}$. Let N be a fuzzy norm on X and N^* be A fuzzy antinorm on X. The (X, N, N^*) will be referred to as a fuzzy normed linear space.

We assume that
$(N6)$ $N(x,t) > 0$ for all $t > 0$ implies $x = 0$,
(N^*6) $N^*(x,t) < 1$ for all $t > 0$ implies $x = 0$,
$(N7)$ For $x \neq 0$, $N(x,.)$ is a continuous function of $\mathbb{R}$ and strictly increasing on the subset $\{t : 0 < N(x,t) < 1\}$ of $\mathbb{R}$,
(N^*7) For $x \neq 0$, $N^*(x,.)$ is a continuous function of $\mathbb{R}$ and strictly decreasing on the subset $\{t : 0 < N^*(x,t) < 1\}$ of $\mathbb{R}$.

Theorem 2. *(Bag and Samanta [1]) Let N be a fuzzy norm on a linear space X satisfying $(N6)$. Define $\|x\|_\alpha = \inf\{t : N(x,t) \geq \alpha\}$. Then $\{\|.\|_\alpha : \alpha \in (0,1)\}$ is an ascending family of norms on X. We call these norms as $\alpha-$norm on X corresponding to the norm N on X.*

Theorem 3. *(Bag and Samanta [1]) Let N^* be a fuzzy antinorm on a linear space X satisfying (N^*6). Define $\|x\|^*_\alpha = \inf\{t > 0 : N^*(x,t) < \alpha\}$. Then $\{\|.\|^*_\alpha : \alpha \in (0,1)\}$ is an decreasing family of norms on X.*

Theorem 4. *(Bag and Samanta [1]) Let $(X, \|.\|)$ be a fuzzy normed linear space and $[\|x\|]_\alpha = [\|x\|^-_\alpha, \|x\|^+_\alpha]$, for all $\alpha \in (0,1]$. Let N and N^* be two functions on $X \times \mathbb{R}$ defined by*

$$N(x,t) = \begin{cases} \sup\{\alpha \in (0,1] : \|x\|^-_\alpha \leq t\}, & (x,t) \neq (0,0), \\ 0, & (x,t) = (0,0), \end{cases}$$

and

$$N^*(x,t) = \begin{cases} \inf\{\alpha \in (0,1] : \|x\|^+_\alpha \leq t\}, & (x,t) \neq (0,0), \\ 1, & (x,t) = (0,0). \end{cases}$$

Then N is a fuzzy norm, N^* is a fuzzy antinorm and they satisfy the following conditions:

(i) N satisfies $(N6)$ condition,
(ii) N^* satisfies (N^*6) condition,
(iii) for each $x \neq 0$, there is $r_x > 0$ such that $N(x,t) = 1$ for all $t \geq r_x$,
(iv) for each $x \neq 0$, there is $t_x > 0$ such that $N(x, t_x) = 0$,
(v) if $N^*(x,t) < 1$, then $\lim_{s \searrow t} N(x,t) = 1$.

Theorem 5. *(Bag and Samanta [1]) Let (X, N, N^*) be a fuzzy normed linear space such that N, N^* satisfying in conditions ((i)-(v)) of Theorem 4. Define*

$$\|x\|^*_\alpha = \inf\{t > 0 : N^*(x,t) < \alpha\},$$

and

$$\|x\|_\alpha = \inf\{t > 0 : N(x,t) \geq \alpha\}.$$

Then there is a fuzzy norm $\|.\|$ on X such that $[\|x\|]_\alpha = [\|x\|_\alpha, \|x\|^*_\alpha]$ for all $\alpha \in (0,1]$ and $x \in X$.

Theorem 6. *(Bag and Samanta [1]) Let $(X, \|.\|)$ be a fuzzy normed linear space such that $\|.\|$ satisfies in the condition $(F5)$, N and N^* be two functions on $X \times \mathbb{R}$ defined in Theorem 4, and $\|.\|'$ be the fuzzy norm defined in Theorem 5. Then $\|.\| = \|.\|'$.*

Definition 10. (Xiao and Zhu [12]) Let $(X, \|.\|)$ be a fuzzy normed linear space.
i) A sequence $\{x_n\} \subseteq X$ is said to be convergent to $x \in X$ ($\lim_{n\to\infty} x_n = x$), if $\lim_{n\to\infty} \|x_n - x\|^+_\alpha = 0$, for all $\alpha \in (0,1]$.
ii) A sequence $\{x_n\} \subseteq X$ is called Cauchy, if $\lim_{m,n\to\infty} \|x_n - x_m\|^+_\alpha = 0$, for all $\alpha \in (0,1]$.

Definition 11. (Bag and Samanta [4]) Let (X, N, N^*) be a fuzzy normed linear space.
i) A sequence $\{x_n\} \subseteq X$ is said to be convergent to $x \in X$ ($\lim_{n\to\infty} x_n = x$), if $\lim_{n\to\infty} N(x_n - x, t) = 1$ and $\lim_{n\to\infty} N^*(x_n - x, t) = 0$, for all $t > 0$.
ii) A sequence $\{x_n\} \subseteq X$ is called Cauchy, if $\lim_{n,m\to\infty} N(x_n - x_m, t) = 1$ and $\lim_{n,m\to\infty} N^*(x_n - x_m, t) = 0$, for all $t > 0$.

Definition 12. (Xiao and Zhu [12]) Let $(X, \|.\|)$ be a fuzzy normed linear space. A subset A of X is said to be complete, if every Cauchy sequence in A converges in A.

Definition 13. (Xiao and Zhu [12]) Let $(X, \|.\|)$ and $(Y, \|.\|)$ be fuzzy normed linear spaces. A function $\varphi : X \longrightarrow Y$ is said to be continuous at $x \in X$, if $\lim_{n\to\infty} \varphi(x_n) = \varphi(x)$ whenever $\{x_n\} \subseteq X$ and $\lim_{n\to\infty} x_n = x$.

3 Some properties of fuzzy norm of linear operators

Definition 14. Let $(X, \|.\|)$ and $(Y, \|.\|)$ be fuzzy normed linear spaces. Furthermore, let $T : X \longrightarrow Y$ be a linear operator. The operator T is said to be strongly fuzzy bounded, if there is a real number $M > 0$ such that

$$\|Tx\| \leq M\|x\|, \text{ for all } x \in X.$$

Definition 15. (Hasankhani, Saheli and Nazari [7]) Let $(X, \|.\|)$ and $(Y, \|.\|)$ be fuzzy normed linear spaces. Furthermore, let $T : X \longrightarrow Y$ be a linear operator. The operator T is said to be fuzzy bounded, if there is a fuzzy real number η such that

$$\|Tx\| \leq \eta\|x\|, \text{ for all } x \in X.$$

The set of all fuzzy bounded linear operators, $T : X \longrightarrow Y$, is denoted by $B(X, Y)$.

Definition 16. (Hasankhani, Saheli and Nazari [7]) Let $(X, \|.\|)$, $(Y, \|.\|)$ be fuzzy normed linear spaces and $T : X \longrightarrow Y$ a fuzzy bounded linear operator. We define $\|T\|$ by,

$$[\|T\|]_\alpha = [\sup_{\beta<\alpha} \sup_{\|x\|_\beta^- \leq 1} \|Tx\|_\beta^-, \ \inf\{\eta_\alpha^+ : \|Tx\| \leq \eta\|x\|\}], \text{ for all } \alpha \in (0, 1].$$

Then $\|T\|$ is called the fuzzy norm of the operator T.

Notation 1. *We write* $\|T\|_\alpha^- = \sup_{\beta<\alpha} \sup_{\|x\|_\beta^- \leq 1} \|Tx\|_\beta^-$ *and* $\|T\|_\alpha^+ = \inf\{\eta_\alpha^+ : \|Tx\| \leq \eta\|x\|\}$, *i.e.* $[\|T\|]_\alpha = [\|T\|_\alpha^-, \|T\|_\alpha^+]$, *for all* $\alpha \in (0, 1]$.

Theorem 7. *(Hasankhani, Saheli and Nazari [7]) The vector space* $B(X, Y)$ *equipped with the norm defined in Definition 16 is a fuzzy normed linear space.*

Lemma 5. *(Hasankhani, Saheli and Nazari [7]) Let* $T : X \longrightarrow Y$ *be a fuzzy bounded linear operator and* $(X, \|.\|)$, $(Y, \|.\|)$ *be fuzzy normed linear spaces. Then* $\|Tx\| \leq \|T\|\|x\|$, *for all* $x \in X$.

Theorem 8. *(Hasankhani, Saheli and Nazari [7]) Let* $T : X \longrightarrow Y$ *be a fuzzy bounded linear operator and* $(X, \|.\|)$, $(Y, \|.\|)$ *be fuzzy normed linear spaces. Then* $\|T\| \leq \eta$ *whenever* $\|Tx\| \leq \eta\|x\|$ $(\eta \in F(\mathbb{R}))$.

Theorem 9. *(Hasankhani, Saheli and Nazari [7]) Let* $(Y, \|.\|)$ *be a complete fuzzy normed linear space and* $(X, \|.\|)$ *be a fuzzy normed linear space. Then* $B(X, Y)$ *is complete fuzzy normed linear space.*

Theorem 10. *(Hasankhani, Saheli and Nazari [7]) Let* $(X, \|.\|)$ *be a finite dimensional fuzzy normed linear space. Then every linear operator on* X *is fuzzy bounded.*

Definition 17. Let (X, N, N^*) and (Y, N, N^*) be fuzzy normed linear spaces. Furthermore, let $T : X \longrightarrow Y$ be a linear operator. The operator T is said to be strongly fuzzy bounded, if there is a real number $M > 0$ such that

$$N(Tx, t) \geq N(x, t/M) \text{ and } N^*(Tx, t) \leq N^*(x, t/M),$$

for all $x \in X$ and all $t \in \mathbb{R}$.

Definition 18. Let (X, N, N^*) and (Y, N, N^*) be fuzzy normed linear spaces. Furthermore, let $T : X \longrightarrow Y$ be a linear operator. The operator T is said to be weakly fuzzy bounded, if there is a fuzzy real number $\eta > 0$ such that

$$\begin{aligned} N(x, t) \geq \alpha \text{ implies that } N(Tx, \eta_\alpha^- t) &\geq \alpha \text{ and} \\ N^*(x, t) \leq \alpha \text{ implies that } N^*(Tx, \eta_\alpha^+ t) &\leq \alpha, \end{aligned}$$

for all $x \in X$, $t \in \mathbb{R}$ and all $\alpha \in (0, 1)$.

Definition 19. Let (X, N, N^*) and (Y, N, N^*) be fuzzy normed linear spaces. Furthermore, let $T : X \longrightarrow Y$ be a linear operator. The operator T is said to be fuzzy continuous at $x_0 \in X$, if for given $(\epsilon, \alpha) \in (0, \infty) \times (0, 1)$ there is $(\delta, \beta) \in (0, \infty) \times (0, 1)$ such that

$$\begin{aligned} N(x - x_0, \delta) > \beta \text{ implies that } N(Tx - Tx_0, \epsilon) &> \alpha \text{ and} \\ N^*(x - x_0, \delta) < \beta \text{ implies that } N^*(Tx - Tx_0, \epsilon) &< \alpha, \end{aligned}$$

for all $x \in X$. T is said fuzzy continuous on X if T be fuzzy continuous at each point of X.

Definition 20. Let (X, N, N^*) and (Y, N, N^*) be fuzzy normed linear spaces. Furthermore, let $T : X \longrightarrow Y$ be a linear operator. The operator T is said to be strongly fuzzy continuous at $x_0 \in X$, if for each $\epsilon > 0$ there is a real number $\delta > 0$ such that

$$N(Tx - Tx_0, \epsilon) \geq N(x - x_0, \delta) \text{ and } N^*(Tx - Tx_0, \epsilon) \leq N^*(x - x_0, \delta),$$

for all $x \in X$. T is said strongly fuzzy continuous on X if T be strongly fuzzy continuous at each point of X.

Definition 21. Let (X, N, N^*) and (Y, N, N^*) be fuzzy normed linear spaces. Furthermore, let $T : X \longrightarrow Y$ be a linear operator. The operator T is said to be weakly fuzzy continuous at $x_0 \in X$, if for given $\epsilon > 0$ there is a fuzzy real number $\delta > 0$ such that

$$N(x - x_0, \delta_\alpha^+) \geq \alpha \text{ implies that } N(Tx - Tx_0, \epsilon) \geq \alpha \text{ and}$$
$$N^*(x - x_0, \delta_\alpha^-) \leq \alpha \text{ implies that } N^*(Tx - Tx_0, \epsilon) \leq \alpha,$$

for all $x \in X$ and all $\alpha \in (0, 1)$.
T is said weakly fuzzy continuous on X if T be weakly fuzzy continuous at each point of X.

Definition 22. Let (X, N, N^*) and (Y, N, N^*) be fuzzy normed linear spaces. Furthermore, let $T : X \longrightarrow Y$ be a linear operator. The operator T is said to be sequentially fuzzy continuous at $x_0 \in X$, if for any sequence $\{x_n\} \subseteq X$ with $x_n \longrightarrow x_0$ implies $Tx_n \longrightarrow Tx_0$.
T is said sequentially fuzzy continuous on X if T be sequentially fuzzy continuous at each point of X.

Theorem 11. *Let (X, N, N^*) and (Y, N, N^*) be fuzzy normed linear spaces. Furthermore, let $T : X \longrightarrow Y$ be a strongly fuzzy continuous linear operator. Then T is sequentially continuous.*

Proof. Proof is similar to proof Theorem 3.1 in [4]. □

Theorem 12. *Let (X, N, N^*) and (Y, N, N^*) be fuzzy normed linear spaces. Furthermore, let $T : X \longrightarrow Y$ be a linear operator. Then T is sequentially fuzzy continuous if and only if it is fuzzy continuous.*

Proof. Let T be a fuzzy continuous at $x_0 \in X$. Suppose that $\{x_n\}$ is a sequence in X such that $x_n \longrightarrow x_0$. Let $\epsilon > 0$ be given. assume that $\alpha \in (0, 1)$. Since T is fuzzy continuous at x_0, there are $\delta > 0$ and $\beta \in (0, 1)$ such that for all $x \in X$

$$N(x - x_0, \delta) > \beta \text{ implies that } N(Tx - Tx_0, \epsilon) > \alpha \text{ and}$$
$$N^*(x - x_0, \delta) < \beta \text{ implies that } N^*(Tx - Tx_0, \epsilon) < \alpha.$$

Since $x_n \longrightarrow x_0$, there is a positive integer N such that

$$N(x_n - x_0, \delta) > \beta \text{ and } N^*(x_n - x_0, \delta) < \beta \text{ for all } n \geq N.$$

Then

$$N(Tx_n - Tx_0, \epsilon) > \alpha \text{ and } N^*(Tx_n - Tx_0, \epsilon) < \alpha \text{ for all } n \geq N.$$

Hence

$$\lim_{n\to\infty} N(Tx_n - Tx_0, \epsilon) = 1 \text{ and } \lim_{n\to\infty} N^*(Tx_n - Tx_0, \epsilon) = 0.$$

Thus T is sequentially fuzzy continuous.
Conversely, let T be a sequentially fuzzy continuous at $x_0 \in X$. Suppose that T is not fuzzy continuous at x_0. Hence there is $(\epsilon, \alpha) \in (0, \infty) \times (0, 1)$ such that for any $(\delta, \beta) \in (0, \infty) \times (0, 1)$, there exists $y \in X$ such that

$$\begin{aligned} N(y - x_0, \delta) &> \beta \text{ but } N(Ty - Tx_0, \epsilon) \leq \alpha \text{ or} \\ N^*(y - x_0, \delta) &< \beta \text{ but } N^*(Ty - Tx_0, \epsilon) \geq \alpha. \end{aligned}$$

Thus for $\beta = 1 - (1/n + 1)$ and $\delta = 1/n + 1$, there is $y_n \in X$ such that

$$\begin{aligned} N(y_n - x_0, \delta) &> \beta \text{ but } N(Ty_n - Tx_0, \epsilon) \leq \alpha \text{ or} \\ N^*(y_n - x_0, \delta) &< \beta \text{ but } N^*(Ty_n - Tx_0, \epsilon) \geq \alpha. \end{aligned}$$

So there is subsequence $\{y_{n_k}\}$ such that $N(Ty_{n_k} - Tx_0, \epsilon) \leq \alpha$, for all $k \geq 1$, or $N^*(Ty_{n_k} - Tx_0, \epsilon) \geq \alpha$, for all $k \geq 1$. Hence

$$\lim_{k\to\infty} N(Ty_{n_k} - Tx_0, \epsilon) \neq 1 \text{ or } \lim_{k\to\infty} N^*(Ty_{n_k} - Tx_0, \epsilon) \neq 0.$$

Since $y_n \longrightarrow x_0$ it follows that $y_{n_k} \longrightarrow x_0$, which is contradiction. Thus T is fuzzy continuous at x_0. □

Theorem 13. *Let (X, N, N^*) and (Y, N, N^*) be fuzzy normed linear spaces. Furthermore, let $T : X \longrightarrow Y$ be a linear operator. Then T is strongly fuzzy continuous if and only if it is strongly fuzzy bounded.*

Proof. Proof is similar to proof of Theorem 3.5. in [4] □

Theorem 14. *Let (X, N, N^*) and (Y, N, N^*) be fuzzy normed linear spaces. Furthermore, let $T : X \longrightarrow Y$ be a linear operator. Then T is weakly fuzzy continuous if and only if it is weakly fuzzy bounded.*

Proof. Let T be weakly fuzzy bounded. Then there is a fuzzy real number $\eta > 0$ such that

$$\begin{aligned} N(x, t) &\geq \alpha \text{ implies that } N(Tx, \eta_\alpha^- t) \geq \alpha \text{ and} \\ N^*(x, t) &\leq \alpha \text{ implies that } N^*(Tx, \eta_\alpha^+ t) \leq \alpha, \end{aligned}$$

for all $x \in X$, $t \in \mathbb{R}$ and all $\alpha \in (0, 1)$.
Let $\epsilon > 0$ be given. Suppose that $\delta = \epsilon/\eta$. Assume that $N(x - x_0, \delta_\alpha^+) \geq \alpha$, hence $N(Tx - Tx_0, \epsilon) \geq \alpha$. And let $N^*(x - x_0, \delta_\alpha^-) \leq \alpha$, thus $N^*(Tx -$

$Tx_0, \epsilon) \leq \alpha$.
So T is weakly fuzzy continuous.
Conversely, let T be weakly fuzzy continuous. For $\epsilon = 1$ there is a fuzzy real number $\delta > 0$ such that

$$N(x, \delta_\alpha^+) \geq \alpha \text{ implies that } N(Tx, \epsilon) \geq \alpha \text{ and}$$
$$N^*(x, \delta_\alpha^-) \leq \alpha \text{ implies that } N^*(Tx, \epsilon) \leq \alpha,$$

for all $x \in X$ and all $\alpha \in (0, 1)$.
Let $\eta = 1/\delta$. Assume that $N(x, t) \geq \alpha$ then $N(\delta_\alpha^+ x/t, \delta_\alpha^+) \geq \alpha$. So $N((\delta_\alpha^+/t)Tx, 1) \geq \alpha$. Hence $N(Tx, \eta_\alpha^- t) \geq \alpha$.
Suppose that $N^*(x, t) \leq \alpha$, then $N^*(\delta_\alpha^- x/t, \delta_\alpha^-) \leq \alpha$. Hence $N^*((\delta_\alpha^-/t)Tx, 1) \leq \alpha$. So $N^*(Tx, \eta_\alpha^+ t) \leq \alpha$. □

4 Comparative study between fuzzy norm of linear operators

Theorem 15. *Let N be a fuzzy norm and N^* be a fuzzy antinorm on a linear space X satisfying (N7), (N^*7) and the conditions ((i)-(v)) of Theorem 4 and let $\|.\|$ be a fuzzy norm defined in Theorem 5. Define two functions N' and N'^* from $X \times \mathbb{R}$ to $[0, 1]$ by*

$$N'(x,t) = \begin{cases} \sup\{\alpha \in (0,1] : \|x\|_\alpha^- \leq t\} & , \quad (x,t) \neq (0,0) \\ 0 & , \quad (x,t) = (0,0), \end{cases}$$

and

$$N'^*(x,t) = \begin{cases} \inf\{\alpha \in (0,1] : \|x\|_\alpha^+ \leq t\} & , \quad (x,t) \neq (0,0) \\ 1 & , \quad (x,t) = (0,0). \end{cases}$$

Then $N = N'$ and $N^* = N'^*$.

Proof. By Theorem 5, we have

$$\|x\|_\alpha^- = \inf\{t > 0 : N(x,t) \geq \alpha\}.$$

Let $(x, t) \neq (0, 0)$. For $\epsilon > 0$ there exists $\beta \in (0, 1]$ such that $N'(x, t) - \epsilon \leq \beta$ and $\|x\|_\beta^- \leq t$. Hence

$$\inf\{s > 0 : N(x,s) \geq \beta\} \leq t.$$

Then there is $t \leq s_n < t + 1/n$ such that $N(x, s_n) \geq \beta$. As $n \longrightarrow \infty$, thus $N(x, t) \geq \beta$. So $N'(x, t) - \epsilon \leq \beta \leq N(x, t)$. Hence $N'(x, t) \leq N(x, t)$.

Let $N(x,t) = \alpha$, this implies that $\|x\|_\alpha^- \leq t$. So $N'(x,t) \geq \alpha$, hence $N(x,t) \leq N'(x,t)$. Thus $N = N'$
Similarly, by Theorem 5, we have

$$\|x\|_\alpha^+ = \inf\{t > 0 : N^*(x,t) < \alpha\}.$$

Let $(x,t) \neq (0,0)$. For $\epsilon > 0$ there exists $\beta \in (0,1]$ such that $N'^*(x,t) + \epsilon \geq \beta$ and $\|x\|_\beta^+ \leq t$. Hence

$$\inf\{s > 0 : N^*(x,s) < \beta\} \leq t.$$

Then there is $t \leq s_n < t + 1/n$ such that $N^*(x,s_n) < \beta$. As $n \longrightarrow \infty$, thus $N^*(x,t) \leq \beta$. So $N'^*(x,t) + \epsilon \geq \beta \geq N^*(x,t)$. Hence $N'^*(x,t) \geq N^*(x,t)$. Let $N^*(x,t) = \alpha < \beta \leq 1$ this implies that $\|x\|_\beta^+ \leq t$. So $N'^*(x,t) \leq \beta$, thus $N'^*(x,t) \leq \alpha$. Hence $N^*(x,t) \geq N'^*(x,t)$. So $N^* = N'^*$. □

Theorem 16. *Let N^* be a fuzzy antinorm on linear space X satisfying (N^*7) and*

$$\|x\|_\alpha^* = \inf\{t > 0 : N^*(x,t) < \alpha\}.$$

Furthermore, let $\alpha_n, \alpha \in (0,1)$ and $\alpha_n \searrow \alpha$. Then $\|x\|_{\alpha_n}^* \nearrow \|x\|_\alpha^*$.

Proof. We have $\alpha \leq \alpha_n$, hence $\|x\|_{\alpha_n}^* \leq \|x\|_\alpha^*$. Then $\lim_{n\longrightarrow\infty} \|x\|_{\alpha_n}^* \leq \|x\|_\alpha^*$. If

$$\lim_{n\longrightarrow\infty} \|x\|_{\alpha_n}^* < k < \|x\|_\alpha^*,$$

then $\|x\|_{\alpha_n}^* < k$, for all $n \in \mathbb{N}$. So $N^*(x,k) < \alpha_n$, for all $n \in \mathbb{N}$. As $n \longrightarrow \infty$, we obtain that $N^*(x,k) \leq \alpha$.
case 1: $N^*(x,k) < \alpha$, hence $\|x\|_\alpha^* \leq k$. This is contradiction.
case 2: $N^*(x,k) = \alpha$. Since $N^*(x,.)$ is strictly decreasing on the subset $\{t : 0 < N^*(x,t) < 1\}$ of $\mathbb{R}$, there is $M > 0$ such that $0 < N^*(x,k+M) < N^*(x,k) = \alpha$. Then $\|x\|_\alpha^* \leq k + \epsilon$ for all $0 < \epsilon < M$. Hence $\|x\|_\alpha^* \leq k$. This is contradiction. Thus

$$\lim_{n\longrightarrow\infty} \|x\|_{\alpha_n}^* = \|x\|_\alpha^*.$$

□

Theorem 17. *Let $(X, \|.\|)$, $(Y, \|.\|)$ be fuzzy normed linear spaces and $T : (X, \|.\|) \longrightarrow (Y, \|.\|)$ a strongly fuzzy bounded linear operator. Furthermore, let (X, N, N^*) and (Y, N, N^*) be fuzzy normed linear spaces defined in Theorem 4. Then $T : (X, N, N^*) \longrightarrow (Y, N, N^*)$ is strongly fuzzy bounded.*

Proof. Let $T : (X, \|.\|) \longrightarrow (Y, \|.\|)$ is a strongly fuzzy bounded linear operator. Then there exists a real number $M > 0$ such that

$$\|Tx\| \leq M\|x\|, \text{ for all } x \in X.$$

Let $(x, t) \neq (0, 0)$. We have

$$\begin{aligned} N(Tx, t) &= \sup\{\alpha \in (0, 1] : \|Tx\|_\alpha^- \leq t\} \\ &\geq \sup\{\alpha \in (0, 1] : \|x\|_\alpha^- \leq t/M\} \\ &= N(x, t/M). \end{aligned}$$

Let $(x, t) = (0, 0)$ then $N(Tx, t) = 0 = N(x, t)$. Hence

$$N(Tx, t) \geq N(x, t/M) \text{ for all } x \in X \text{ and all } t \in \mathbb{R}.$$

Let $(x, t) \neq (0, 0)$. We have

$$\begin{aligned} N^*(Tx, t) &= \inf\{\alpha \in (0, 1] : \|Tx\|_\alpha^+ \leq t\} \\ &\leq \inf\{\alpha \in (0, 1] : \|x\|_\alpha^+ \leq t/M\} \\ &= N^*(x, t/M). \end{aligned}$$

Let $(x, t) = (0, 0)$ then $N^*(Tx, t) = 1 = N^*(x, t)$. Hence

$$N^*(Tx, t) \leq N^*(x, t/M) \text{ for all } x \in X \text{ and all } t \in \mathbb{R}.$$

So $T : (X, N, N^*) \longrightarrow (Y, N, N^*)$ is strongly fuzzy bounded. □

Theorem 18. *Let (X, N, N^*) and (Y, N, N^*) be fuzzy normed linear spaces such that N and N^* satisfying conditions ((i)-(v)) of Theorem 4 and $T : (X, N, N^*) \longrightarrow (Y, N, N^*)$ a strongly fuzzy bounded linear operator. Furthermore, let $(X, \|.\|)$ and $(Y, \|.\|)$ be fuzzy normed linear spaces defined in Theorem 5. Then $T : (X, \|.\|) \longrightarrow (Y, \|.\|)$ is strongly fuzzy bounded.*

Proof. We have

$$\|x\|_\alpha^- = \inf\{t > 0 : N(x, t) \geq \alpha\},$$

and

$$\|Tx\|_\alpha^- = \inf\{t > 0 : N(Tx, t) \geq \alpha\}.$$

Hence for $\epsilon > 0$ there exists $t_0 > 0$ such that $N(x, t_0) \geq \alpha$ and $\|x\|_\alpha^- + \epsilon \geq t_0$. Since $T : (X, N, N^*) \longrightarrow (Y, N, N^*)$ is strongly fuzzy bounded, we have

$$N(Tx, Mt_0) \geq N(x, t_0) \geq \alpha.$$

So $\|Tx\|_\alpha^- \leq Mt_0$, thus $\|Tx\|_\alpha^- \leq M\|x\|_\alpha^-$.
Now we have

$$\|x\|_\alpha^+ = \inf\{t > 0 : N^*(x,t) < \alpha\},$$

and

$$\|Tx\|_\alpha^+ = \inf\{t > 0 : N^*(Tx,t) < \alpha\}.$$

Hence for $\epsilon > 0$ there exists $t_1 > 0$ such that $N^*(x,t_1) < \alpha$ and $\|x\|_\alpha^- + \epsilon \geq t_1$. Since $T : (X,N,N^*) \longrightarrow (Y,N,N^*)$ is strongly fuzzy bounded, we have

$$N^*(Tx, Mt_1) \leq N^*(x,t_1) < \alpha.$$

So $\|Tx\|_\alpha^- \leq Mt_1$, thus $\|Tx\|_\alpha^+ \leq M\|x\|_\alpha^+$.
Hence $T : (X,\|.\|) \longrightarrow (Y,\|.\|)$ is strongly fuzzy bounded. □

Theorem 19. *Let (X,N,N^*) and (Y,N,N^*) be fuzzy normed linear spaces such that N and N^* satisfying (N^*7) and conditions ((i)-(v)) of Theorem 4. Suppose that $T : (X,N,N^*) \longrightarrow (Y,N,N^*)$ is a weakly fuzzy bounded linear operator. Furthermore, let $(X,\|.\|)$ and $(Y,\|.\|)$ be fuzzy normed linear spaces defined in Theorem 5. Then $T : (X,\|.\|) \longrightarrow (Y,\|.\|)$ is fuzzy bounded.*

Proof. Let $\beta \in (0,1)$ and $x \in X$. Since operator T is weakly fuzzy bounded, there is a fuzzy real number $\eta > 0$ such that

$$N(x,t) \geq \alpha \text{ implies that } N(Tx,\eta_\alpha^- t) \geq \alpha \text{ and}$$
$$N^*(x,t) \leq \alpha \text{ implies that } N^*(Tx,\eta_\alpha^+ t) \leq \alpha,$$

for all $x \in X$, $t \in \mathbb{R}$ and all $\alpha \in (0,1)$.
If $N(x,t) \geq \beta$ then $N(Tx,\eta_\alpha^- t) \geq \beta$. So $\|Tx\|_\beta^- \leq \eta_\beta^- t$. Hence

$$\|Tx\|_\beta^- \leq \eta_\beta^- \|x\|_\beta^-.$$

If $N^*(x,t) < \beta$ then $N^*(Tx,\eta_\alpha^+ t) \leq \beta$. So there exists $\beta_n \in (0,1]$ such that $\beta_n < \beta + 1/n$ and $\|Tx\|_{\beta_n}^+ \leq \eta_\beta^+ t$, this implies that $\|Tx\|_\beta^+ \leq \eta_\beta^+ t$. Hence

$$\|Tx\|_\beta^+ \leq \eta_\alpha^+ \|x\|_\beta^+.$$

Thus

$$\|Tx\| \leq \eta\|x\|, \text{ for all } x \in X.$$

Then $T : (X,\|.\|) \longrightarrow (Y,\|.\|)$ is fuzzy bounded. □

Theorem 20. *Let $(X, \|.\|)$ and $(Y, \|.\|)$ be fuzzy normed linear spaces such that $\|.\|$ satisfying $(F5)$ and $T : (X, \|.\|) \longrightarrow (Y, \|.\|)$ a fuzzy bounded linear operator. Furthermore, let (X, N, N^*) and (Y, N, N^*) be fuzzy normed linear spaces defined in Theorem 4. Then $T : (X, N, N^*) \longrightarrow (Y, N, N^*)$ is weakly fuzzy bounded.*

Proof. $T : (X, \|.\|) \longrightarrow (Y, \|.\|)$ is a fuzzy bounded linear operator. Then there exists a fuzzy real number $\eta > 0$ such that

$$\|Tx\| \leq \eta\|x\|, \text{ for all } x \in X.$$

Let $\alpha \in (0, 1)$ and $N(x, t) \geq \alpha$. Then $\|x\|_\alpha^- \leq t$, so $\eta_\alpha^- \|x\|_\alpha^- \leq \eta_\alpha^- t$ and hence $\|Tx\|_\alpha^- \leq \eta_\alpha^- t$. Thus $N(Tx, \eta_\alpha^- t) \geq \alpha$.
Let $N^*(x, t/\eta_\alpha^+) \leq \alpha$. Then there is $\beta_n \in (0, 1]$ such that $\beta_n < \alpha + 1/n$ and $\|x\|_{\beta_n}^+ \leq t/\eta_\alpha^+$, so $\eta_\alpha^+ \|x\|_\alpha^+ \leq t$ and hence $\|Tx\|_\alpha^+ \leq t$.
Thus $N^*(Tx, t) \leq \alpha$. So $T : (X, N, N^*) \longrightarrow (Y, N, N^*)$ is weakly fuzzy bounded. □

Theorem 21. *Let (X, N, N^*) and (Y, N, N^*) be fuzzy normed linear spaces such that N and N^* satisfying (N^*7), $(N7)$ and conditions ((i)-(v)) of Theorem 4. Suppose that X is a finite dimensional vector space. Then $T : (X, N, N^*) \longrightarrow (Y, N, N^*)$ is a weakly fuzzy bounded linear operator.*

Proof. Let $(X, \|.\|)$ and $(Y, \|.\|)$ be fuzzy normed linear spaces defined in Theorem 5. Then by Theorem 10, $T : (X, \|.\|) \longrightarrow (Y, \|.\|)$ is fuzzy bounded. By Theorem 16, fuzzy normed linear spaces X and Y satisfy in condition $(F5)$.
Let (X, N', N'^*) and (Y, N', N'^*) be fuzzy normed linear spaces defined in Theorem 4. Then by Theorem 20, $T : (X, N', N'^*) \longrightarrow (Y, N', N'^*)$ is weakly fuzzy bounded.
By Theorem 15, $N = N'$ and $N^* = N'^*$. Hence $T : (X, N, N^*) \longrightarrow (Y, N, N^*)$ is weakly fuzzy bounded. □

Theorem 22. *Let (X, N, N^*) and (Y, N, N^*) be fuzzy normed linear spaces such that N and N^* satisfying (N^*7), $(N7)$ and conditions ((i)-(v)) of Theorem 4. Furthermore, let*

$$B_N(X, Y) = \{T : X \longrightarrow Y : T \textit{ is a weakly fuzzy bounded linear operator}\}.$$

Then there are fuzzy norm N and fuzzy antinorm N^* on $B_N(X, Y)$.

Proof. Let $T \in B_N(X, Y)$ that is $T : (X, N, N^*) \longrightarrow (Y, N, N^*)$ is a weakly fuzzy bounded linear operator. Suppose that $(X, \|.\|)$ and $(Y, \|.\|)$ be fuzzy

normed linear spaces defined in Theorem 5. Then by Theorem 10, $T : (X, \|.\|) \longrightarrow (Y, \|.\|)$ is fuzzy bounded.
Let $\|.\|$ be a fuzzy norm defined in Definition 16, on $B(X,Y)$. Hence there are fuzzy norm N and fuzzy antinorm N^* on $B_N(X,Y)$ defined by

$$N(T,t) = \begin{cases} \sup\{\alpha \in (0,1] : \|T\|_\alpha^- \leq t\} & , \quad (T,t) \neq (0,0) \\ 0 & , \quad (T,t) = (0,0), \end{cases}$$

and

$$N^*(T,t) = \begin{cases} \inf\{\alpha \in (0,1] : \|T\|_\alpha^+ \leq t\} & , \quad (T,t) \neq (0,0) \\ 1 & , \quad (x,t) = (0,0). \end{cases}$$

□

Theorem 23. *Let (X, N, N^*) and (Y, N, N^*) be fuzzy normed linear spaces. Then $B_N(X,Y)$ is a linear space.*

Proof. Proof is similar to Theorem 5.1. in [4]. □

Theorem 24. *Let (X, N, N^*) be a fuzzy normed linear space such that N, N^* satisfying in conditions ((i)-(v)) of Theorem 4. And Suppose that $\{x_n\} \subseteq X$ is a sequence converges to x in fuzzy normed linear space (X, N, N^*). Furthermore, let $(X, \|.\|)$ be fuzzy normed linear spaces defined in Theorem 5. Then $\{x_n\}$ converges to x in fuzzy normed linear space $(X, \|.\|)$.*

Proof. Let sequence $\{x_n\} \subseteq X$ converges to x in fuzzy normed linear space (X, N, N^*). Then $\lim_{n\to\infty} N(x_n - x, t) = 1$ and $\lim_{n\to\infty} N^*(x_n - x, t) = 0$, for all $t > 0$.
If $x_n \nrightarrow x$ in fuzzy normed linear space $(X, \|.\|)$, then there is $\alpha \in (0,1]$ such that $\|x_n - x\|_\alpha^+ \nrightarrow 0$. Hence there exist $\epsilon > 0$ and subsequence $\{x_{n_k}\}$ such that $\|x_{n_k} - x\|_\alpha^+ \geq \epsilon$ for all $k > 0$.
Let $0 < t_0 < \epsilon$. We have $\lim_{n\to\infty} N^*(x_{n_k} - x, t_0) = 0$. Then there is $n_m > 0$ such that $N^*(x_{n_m} - x, t_0) < \alpha$. So $\|x_{n_m} - x\|_\alpha^+ \leq t_0 < \epsilon$. It is contradiction. Thus $x_n \to x$ in fuzzy normed linear space $(X, \|.\|)$. □

Theorem 25. *Let $(X, \|.\|)$ be a fuzzy normed linear space and suppose that $\{x_n\} \subseteq X$ is a sequence converges to x in fuzzy normed linear space $(X, \|.\|)$. Furthermore, let (X, N, N^*) be a fuzzy normed linear spaces defined in Theorem 4. Then $\{x_n\}$ converges to x in fuzzy normed linear space (X, N, N^*).*

Proof. Let sequence $\{x_n\} \subseteq X$ converges to x in fuzzy normed linear space $(X, \|.\|)$. Then $\|x_n - x\|_\alpha^+ \to 0$ for all $\alpha \in (0, 1]$.
Let $t > 0$ and $\epsilon > 0$ be given. Suppose that $\alpha_0 = min\{1, \epsilon/2\}$. We have $\|x_n - x\|_{\alpha_0}^+ \to 0$. Hence there is $N > 0$ such that $\|x_n - x\|_{\alpha_0}^+ < t$ for all $n \geq N$. Then $N^*(x_n - x, t) \leq \alpha_0 < \epsilon$ for all $n \geq N$. So $\lim_{n\to\infty} N^*(x_n - x, t) = 0$.
Let $t > 0$, and let $\epsilon > 0$ be given. Suppose that $\alpha_0 = 1 - \epsilon/2$. If $\alpha_0 \leq 0$, then $\alpha_0 \leq N(x_n - x, t)$ for all $n > 0$. Hence $1 - \epsilon/2 \leq N(x_n - x, t)$ for all $n > 0$. So $0 \leq 1 - N(x_n - x, t) \leq \epsilon/2 < \epsilon$ for all $n > 0$.
If $\alpha_0 > 0$, we have $\|x_n - x\|_{\alpha_0}^- \to 0$. Hence there is $N > 0$ such that $\|x_n - x\|_{\alpha_0}^- < t$ for all $n \geq N$. Then $N(x_n - x, t) \geq \alpha_0 = 1 - \epsilon/2$ for all $n \geq N$. So $0 < 1 - N(x_n - x, t) \leq \epsilon/2 < \epsilon$ for all $n \geq N$. Thus $\lim_{n\to\infty} N(x_n - x, t) = 1$.
Hence $x_n \to x$ in fuzzy normed linear space (X, N, N^*). □

Theorem 26. *Let (X, N, N^*) and (Y, N, N^*) be fuzzy normed linear spaces such that N and N^* satisfying (N^*7), ($N7$) and conditions ((i)-(v)) of Theorem 4, and (Y, N, N^*) a complete fuzzy normed linear space. Furthermore, let $(B_N(X, Y), N, N^*)$ be a fuzzy normed linear spaces defined in Theorem 22. Then $(B_N(X, Y), N, N^*)$ is a complete fuzzy normed linear space.*

Proof. Suppose that $(X, \|.\|)$ and $(Y, \|.\|)$ be fuzzy normed linear spaces defined in Theorem 5. Let $T \in B_N(X, Y)$ that is $T : (X, N, N^*) \longrightarrow (Y, N, N^*)$ is a weakly fuzzy bounded linear operator. Then by Theorem 10, $T : (X, \|.\|) \longrightarrow (Y, \|.\|)$ is fuzzy bounded. Hence $B(X, Y) \supseteq B_N(X, Y)$.
Let (X, N', N'^*), (Y, N', N'^*) be fuzzy normed linear spaces defined in Theorem 4. Suppose that $T \in B(X, Y)$, then by Theorem 20, $T : (X, N', N'^*) \longrightarrow (Y, N', N'^*)$ is weakly fuzzy bounded.
By Theorem 15, $N = N'$ and $N^* = N'^*$. Hence $T : (X, N, N^*) \longrightarrow (Y, N, N^*)$ is weakly fuzzy bounded, so $T \in B_N(X, Y)$. Thus $B(X, Y) = B_N(X, Y)$.
(Y, N, N^*) is a complete fuzzy normed linear space, then by Theorem 24, $(Y, \|.\|)$ is a complete fuzzy normed linear space. By Theorem 9, $(B(X, Y), \|.\|)$ is a complete fuzzy normed linear space. Then By Theorem 25, $(B_N(X, Y), N, N^*)$ is a complete fuzzy normed linear space. □

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