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On the numerical solution of Urysohn integral equation using Legendre approximation

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Abstract. Urysohn integral equation is one of the most applicable topics in both pure and applied mathematics. The main objective of this paper is to solve the Urysohn type Fredholm integral equation. To do this, we approximate the solution of the problem by substituting a suitable truncated series of the well known Legendre polynomials instead of the known function. After discretization of the problem on the given integral interval, by using the proposed procedure the original integral equation is converted to a linear algebraic system. Now, the solution of the resulting system yields the unknown Legendre coefficients. Finally, two numerical examples are given to show the effectiveness of the proposed method.

Keywords: Fredholm Urysohn integral equations, Legendre collocation matrix method, Legendre polynomials.
AMS Subject Classification: 65A05, 45G10.

1 Introduction

Nonlinear integral equations are encountered in various fields of science and numerous application problems. So the exact solutions of these equations play an important role in the proper understanding of qualitative features of many phenomena and processes in various areas of natural sciences. For

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example, lots of equations of physics, chemistry, and biology contain functions or parameters which are obtained from experiments and hence are not strictly fixed [5]. Therefore, it is expedient to choose the structure of these functions so that it would be easier to analyze and solve the equation. On the other hand, various kinds of nonlinear integral equations usually can not be solved explicitly, so it is required to obtain approximate solutions. Therefore, many different numerical methods have been offered to obtain the solution of these kinds of mathematical problems. Some well known numerical methods are reviewed as follows. In [1], the numerical solution of an integral equation has been derived by using a combination of spline-collocation method and the Legendre interpolation. The Legendre polynomials are mostly used to solve several problems of integral equations. For example, the Legendre pesudo-spectral method is used to solve the delay and the diffusion differential equations (see [1, 2]). In [3] the Chebyshev polynomials are used to introduce an efficient modification of homotopy perturbation method. The main purpose of the present study is to consider the numerical solution of Urysohn integral equation based on the Legendre approximation. Nonlinear integral equations with constant integration limits can be represented in the form

$$F(t, x(t)) = \int_a^b K(t, s, x(s))ds, \quad \alpha \leq t \leq \beta,$$

where $K(t, s, x(s))$ is the kernel of the integral equation, $x(s)$ is the unknown function. Usually all functions in this equation are assumed to be continuous and the case of $\alpha \leq t \leq \beta$ is considered. The above form does not cover all possible forms of nonlinear integral equations with constant integration limits. This kind of nonlinear integral equation with constant limits of integration is called an integral equation of the Urysohn type. If the above integral equation can be rewritten in the form

$$f(x) = \int_a^b K(t, s, x(s))ds,$$

then it is called an Urysohn equation of the first kind. Similarly, the equation

$$x(t) = f(t) + \int_0^1 K(t, s, x(s))ds, \qquad 0 \leq t, s \leq 1, \tag{1}$$

is called an Urysohn equation of the second kind. The main objective of this paper is to solve the Urysohn type Fredholm integral equation Eq. (1). This method is based on replacement of the unknown function by the truncated series of the well known Legendre expansion of functions. The proposed method converts the equation to a matrix equation, by means of collocation points on the interval $[-1, 1]$ which corresponds to system of algebraic equations with Legendre coefficients. Thus, by solving the matrix equation, Legendre coefficients are obtained.

The layout of the article is as follows. In Section 2, we give basic definitions, assumptions and preliminaries of the Legendre polynomials. In Section 3 we introduce our method. Numerical examples are given in Section 4. Finally, concluding remarks are given in Section 5.

2 A brief review of the Legendre polynomials

Orthogonal polynomials are widely used in applications to a variety of fields in mathematics, mathematical physics, engineering and computer science. One of the most common set of this kind of polynomials are the set of Legendre polynomials $P_0(t), P_1(t), \ldots, P_N(t)$ which are orthogonal with respect to the weight function $w(t) = 1$ on $[-1, 1]$. The Legendre polynomials $P_n(t)$ satisfy the Legendre differential equation

$$(1-t^2)u''(t) - 2tu'(t) + n(n+1)u(t) = 0, \quad -1 \le t \le 1, \quad n \ge 0,$$

and are given by the following relation

$$P_n(t) = \frac{1}{2^n} \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k (2n-2k)!}{(n-k)!((n-2k)k)!} t^{n-2k}, \quad n = 0, 1, 2, \ldots. \tag{2}$$

Also, the recurrence formula associated with Legendre polynomials is given by the relations

$$\begin{aligned} P_0(t) &= 1, \\ P_1(t) &= \frac{1}{2}(3t^2 - 1), \\ (n+1)P_{n+1}(t) &= (2n+1)tP_n(t) - nP_{n-1}(t), \quad n \ge 1. \end{aligned} \tag{3}$$

Legendre polynomials occur in the solution of Laplace equation of the potential, $\Delta\Phi(x) = 0$ in a charge-free region of space, using the method of separation of variables, where the boundary conditions have axial symmetry. In fact, the solution is given by

$$\Phi(r, \theta) = \sum_{l=0}^{\infty} [A_l r^l + B_l r^{-(l+1)}] P_l(\cos\theta),$$

where A_l and B_l are to be determined according to the boundary condition of each problem. They also appear when solving Schrodinger equation in three dimensions for a central force.

3 Description of the method

We consider the Fredholm Urysohn integral equation (1). The function $x(t)$ may be expanded by a infinite series of Legendre polynomials as follows

$$x(t) = \sum_{n=0}^{\infty} c_n P_n(t), \tag{4}$$

where $c_n = (x(t), P_n(t))$. We consider a truncated series of Eq. (4), as

$$x_N(t) = \sum_{n=0}^{N} c_n P_n(t) = C^T P(t), \tag{5}$$

where C and P are two vectors given by

$$C = [c_0 \ c_1 \ c_2 \ \dots \ c_N], \quad P(t) = [P_0(t) \ P_1(t) \ P_1(t) \ \dots \ P_N(t)]^T. \tag{6}$$

Then, by substituting the $x_n(t)$ into Eq. (1) we get

$$C^T P(t) = f(t) + \int_0^1 K(t, s, C^T P(t))ds, \tag{7}$$

Now, to use the Legendre collocation method which is a matrix method based on the Legendre collocation points depended by

$$t_j = -1 + \frac{2}{N}, \quad j = 0, 1, 2, \dots, N, \tag{8}$$

we collocate Eq. (7) with the points (8) to obtain

$$C^T P(t_j) = f(t_j) + \int_0^1 K(t_j, s, C^T P(s))ds. \tag{9}$$

The integral terms in Eq. (9) can be found using composite trapezoidal integration technique as

$$\int_0^1 K(t_j, s, C^T P(s))ds \approx \frac{h}{2}\left(\Omega(s_0) + \Omega(s_m) + 2\sum_{k=1}^{m-1} \Omega(s_k)\right), \tag{10}$$

where $\Omega(s) = K(t_j, s, C^T P(s))$ and $h = \frac{1}{m}$, for an arbitrary integer m, $s_i = ih$, $i = 0, 1, \dots, m$. Therefore, Eq. (8) together with Eq. (9) gives an $(N+1) \times (N+1)$ systems of linear or nonlinear algebraic equations, which can be solved for c_k, $k = 0, 1, 2, \dots, N$. Hence, the unknown function $x_N(t)$ can be found.

4 Numerical examples

In this section, two numerical examples are presented based on Legendre approximate method to illustrate the effectiveness of the proposed method. All of the results have been obtained by the MATLAB Software.

Example 1. Consider the following Fredholm Urysohn integral equation (see [4])

$$x(t) = e^{t+1} - \int_0^1 e^{(t-2s)} x^3(s) ds, \tag{11}$$

where $f(t) = e^{t+1}$ and $K(t, s, x(s)) = e^{(t-2s)} x^3(s)$. It is easy to verify that the exact solution of the equation is $x(t) = e^t$. We apply the suggested method with $N = 4$, and approximate the solution $x(t)$ as follows

$$x_4(t) = \sum_{i=0}^{4} c_i P_i(t) = C^T P(t). \tag{12}$$

By the procedure presented in the pervious section and using Eq. (9) we have

$$\sum_{i=0}^{4} c_i P_i(t_j) - e^{t_j+1} - \frac{h}{2}(\Omega(s_0) + \Omega(s_m) + 2\sum_{k=1}^{m-1} \Omega(s_k)) = 0, \quad j = 0, 1, 2, 3, 4, \tag{13}$$

where

$$\begin{aligned} \Omega(s_0) &= e^{t_j-2s_0}(\sum_{i=0}^{4} c_i P_i(s_0))^3, \\ \Omega(s_m) &= (e^{t_j-2s_m})(\sum_{i=0}^{4} c_i P_i(s_m))^3, \\ \Omega(s_k) &= (e^{t_j-2s_k})(\sum_{i=0}^{4} c_i P_i(s_k))^3, \end{aligned}$$

in which $s_{l+1} = s_l + h,\ l = 0, 1, \ldots, m,\ s_0 = 0$ and $h = \frac{1}{m}$. Eq. (14) represents a system of $(N+1)$ nonlinear algebraic equations with unknowns c_i. By using the Newton iterative method and initial guess $c_i = 0$, we obtain

$$c_0 = 0.7733, \quad c_1 = 4.2157, \quad c_2 = -3.7310, \quad c_3 = 1.8874, \quad c_4 = -0.4271.$$

Table 1: Exact and approximate solution and error for Example 1.

Nodes	Exact solution	Approximate solution	Error
0	1	1	8.8818e-015
1	1.1052	1.0953	-0.0098853
2	1.2214	1.2177	0.0037178
3	1.3499	1.3525	0.0026618
4	1.4918	1.4954	0.0035494
5	1.6487	1.6487	1.2212e-014
6	1.8221	1.8186	-0.0035499
7	2.0138	2.0111	-0.0026627
8	2.2255	2.2293	0.0037196
9	2.4596	2.4695	0.0098917
10	2.7183	2.7183	-1.6875e-014

Therefore, the approximation solution of this example is given by

$$\begin{aligned} x_4(t) &= \sum_{n=0}^{4} c_i P_i(t) = 0.7733 P_0(t) + 4.2157 P_1(t) - 3.7310 P_2(t) \\ &+ 1.8874 P_3(t) - 0.4271 P_4(t). \end{aligned}$$

Numerical results are given in Table 1. In this table, the exact and the computed solutions together with the related absolute errors at points $x_i = 0.1i$, $i = 0, 1, \ldots, 10$ have been given. As seen the the computed solution is in good agreement with the exact solution.

The behavior of the approximate solution using the Legendre approximate method with $N = 4$ and exact solution are presented in Fig. 1. It is clear that the proposed method can be considered as an efficient method to solve the linear integral equations.

Example 2. Consider the following integral equation (see [5])

$$x(t) = t^3 - (6 - 2e)e^t + \int_0^1 e^{(t-2s)} x(s) ds, \tag{14}$$

where $f(t) = t^3 - (6 - 2e)e^t$ and $K(t, s, x(s)) = e^{(t-2s)} x(s)$, such that the exact solution of the equation is $x(t) = t^3$. We apply the suggested method with $N = 4$, and approximate the solution $x(t)$ as follows

$$x_4(t) = \sum_{i=0}^{4} c_i P_i(t) = C^T P(t). \tag{15}$$

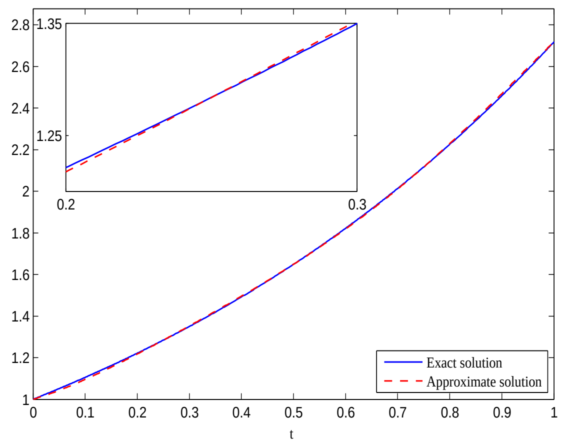

Figure 1: Approximate and exact solution for Example 1.

By the same procedure in the previous section and using Eq. (9) we have

$$\sum_{i=0}^{4} c_i P_i(t_j) - (t_j^3 - (6-2e)e^{t_j}) - \frac{h}{2}(\Omega(s_0) + \Omega(s_m) + 2\sum_{k=1}^{m-1}\Omega(s_k)) = 0, \quad j = 0,1,2,3,4, \tag{16}$$

where

$$\Omega(s_0) = e^{(s_0 - t_j)}(\sum_{i=0}^{4} c_i P_i(s_0))$$

$$\Omega(s_m) = e^{(s_m - t_j)}(\sum_{i=0}^{4} c_i P_i(s_m))$$

$$\Omega(s_k) = e^{(s_k - t_j)}(\sum_{i=0}^{4} c_i P_i(s_k))$$

in which $s_{l+1} = s_l + h$, $l = 0, 1, \ldots m$, $s_0 = 0$ and $h = \frac{1}{m}$. Eq. (14) gives a system of $(N+1)$ nonlinear algebraic equations with unknowns c_i. By using the Newton iterative method and the initial guess $c_i = 0$, we obtain

$$c_0 = 0.05842, \quad c_1 = 1.2089, \quad c_2 = -0.7473, \quad c_3 = 0.8571, \quad c_4 = -0.1342.$$

Therefore, the approximation solution of this example by using

$$x_N(t) = \sum_{n=0}^{N} c_i P_i(t),$$

Table 2: Exact and approximate solution and error for Example 2.

Nodes	Exact solution	Approximate solution	Error
0	0	-5.5511e-017	-5.5511e-017
1	0.001	-0.0020914	-0.0030914
2	0.008	0.0068374	-0.0011626
3	0.027	0.027832	0.00083229
4	0.064	0.06511	0.0011097
5	0.125	0.125	-1.6237e-015
6	0.216	0.21489	-0.0011097
7	0.343	0.34217	-0.00083229
8	0.512	0.51316	0.0011626
9	0.729	0.73209	0.0030914
10	1	1	1.7764e-015

is given by

$$x_4(t) = 0.05842P_0(t) + 1.2089P_1(t) - 0.7473P_2(t) + 0.8571P_3(t) - 0.1342P_4(t).$$

Numerical results are given in Table 2. In this table, the exact and the computed solutions together with the related absolute errors at points $x_i = 0.1i$, $i = 0, 1, \ldots, 10$ have been given. As observed the method provides a suitable solution to the problem. The behavior of the approximate solution using the Legendre approximate method with $N = 4$ and exact solution are presented in Fig. 2.

5 Conclusion

An approximate method for the solution of linear and nonlinear Fredholm Urysohn integral equations in the most general form has been proposed and investigated. A comparison of the exact and computed solutions reveals that the presented method is effective and convenient. The numerical results show that the accuracy can be improved by increasing N.

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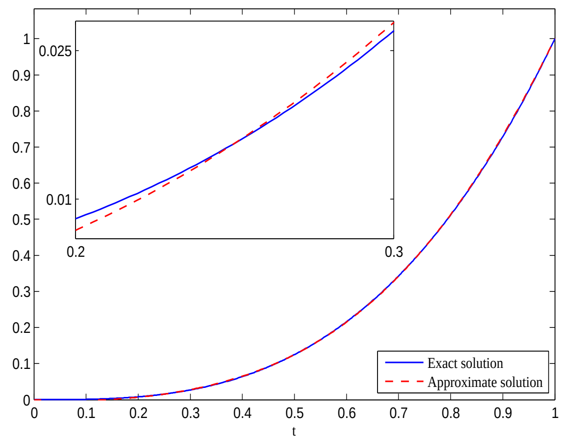

Figure 2: Approximate and exact solution for Example 2.

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