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Homotopy perturbation method for solving fractional Bratu-type equation

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Abstract. In this paper, the homotopy perturbation method (HPM) is applied to obtain an approximate solution of the fractional Bratu-type equations. The convergence of the method is also studied. The fractional derivatives are described in the modified Riemann-Liouville sense. The results show that the proposed method is very efficient and convenient and can readily be applied to a large class of fractional problems.

Keywords: Fractional Bratu-type problem, Homotopy Perturbation Method, Jumarie's fractional derivative.
AMS Subject Classification: 26A33, 34A08.

1 Introduction

The Bratu-type equation was used to model a combustion problem in a numerical slab [2]. The Bratu's problem is also used in a large variety of applications such as the fuel ignition model, the model of the thermal reaction process, the Chandrasekhar model [4], and many other applications (see [14] and references therein).

Recently, the study of fractional differential equations has been an important topic. Fractional models have been shown by many scientists to adequately describe the operation of variety of physical and biological processes and systems [13, 11, 12]. Consequently, considerable attention has

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been given to the solution of fractional ordinary differential equations, integral equations and fractional partial differential equations of physical interest. Since most fractional differential equations do not have exact analytic solutions, approximation and numerical techniques, therefore, are used extensively.

Adomian decomposition method (ADM) and modified variational iteration method (MVIM) was used by authors for solving fractional Bratu-type equations [6, 7]. The homotopy perturbation method (HPM) proposed by He [8] is an approach which searches for an analytical approximate solution of linear and nonlinear problems [1, 3, 5, 9]. In this paper, we use HPM to construct an approximate solution to the fractional Bratu's initial value problems of the form

$$\begin{cases} D_x^{2\alpha} u + \lambda e^u = 0, \quad 0 < \alpha \le 1, \quad 0 < x < 1, \\ u(0) = u^{(\alpha)}(0) = 0, \quad \lambda \text{ is a constant}, \end{cases} \tag{1}$$

where $D^{2\alpha} = D^\alpha D^\alpha$ and $D_x^\alpha u = \mathrm{d}^\alpha u / \mathrm{d}x^\alpha$ denotes Jumarie's fractional derivation. We also compare the computed solutions by the homotopy perturbation method with those provided by the ADM [6] and MVIM [7].

This paper is organized as follows. In Section 2 we present the basic concepts of fractional derivatives and HPM. Section 3 is devoted to the analysis of HPM to the fractional Bratu-type equations. Three examples are given in Section 4. Finally, the paper is ended with some concluding remarks.

2 Basic definitions of the fractional calculus

In this section, we first give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 1. ([15]) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$, $x \longrightarrow f(x)$, denote a continuous (but not necessarily differentiable) function, and let the partition $h > 0$ in the interval $[0, 1]$. Through the fractional Riemann Liouville integral

$${}_0I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} f(\xi) \mathrm{d}\xi, \quad \alpha > 0, \tag{2}$$

the modified Riemann-Liouville derivative is defined as

$${}_0D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_0^x (x-\xi)^{n-\alpha} (f(\xi) - f(0)) \mathrm{d}\xi, \tag{3}$$

where $x \in [0, 1], n - 1 \le \alpha < n$ and $n \ge 1$.

Jumarie's derivative is defined through the fractional difference

$$\Delta^{\alpha} f(x) = (FW - 1)^{\alpha} f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f[x + (\alpha - k)h], \tag{4}$$

where $FW f(x) = f(x+h)$. Then the fractional derivative is defined as the following limit,

$$f^{(\alpha)}(x) = \lim_{h \to 0} \frac{\Delta^{\alpha} f(x)}{h^{\alpha}}. \tag{5}$$

The proposed modified Riemann-Liouville derivative as shown in Eq. (3) is strictly equivalent to Eq. (5). For more information see [15] and [10].

3 Analysis of HPM to the fractional Bratu-type equations

In this section, we fist present a brief review of HPM, investigate the uniqueness of a solution to Eq. (1) and then present the application of HPM to solve Eq. (1) and its convergence.

3.1 Homotopy perturbation method

Consider the nonlinear differential equations in this form

$$L(u) + N(u) = f(r), \quad r \in \ \Omega, \tag{6}$$

with boundary conditions

$$B\left(u, \frac{\partial u}{\partial \mathbf{n}}\right) = 0, \quad r \in \Gamma.$$

The He's homotopy perturbation technique defines the homotopy $H(v,p) : \Omega \times [0,1] \to \mathbb{R}$ which satisfies

$$H(v,p) = (1-p)\ [L(v) - L(u_0)] + p\ [L(v) + N(v) - f(r)] = 0, \tag{7}$$

where $r \in \Omega$ and $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation which satisfies the boundary conditions. The approximate solution of Eq. (6), therefore, can be readily obtained via

$$u = \lim_{p \to 1} (v_0 + p\ v_1 + p^2 v_2 + \cdots) = v_\circ + v_1 + v_2 + \cdots, \tag{8}$$

by considering its few first terms.

3.2 Uniqueness and convergence

Theorem 1. *(Uniqueness) If Eq. (1) has a solution, then it is unique whenever* $0 < a < 1$*, where*

$$a = \frac{L|\lambda|}{\Gamma(1+\alpha)} x^{\alpha},$$

and L *is a Lipschitz constant.*

Proof. Eq. (1) can be written in the form

$$u(x) = \frac{-\lambda}{\Gamma(1+\alpha)} \int_0^x F(t, u(t))(\mathrm{d}t)^{\alpha},$$

where

$$F(x, u(x)) = \frac{1}{\Gamma(1+\alpha)} \int_0^x e^{u(t)}(\mathrm{d}t)^{\alpha},$$

such that the nonlinear term $F(x, u(x))$ is Lipschitz continuous with

$$\|F(x,u) - F(x,v)\| \leq L\|u - v\|.$$

The Lipschitz constant L can be computed as follows. Using the maximum norm $\|F\| = \max_{0 \leq x \leq 1} |F(x, u(x))|$, we can write

$$\begin{aligned} |e^u - e^v| &\leq \sum_{k=0}^{\infty} \frac{|u^k - v^k|}{k!} \\ &= \sum_{k=1}^{\infty} \frac{1}{k!} |u - v||u^{k-1} + u^{k-2}v + \cdots + uv^{k-2} + v^{k-1}|, \end{aligned}$$

since the series is convergent, $|u^{k-1} + u^{k-2}v + \cdots + uv^{k-2} + v^{k-1}|$ is bounded for every k and we have

$$|u^{k-1} + u^{k-2}v + \cdots + uv^{k-2} + v^{k-1}| \leq M, \quad k = 1, 2, 3, \ldots.$$

Therefore

$$|e^u - e^v| \leq |u - v| M \sum_{k=1}^{\infty} \frac{1}{k!} = M(e-1)|u - v|.$$

Hence we have

$$\begin{aligned} |F(x,u) - F(x,v)| &\leq \frac{1}{\Gamma(1+\alpha)} \int_0^x |e^u - e^v|(\mathrm{d}t)^{\alpha} \\ &\leq \frac{M(e-1)}{\Gamma(1+\alpha)} \int_0^x |u - v|(\mathrm{d}t)^{\alpha}, \end{aligned}$$

So one obtains

$$\begin{aligned}\|F(x,u)-F(x,v)\| &\leq \frac{M(e-1)}{\Gamma(1+\alpha)}\|u-v\|\int_0^x (\mathrm{d}t)^\alpha = \frac{M(e-1)x^\alpha}{\Gamma(1+\alpha)}\|u-v\| \\ &\leq \frac{M(e-1)}{\Gamma(1+\alpha)}\|u-v\|.\end{aligned}$$

Then, we can choose $L = \frac{M(e-1)}{\Gamma(1+\alpha)}$.

Now, let u and u^* be two different solutions for Eq. (1). Then,

$$\begin{aligned}\|u-u^*\| &= \left\|\frac{-\lambda}{\Gamma(1+\alpha)}\int_0^x F(t,u(t))(\mathrm{d}t)^\alpha + \frac{\lambda}{\Gamma(1+\alpha)}\int_0^x F(t,u^*(t))(\mathrm{d}t)^\alpha\right\| \\ &\leq \frac{|\lambda|}{\Gamma(1+\alpha)}\int_0^x \|F(u)-F(u^*)\|(\mathrm{d}t)^\alpha \leq \frac{L|\lambda|}{\Gamma(1+\alpha)}\|u-u^*\|x^\alpha.\end{aligned}$$

This implies that

$$\|u-u^*\|(1-\frac{L|\lambda|}{\Gamma(1+\alpha)}x^\alpha) \leq 0,$$

i.e. $\|u-u^*\|(1-a)\leq 0$ where $a = \frac{L|\lambda|}{\Gamma(1+\alpha)}x^\alpha$. As $0<a<1, \|u-u^*\| = 0$, implies $u = u^*$. □

Lemma 1. *If $f(u(x)) = e^{u(x)}$, $x_0 = 0$ and $F(k)$ is coefficient of Maclaurin series of order fractional of $f(u(x))$, then*

$$F(k) = \begin{cases} e^{U(0)}, & k = 0, \\ \dfrac{\Gamma(\alpha(k-1)+1)}{\Gamma(\alpha k+1)}\displaystyle\sum_{i=1}^{k}\frac{\Gamma(\alpha i+1)}{\Gamma(\alpha(i-1)+1)}U(i)F(k-i), & k\geq 1, \end{cases} \tag{9}$$

where

$$U(i) = \frac{1}{\Gamma(\alpha i+1)}D_x^{\alpha i}u(x)\big|_{x=0} \qquad i = 1,2,\ldots,k.$$

Proof. For $k = 0$, we have $F(0) = e^{u(0)} = e^{U(0)}$. Put

$$f(u(x)) = e^{u(x)} = \sum_{k=0}^{\infty}F(k)x^{k\alpha}, \quad u(x) = \sum_{k=0}^{\infty}U(k)x^{k\alpha}.$$

By differentiation of order α from $f(u(x))$ with respect to x, we get

$$D_x^\alpha f(u(x)) = D_x^\alpha u(x)f(u(x)), \tag{10}$$

where

$$D_x^{\alpha} f(u(x)) = D_x^{\alpha} \sum_{k=0}^{\infty} F(k) x^{k\alpha} = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} F(k+1) x^{k\alpha}, \quad (11)$$

and

$$D_x^{\alpha} u(x) = D_x^{\alpha} \sum_{k=0}^{\infty} U(k) x^{k\alpha} = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} U(k+1) x^{k\alpha}. \quad (12)$$

Substituting Eq. (11) and Eq. (12) in Eq. (10), we deduce

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} F(k+1) x^{k\alpha} = \\ &\qquad \Big(\sum_{k=0}^{\infty} \frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} U(k+1) x^{k\alpha} \Big) \Big(\sum_{k=0}^{\infty} F(k) x^{k\alpha} \Big). \end{aligned} \quad (13)$$

Comparing the terms with the same power of $x^{k\alpha}$, we have

$$\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} F(k+1) = \sum_{i=0}^{k} \frac{\Gamma(\alpha(i+1)+1)}{\Gamma(\alpha i+1)} U(i+1) F(k-i). \quad (14)$$

From Eq. (14), we get

$$F(k+1) = \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)} \sum_{i=0}^{k} \frac{\Gamma(\alpha(i+1)+1)}{\Gamma(\alpha i+1)} U(i+1) F(k-i).$$

Replacing $k+1$ by k and $i+1$ by i, it follows

$$F(k) = \frac{\Gamma(\alpha(k-1)+1)}{\Gamma(\alpha k+1)} \sum_{i=1}^{k} \frac{\Gamma(\alpha i+1)}{\Gamma(\alpha(i-1)+1)} U(i) F(k-i), \quad k \geq 1,$$

which completes the proof. □

We are now ready to present our method to solve Eq. (1). By using HPM for Eq. (1), we obtain

$$(1-p) D_x^{2\alpha} u + p(D_x^{2\alpha} u + \lambda e^u) = 0,$$

or

$$\begin{aligned} &D_x^{2\alpha}(u_0 + pu_1 + p^2 u_2 + \cdots) + \lambda p(1 + \{u_0 + pu_1 + p^2 u_2 + \cdots\} \\ &\qquad + \{u_0 + pu_1 + p^2 u_2 + \cdots\}^2/2! + \cdots) = 0. \end{aligned}$$

Therefore, we find for Eq. (1)

$$u_0(x) = 0, \quad u_1(x) = \frac{(-\lambda)x^{2\alpha}}{\Gamma(\alpha+1)}, \quad u_2(x) = \frac{(-\lambda)^2 x^{4\alpha}}{\Gamma(4\alpha+1)}, \cdots \tag{15}$$

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots = \sum_{k=0}^{\infty} u_k(x).$$

Theorem 2. *(Convergence) If the series* $\sum_{k=0}^{\infty} u_k(x)$ *is convergent to* $s(x)$*, then it must be the exact solution of Eq. (1).*

Proof. By using Lemma 1 and Eq. (15), Eq. (1) can be written as

$$u_k(x) = \begin{cases} 0, & k = 0 \\ U(2k)x^{2k\alpha}, & k \geq 1, \end{cases}$$

where

$$U(2k) = \begin{cases} 0, & k = 0, \\ \frac{-\lambda\Gamma(2\alpha(k-1)+1)}{\Gamma(2k\alpha+1)} F(2(k-1)), & k \geq 1, \end{cases}$$

and from Eq. (9)

$$F(2(k-1))$$

$$= \begin{cases} 1, & k = 1, \\ \frac{\Gamma(\alpha(2k-3)+1)}{\Gamma(2\alpha(k-1)+1)} \sum_{i=1}^{k-1} \frac{\Gamma(2\alpha i+1)}{\Gamma(\alpha(2i-1)+1)} U(2i)F(2(k-i-1)), & k \geq 2. \end{cases}$$

Therefore, for Eq. (1), we can write

$$s(x) = \sum_{k=0}^{\infty} U(2k)x^{2k\alpha},$$

and

$$e^{s(x)} = \sum_{k=0}^{\infty} F(2k)x^{2k\alpha}.$$

Now to complete the proof, we show that $s(x)$ satisfies Eq. (1). Putting $s(x)$ in Eq. (1), results in

$$\begin{aligned} D_x^{2\alpha}s(x)+\lambda e^{s(x)} &= D_x^{2\alpha}\Big(\sum_{k=0}^{\infty}U(2k)x^{2k\alpha}\Big)+\lambda\sum_{k=0}^{\infty}F(2k)x^{2k\alpha} \\ &= \sum_{k=1}^{\infty}U(2k)\frac{\Gamma(2k\alpha+1)}{\Gamma(2\alpha(k-1)+1)}x^{2\alpha(k-1)}+\lambda\sum_{k=0}^{\infty}F(2k)x^{2k\alpha} \\ &= 0, \end{aligned}$$

and this completes the proof. □

4 Numerical examples

In this section we present some numerical results to show the effectiveness of the method. In comparisons, three iterations of MVIM, and approximate solution $u(x) \approx u_0(x) + u_1(x) + u_2(x)$ of ADM and HPM are used.

Example 1. Consider the initial value problem

$$\begin{cases} D_x^{2\alpha}u - 2e^u = 0, \quad 0 < \alpha \le 1, \quad 0 < x < 1, \\ u(0) = u^{(\alpha)}(0) = 0. \end{cases} \tag{16}$$

Here $\lambda = -2$. The exact solution of Eq. (16) in $\alpha = 1$ is $u(x) = -2\ln(\cos x)$. From Eq. (15) the approximate solution of Eq. (16) is

$$u(x) = u_0 + u_1 + u_2 + \cdots = \frac{2}{\Gamma(2\alpha+1)}x^{2\alpha} + \frac{4}{\Gamma(4\alpha+1)}x^{4\alpha} + \cdots .$$

The exact solution for $\alpha = 1$ and the approximate solutions for $\alpha = 0.5, 0.6, \ldots, 1$ are shown in Fig. 1. The comparison of approximate solutions provided by MVIM, ADM and HPM is shown in Fig. 2.

Example 2. Consider the initial value problem

$$\begin{cases} D_x^{2\alpha}u - \pi^2 e^u = 0, \quad 0 < \alpha \le 1, \quad 0 < x < 1, \\ u(0) = 0, \ u^{(\alpha)}(0) = \pi. \end{cases} \tag{17}$$

Here $\lambda = -\pi^2$. The exact solution of Eq. (17) in $\alpha = 1$ is $u(x) = -\ln(1 - \sin \pi x)$. We can take an initial approximation $u_0(x) = \frac{\pi x^\alpha}{\Gamma(\alpha+1)}$. The approximate solution of Eq. (17) obtained by HPM is

$$u(x) = u_0 + u_1 + u_2 + \cdots = \frac{\pi}{\Gamma(\alpha+1)}x^{\alpha} + \frac{\pi^2}{\Gamma(2\alpha+1)}x^{2\alpha} + \cdots , \tag{18}$$

The exact solution for $\alpha = 1$ and the approximate solutions for $\alpha = 0.5, 0.6, \ldots, 1$ are shown in Fig. 3. The comparison of approximate solutions obtained by MVIM, ADM and HPM is shown in Fig. 4.

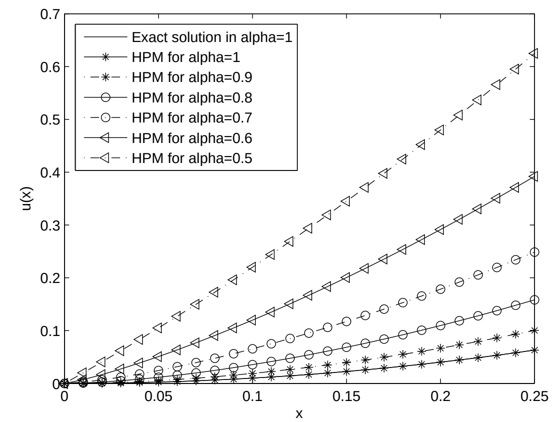

Figure 1: The exact solution in $\alpha = 1$ and the approximate solutions of HPM for Example 1.

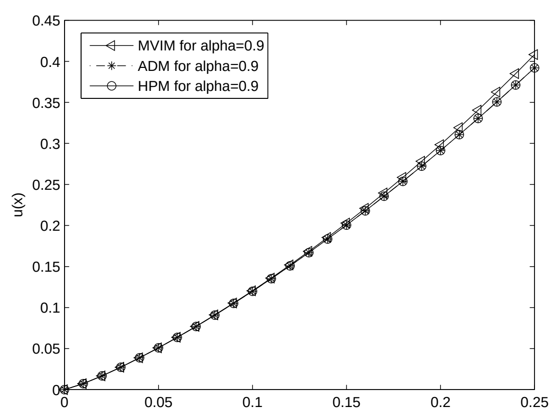

Figure 2: Comparison of the approximate solutions provided by MVIM, ADM and HPM in $\alpha = 0.9$ for Example 1.

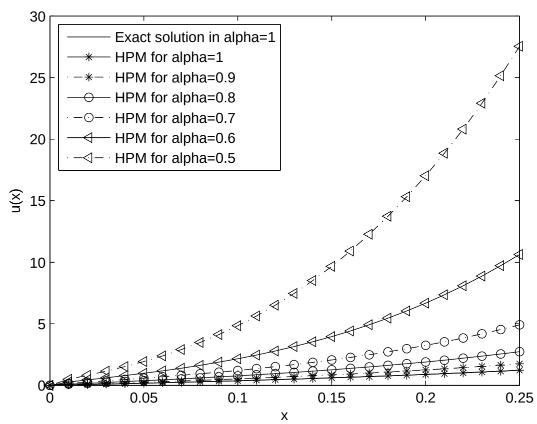

Figure 3: The exact solution in $\alpha = 1$ and approximate solutions of HPM for Example 2.

Example 3. Consider the initial value problem

$$\begin{cases} D_x^{2\alpha} u + \pi^2 e^{-u} = 0, \quad 0 < \alpha \leq 1, \quad 0 < x < 1, \\ u(0) = 0, \ u^{(\alpha)}(0) = \pi. \end{cases} \tag{19}$$

Here $\lambda = \pi^2$. The exact solution of Eq. (19) in $\alpha = 1$ is

$$u(x) = \ln(1 + \sin \pi x).$$

We can take an initial approximation $u_0(x) = \frac{\pi x^\alpha}{\Gamma(\alpha+1)}$. The approximate solution of Eq. (19) by HPM is

$$u(x) = u_0 + u_1 + u_2 + \cdots = \frac{\pi}{\Gamma(\alpha+1)} x^\alpha - \frac{\pi^2}{\Gamma(2\alpha+1)} x^{2\alpha} + \cdots \tag{20}$$

The exact solution for $\alpha = 1$ and approximate solutions for $\alpha = 0.5, 0.6, \ldots, 1$ are shown in Fig. 5. The comparison of the approximate solutions computed by MVIM, ADM and HPM is shown in Fig. 6.

5 Conclusion

From the obtained results it is clear that the homotopy perturbation method suggested in this article provide the solutions in terms of convergent series with easily computable components, so it is an efficient and appropriate method for solving fractional Bratu-type equations.

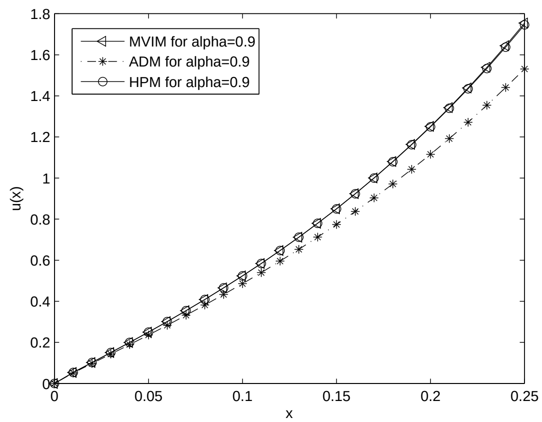

Figure 4: Comparison of approximate solutions MVIM, ADM and HPM in $\alpha = 0.9$ for Example 2.

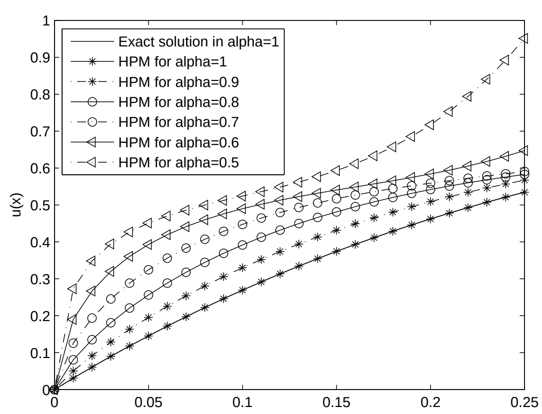

Figure 5: The exact solution in $\alpha = 1$ and approximate solutions of HPM for Example 3.

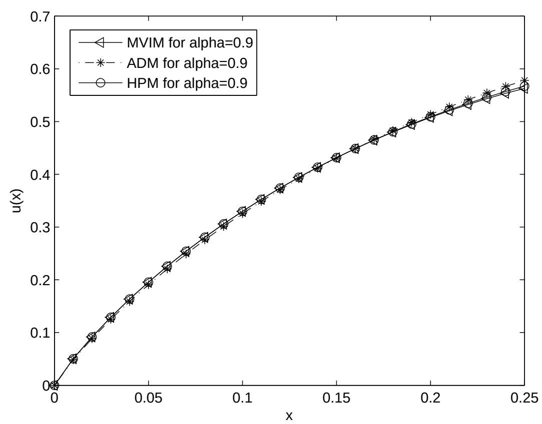

Figure 6: Comparison of approximate solutions obtained by MVIM, ADM and HPM in $\alpha = 0.9$ for Example 3.

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