Journal of Mathematical Modeling
Vol. 11, No. 4, 2023, pp. 745–766. Research Article

A second order numerical method for two-parameter singularly perturbed time-delay parabolic problems

Mekashaw Ali Mohye[†], **Justin B. Munyakazi**[‡*], **Tekle Gemechu Dinka**[†]

[†] *Department of Applied Mathematics, Adama Science and Technology University, Ethiopia*
[‡] *Department of Mathematics and Applied Mathematics, University of the Western Cape, South Africa*
Email(s): alimekashaw12@gmail.com, jmunyakazi@uwc.ac.za, tekgem@yahoo.com

Abstract. In this article, a time delay parabolic convection-reaction-diffusion singularly perturbed problem with two small parameters is considered. We investigate the layer behavior of the solution for both smooth and non-smooth data. A numerical method to solve the problems described is developed using the Crank-Nicolson scheme to discretize the time-variable on a uniform mesh while a hybrid finite difference is applied for the space-variable. The hybrid scheme is a combination of the central, upwind and mid-point differencing on a piecewise uniform mesh of Shishkin type. The convergence analysis shows that the proposed method is uniformly convergent of second order in both space and time. Numerical experiments conducted on some test examples confirm the theoretical results.

Keywords: Singular perturbation, delay differential equation, Shishkin mesh, hybrid scheme, boundary layers, fitted mesh finite difference method, uniform convergence.
AMS Subject Classification 2010: 34A34, 65L05.

1 Introduction

Singularly perturbed differential equations are those in which a small positive number, known as the perturbation parameter, is multiplied with the highest-order derivative term of the differential equation. Many real-world models are represented by parameter-dependent differential equations, the performance of which is determined on the size of the parameter.

Singularly Perturbed Problems (SPPs) arise in the modelling of fluid dynamics, elasticity, quantum mechanics, reaction-diffusion processes, chemical-reactor theory, plasma dynamics, meteorology, diffraction theory, aerodynamics, semi-conductor modelling, hydrodynamics, and a variety of other areas [5,48]. Singular perturbations were first described by Prandtl [41] in a seven-page report presented at

*Corresponding author
Received: 20 April 2023 / Revised: 09 August 2023 / Accepted: 06 September 2023
DOI: 10.22124/jmm.2023.24353.2182

the Third International Congress of Mathematicians in Heidelberg in 1904. However, the term singular perturbations was first used by Friedrichs and Wasow [17] in a paper presented at a seminar on non-linear vibrations at New-York University. In such problems, typically there are thin transition layers where the solution varies rapidly or jumps abruptly, while away from the layers the solution behaves regularly and varies slowly.

To describe the dynamics of various biological systems, many singularly perturbed diffusive models have been developed. A small diffusion parameter is found in many real life applications, for example see [34] where it is pointed out that in blood, haemoglobin molecules have a diffusion coefficient of the order of $10^{-11}m^2/s$ while that for oxygen in blood is of the order of $10^{-9}m^2/s$.

Two most popular approaches for solving SPPs are the numerical methods and the asymptotic ones. The asymptotic technique helps gain insight into the qualitative behavior of the problem and/or its solution and provides only a semi-quantitative information whereas the numerical approach provides quantitative information about a specific member of the family of solutions. In this study, we consider the time-delay parabolic convection-reaction-diffusion singularly perturbed problems with two small perturbation parameters.

We define the domain $\bar{\Lambda} = \Lambda \cup \partial\Lambda$ where $\Lambda = (0,\ 1) \times (0,T)$ and $\partial\Lambda = L_l \cup L_d \cup L_r$ with $L_d = [0,1] \times [-\gamma, 0], L_l = \{0\} \times (0,T]$ and $L_r = \{1\} \times (0,T]$.

The governing equation is the singularly perturbed problem given by

$$\begin{aligned} \mathfrak{L}u(x,\ t) &\equiv \varepsilon u_{xx}(x,\ t) + \mu a(x,t)u_x(x,\ t) - b(x,t)u(x,t) - u_t(x,\ t) \\ &= -c(x,\ t)u(x,\ t-\gamma) + f(x,t); \quad (x,\ t) \in \Lambda \end{aligned} \tag{1}$$

with the following conditions

$$\begin{aligned} &u(x,\ t) = \Phi(x,\ t), \ \text{ for } \ (x,\ t) \in [0,\ 1] \times [-\gamma,\ 0] \\ &u(0,\ t) = \Phi_l(t), \ \ u(1,\ t) = \Phi_r(t) \ \text{ for } \ t \in [0,\ T], \end{aligned} \tag{2}$$

where $0 < \varepsilon \leq 1$ and $0 \leq \mu \leq 1$ are perturbation parameters and γ is a delay parameter.

The coefficients $a(x,\ t)$, $b(x,\ t)$, $c(x,\ t)$, $f(x,\ t)$ for $(x,\ t) \in \overline{\Lambda}$ and $\Phi_l(t)$, $\Phi_r(t)$ and $\Phi_b(x,t)$ are assumed to be sufficiently regular such that $a(x,t) \geq \alpha > 0$, $b(x,t) \geq \beta > 0$, $c(x,\ t) \geq \Upsilon > 0$ for $(x,\ t) \in \overline{\Lambda}$. The regularity and compatibility (at the corners) are

$$\begin{aligned} &u(0,\ 0) = \Phi_l(0), \quad u(1,\ 0) = \Phi_r(0), \quad u(0,-\gamma) = \Phi_l(\gamma), \quad u(1,-\gamma) = \Phi_r(\gamma), \\ &\varepsilon(\Phi_b)_{xx}(0,\ 0) + \mu a(0,0)(\Phi_b)_x(0,\ 0) - b(0,0)(\Phi_b)(0,0) - (\Phi_b)_t(0,\ 0) \\ &\qquad = -c(0,\ 0)(\Phi_b)(0,-\gamma) + f(0,0), \\ &\varepsilon(\Phi_b)_{xx}(1,\ 0) + \mu a(1,0)(\Phi_b)_x(1,\ 0) - b(1,0)(\Phi_b)(1,0) - (\Phi_b)_t(1,\ 0) \\ &\qquad = -c(1,\ 0)(\Phi_b)(1,-\gamma) + f(1,0), \end{aligned}$$

for $\Lambda = (0,\ 1) \times (0,T]$, and so that the data matches at the two corners $(0,0)$ and $(1,0)$.

Further, we assume that the convection coefficient term $a(x,t)$ and the source term $f(x,t)$ have jump (discontinuity) at a point (ξ, ζ) with the bound $|[a](\xi,\zeta)| \leq C$ and $|[f](\xi,\zeta)| \leq C$. In general, due to the presence of a discontinuity in the convection term and source term, problem (1) possesses an interior layer in the neighborhood of the point of discontinuity of the solution $u(x,t)$ [5]. Again, problem (1) exhibits boundary layer(s) near the boundary region due to the presence of small perturbation parameters, ε and μ. If all of the above assumptions are fulfilled, then the presented problem in (1) has a unique solution.

As the parameters ε and μ tend to zero, the solution of governing problem forms boundary layers at $x = 0$ and $x = 1$. When the parameter $\mu = 1$, the given problem in (1) is a convection-diffusion problem [26, 42, 47] and in this case a boundary layer of width $O(\varepsilon)$ will appear in the neighborhood of the edge $x = 0$. Also, when $\mu = 0$, we have a parabolic reaction-diffusion problem [32] and thin boundary layers of width $O(\sqrt{\varepsilon})$ appear in the neighborhood of both $x = 0$ and $x = 1$.

The nature of singularly perturbed two-parameter problems changes according to the values of perturbation parameters ε and μ ; from convection-reaction-diffusion equation to reaction-diffusion when $\mu = 0$ and to convection-diffusion equation when $\mu = 1$. OMalley [36] introduced singularly perturbed two-parameter problems and examined asymptotic expansion for their solutions. He [36, 37] identified that the nature of these problems is quite affected by the choice of the ratio of μ^2 to ε.

Several numerical methods were developed to improve the accuracy of the asymptotic methods proposed by OMalley and his co-researchers. Two-parameter singularly perturbed problems were numerically solved using nonstandard finite difference method for the stationary case in [40] and the time-dependent case in [33]. Other works on numerical solutions of two-parameter singularly perturbed ordinary differential equations with smooth data include [7–12, 19, 22, 23, 25, 30, 38, 46] and [6, 43] for the non-smooth data.

In the past few years, singularly perturbed delay differential equations also attracted numerical analysts [1, 2, 4, 13, 16, 18, 21, 24, 28, 29, 39, 44, 45] those developed several competitive numerical methods.

An almost second order convergence is obtained in [20] using the finite difference scheme on Bakhvalov-type mesh for the singularly perturbed pseudo-parabolic problems with time-delay. A parameter-uniform exponentially fitted method for two-parameter singularly perturbed parabolic problems was developed in [3]. The authors in [3], proposed that their method resolve the two lateral boundary layers of the solution. The physical and numerical aspects of two-parameter SPPs with Robin boundary conditions was explored in [27] which presents a highly-accurate wavelet-based approximation. Recently, [15, 35] developed exponentially fitted scheme.

The relative sizes of the parameters pose a challenge to the development of a reliable numerical method for (1)-(2). The challenge is more serious when the time-delay aspect is included or the data is non-smooth.

From existing literature, developing a parameter uniformly numerical method for two-parameter SPPs is still open for further exploration. In this study, we develop a parameter uniformly convergent numerical scheme to treat a class of second order two-parameter singularly perturbed time dependent problems with time delay. The proposed scheme is a hybrid method for the spatial variable, which blends central, upwind, and mid-point differencing on a piecewise uniform mesh of Shishkin type and an implicit scheme which uses a Crank-Nicolson discretization in time direction. The parameter uniformly convergence analysis of our scheme is proved. The developed scheme requires a priori knowledge about the position and width of the solution boundary layer(s), and the objective of employing this method is to establish a more refined mesh in the layer(s) region.

The rest of this article is organized as follows. First, we discuss the qualitative properties such as the bounds of the solution $u(x,t)$ and its derivative bounds in Section 2. The numerical part of the continuous problem is presented in Section 3. In this section, we also discuss the discretization of time and space domains and the continuous problem. The bounds of the discrete solution is presented in Section 4. We provide numerical examples to confirm the convergence and accuracy of the proposed method in Section 5. In this section, we show that the convergence does not depend on the magnitude of the perturbation parameters. In Section 6, we present the result and conclusions.

Notations: Within this work, the maximum norm is denoted by $\|.\|_\Lambda$, where Λ is any bounded and closed subset of $[0,1]\times[0,T]$. The value C which we have used throughout this paper is a generic positive constant, and it is independent of the perturbation parameters, ε and μ. Further, we denote $\eta = \min_{(x,t)\in\bar{\Lambda}}\{\frac{b(x,t)}{a(x,t)}\} \geq \frac{\beta}{\alpha} > 0$.

2 Qualitative properties of the continuous problem

In this section, we discuss the existence and uniqueness, boundedness of analytical solution and its derivatives also.

Lemma 1 (**The continuous minimum principle** [45]). *Assume* $\varphi(x,t)\in C^{2,1}\bar{\Lambda}$. *If* $\varphi|_{\partial\Lambda}\geq 0$ *and* $\left(\mathfrak{L}_{\varepsilon,\mu}-\frac{\partial}{\partial t}\right)\varphi|_\Lambda \leq 0$, *then* $\varphi|_{\bar{\Lambda}}\geq 0$.

Proof. Let $(x^\star,t^\star)$ be an arbitrary point in a plane, $\Lambda=(0,1)\times(0,T)$ such that $\varphi(x^\star,t^\star)=\min\{\varphi(x,t)\}_{(x^\star,t^\star)\in\bar{\Lambda}}$ and again suppose that $\varphi(x^\star,t^\star)<0$. Clearly, $(x^\star,t^\star)\notin\{0,1\}\times\{0,T\}$ and from the definition of $(x^\star,t^\star)$, we have, $\varphi_{xx}(x^\star,t^\star)\geq 0$, $\nabla\varphi_x(x^\star,t^\star)=0$, $\nabla\varphi_t(x^\star,t^\star)=0$ (Using first and second derivative test for multi-variable functions). Then

$$\left(\mathfrak{L}_{\varepsilon,\mu}-\frac{\partial}{\partial t}\right)\varphi|_\Omega = \underbrace{\varepsilon\varphi_{xx}(x^\star,\ t^\star)}_{\geq 0}+\underbrace{\mu a(x^\star,\ t^\star)\nabla_x\varphi(x^\star,t^\star)}_{=0}\underbrace{-b(x^\star,t^\star)\varphi(x^\star,\ t^\star)}_{\geq 0}\underbrace{-\nabla_t\varphi(x^\star,t^\star)}_{=0}\geq 0,$$

which is a contradiction. So, $\varphi(x^\star,t^\star)|_{\bar{\Lambda}}\geq 0$. Since, $(x^\star,t^\star)$ is an arbitrary point, we have then $\varphi(x,t)\geq 0$ for all $(x,t)\ \in\bar{\Lambda}$. □

Lemma 2 (**Bound of the continuous problem and its derivatives**). *Let* u *be the solution of problem (1)-(2). And suppose* $u=v+w_L+w_R$ *where* v *is the regular component and* w_L *and* w_R *are the left and right singular components, respectively [45]. Again, let* C *be a sufficiently large constant which is independent of the perturbation parameters. Then*

(a) $\|u\|\leq C$

(b) For all non-negative integers i *and* j $(0\leq i+2j\leq 4)$, *the derivatives of the solution* u *of problem (1)- (2) satisfy*

$$\left\|\frac{\partial^{i+j}u}{\partial x^i\partial t^j}\right\|\leq\begin{cases} C\dfrac{1}{(\sqrt{\varepsilon})^i}, & when\ \ \alpha\mu^2\leq\eta\varepsilon,\\ C\left(\dfrac{\mu}{\varepsilon}\right)^i\left(\dfrac{\mu^2}{\varepsilon}\right)^j, & when\ \ \alpha\mu^2\geq\eta\varepsilon,\end{cases}$$

(c) $|w_L(x,t)|\leq Ce^{-\theta_L x}$, $\quad |w_R(x,t)|\leq Ce^{-\theta_R(1-x)}$, *where*

$$\theta_L=\begin{cases}\frac{\sqrt{\eta\alpha}}{\sqrt{\varepsilon}}, & \alpha\mu^2\leq\eta\varepsilon,\\ \frac{\alpha\mu}{\varepsilon}, & \alpha\mu^2\geq\eta\varepsilon,\end{cases}\qquad \theta_R=\begin{cases}\frac{\sqrt{\eta a}}{2\sqrt{\varepsilon}}, & \alpha\mu^2\leq\eta\varepsilon,\\ \frac{\eta}{2\mu}, & \alpha\mu^2\geq\eta\varepsilon.\end{cases}$$

Proof. For the proof of this lemma, one may refer [45]. □

Theorem 1 (**Derivative bounds of the continuous problem**). *For nonnegative integers $i, j \in \mathbb{N}$ satisfying $0 \le i+2j \le 4$, derivative bounds of u are given by [21]:*

$$\left\| \frac{\partial^{i+j} v}{\partial x^i \partial t^j} \right\| \le \begin{cases} C, & \text{when } \alpha\mu^2 \le \eta\varepsilon, \\ C\left(1+\left(\frac{\varepsilon}{\mu}\right)^{3-i}\left(\frac{\mu^2}{\varepsilon}\right)^{j}\right), & \text{when } \alpha\mu^2 \ge \eta\varepsilon. \end{cases}$$

Proof. The details of the proof is in [39, 45]. □

3 Derivation of the numerical Scheme

This section is concerned with the derivation of the proposed numerical scheme, which solves the governing problem. We divide the time domain $[0,\ T]$ using a uniform mesh. Again, we have chosen γ in such a way that $T = k\gamma$ for some positive integer $k > 1$. Also, let Λ^M be the collection of all mesh points in $[0,\ T]$ and Λ^m_γ be the collection of all mesh points $[-\gamma,\ 0]$ as follows:

$$\Lambda^M = \{t_j = j\triangle t,\ \ j = 0,1,2,\ldots,M,\ t_M = T,\ \triangle t = \frac{T}{M}\},$$
$$\Lambda^m_\gamma = \{t_j = j\triangle t,\ \ j = 0,1,2,\ldots,m,\ t_m = \gamma,\ \triangle t = \frac{\gamma}{m}\},$$

where M and m are the number of mesh points in time direction in the interval $[0,\ T]$ and $[-\gamma,\ 0]$, respectively.

To define the mesh of the space variable, we consider two transition parameters τ_1 and τ_2 as follows:

$$\tau_1 = \begin{cases} \min\{\frac{1}{4}, \frac{4\sqrt{\varepsilon}}{\sqrt{\eta\alpha}}\ln N\},\ \text{for } \mu^2 \le \frac{\eta\varepsilon}{\alpha}, \\ \min\{\frac{1}{4}, \frac{4\varepsilon}{\eta\alpha}\ln N\},\ \text{for } \mu^2 \ge \frac{\eta\varepsilon}{\alpha}, \end{cases} \qquad \tau_2 = \begin{cases} \min\{\frac{1}{4}, \frac{4\sqrt{\varepsilon}}{\sqrt{\eta\alpha}}\ln N\},\ \text{for } \mu^2 \le \frac{\eta\varepsilon}{\alpha}, \\ \min\{\frac{1}{4}, \frac{4\mu}{\eta}\ln N\},\ \text{for } \mu^2 \ge \frac{\eta\varepsilon}{\alpha}, \end{cases}$$

such that $0 < \tau_1 < \tau_2 < 1$. Then, the interval $[0,1]$ is divided into three subintervals $[0,\ \tau_1], [\tau_1, 1-\tau_2]$ and $[1-\tau_2, 1]$ and

(i) the interval $[0,\ \tau_1]$ is divided into $\frac{N}{4}$ with the mesh spacing is $h_L = x_i - x_{i-1} = \frac{4\tau_1}{N}$,

(ii) the interval $[\tau_1, 1-\tau_2]$ is divided into $\frac{N}{2}$ with the mesh spacing $h_M = x_i - x_{i-1} = \frac{2(1-\tau_1-\tau_2)}{(N)}$,

(iii) the interval $[1-\tau_2, 1]$ is divided into $\frac{N}{4}$ with the mesh spacing $h_R = x_i - x_{i-1} = \frac{4\tau_2}{N}$.

The mesh points in space are given by $\bar{\Lambda}^N = x_i,\ \ i = 0,1,\ldots,N$, where

$$x_i = \begin{cases} ih_L, & 0 \le i \le N/4, \\ \tau_1 + (i - N/4)h_M, & N/4 < i \le 3N/4, \\ (1-\tau_2) + (i - 3N/4)h_R, & 3N/4 < i \le N. \end{cases}$$

We also define $h_i = x_i - x_{i-1},\ i = 1,\dots,N$ and $\hbar_i = \dfrac{h_i + h_{i+1}}{2},\ i = 1,\dots,N-1$.

After generating the rectangular grid, we develop a finite difference method that uses a hybrid scheme comprised of the Crank-Nicolson discretization of the time variable and a combination of the central difference, upwind, and mid-point schemes in spatial direction. The Taylor's series expansion was applied for $U_t(x_i,t_j)$ at a point $(x_i,t_{j+1/2})$ to derive the Crank-Nicolson scheme in temporal direction as follows:

$$U_i^{j+1} = U_i^{j+1/2} + \frac{\Delta t}{2}\frac{\partial U_i^{j+1/2}}{\partial t} + \left(\frac{\Delta t}{2}\right)^2 \frac{1}{2!}\frac{\partial^2 U_i^{j+1/2}}{\partial t^2} + \left(\frac{\Delta t}{2}\right)^3 \frac{1}{3!}\frac{\partial^3 U_i^{j+1/2}}{\partial t^3} + \cdots, \tag{3}$$

$$U_i^{j} = U_i^{j+1/2} - \frac{\Delta t}{2}\frac{\partial U_i^{j+1/2}}{\partial t} + \left(\frac{\Delta t}{2}\right)^2 \frac{1}{2!}\frac{\partial^2 U_i^{j+1/2}}{\partial t^2} - \left(\frac{\Delta t}{2}\right)^3 \frac{1}{3!}\frac{\partial^3 U_i^{j+1/2}}{\partial t^3} + \cdots. \tag{4}$$

By subtract Eq. 3 from Eq. 4, we eliminate the term $U_i^{j+1/2}$ and then we get

$$\frac{\partial U_i^{j+1/2}}{\partial t} = \frac{U_i^{j+1} - U_i^{j}}{\Delta t} + T_i^{j+1/2},$$

where the term $T_i^{j+1/2}$ is the local truncation error of the scheme given by

$$T^{j+1/2}(x) = \frac{(\Delta t)^3}{24}\frac{\partial^3 U^{j+1/2}(x)}{\partial t^3} + H.O.Ts \text{ (higher order terms)}, \tag{5}$$

and its order is three. Now, using the above discretizations in time direction, we can write the hybrid scheme as follows:

$$\begin{aligned}
\left[L_{\text{cen}}^{N,M}U\right]_i^j &= \frac{1}{2}\left[\varepsilon\delta_x^2 U_i^{j+1} + \mu a_i^{j+1} D_x^0 U_i^{j+1} - b_i^{j+1}U_i^{j+1} + \varepsilon\delta_x^2 U_i^j + \mu a_i^j D_x^0 U_i^j - b_i^j U_i^j\right] - D_t U_i^{j+1/2},\\
\left[L_{\text{up}}^{N,M}U\right]_i^j &= \frac{1}{2}\left[\varepsilon\delta_x^2 U_i^{j+1} + \mu a_i^{j+1} D_x^+ U_i^{j+1} - b_i^{j+1}U_i^{j+1} + \varepsilon\delta_x^2 U_i^j + \mu a_i^j D_x^+ U_i^j - b_i^j U_i^j\right] - D_t U_i^{j+1/2},\\
\left[L_{mp}^{N,M}U\right]_i^j &= \frac{1}{2}\left[\varepsilon\delta_x^2 U_i^{j+1} + \mu \bar{a}_i^{j+1} D_x^+ U_i^{j+1} - \overline{b_i^{j+1}U_i^{j+1}} + \varepsilon\delta_x^2 U_i^j + \mu \bar{a}_i^j D_x^+ U_i^j - \overline{b_i^j U_i^j}\right] - \overline{D_t U_i^{j+1/2}},
\end{aligned} \tag{6}$$

where

$$D_x^+ U_i^j = \frac{U_{i+1}^j - U_i^j}{h_{i+1}}, \quad D_x^0 U_i^j = \frac{U_{i+1}^j - U_{i-1}^j}{2\hbar_i}, \quad D_t U_i^{j+1/2} = \frac{U_i^{j+1} - U_i^j}{\Delta t},$$

$$\delta_x^2 U_i^j = \frac{1}{\hbar_i}\left(\frac{U_{i+1}^j - U_i^j}{h_{i+1}} - \frac{U_i^j - U_{i-1}^j}{h_i}\right), \qquad \overline{U_i^j} = \frac{U_i^j + U_{i+1}^j}{2}.$$

Further, define $\hat{b} = b + \dfrac{1}{\Delta t}$. Then the discretized form of problem (6) is of the form

$$\begin{aligned}
L^{N,M} &= \overset{+}{\sigma}_i U_{i+1}^j + \overset{c}{\sigma}_i U_i^j + \overset{-}{\sigma}_i U_{i-1}^j + \overset{++}{\sigma}_i U_{i+1}^{j+1} + \overset{cc}{\sigma}_i U_i^{j+1} + \overset{--}{\sigma}_i U_{i-1}^{j+1} = F_i^j, \quad (x_i,t_j) \in \Lambda^{N,M},\\
U_i^j &= u(x_i,t_j), \quad (x_i,\ t_j) \in \partial\Lambda^{N,M},
\end{aligned} \tag{7}$$

where

$$[L^{N,M}U]_i^j = \begin{cases} \left[L_{\text{cen}}^{N,M}U\right]_i^j, & \text{if } 1 \le i < N/4, \\ \left[L_{\text{cen}}^{N,M}U\right]_i^j, & \text{if } N/4 < i < 3N/4 \text{ and } \mu h_M \|a\| < 2\varepsilon, \\ \left[L_{\text{mp}}^{N,M}U\right]_i^j, & \text{if } N/4 < i < 3N/4, \mu h_M \|a\| \ge 2\varepsilon \text{ and } h_M \|\hat{b}\| < 2\mu\alpha, \\ \left[L_{\text{up}}^{N,M}U\right]_i^j, & \text{if } N/4 < i < 3N/4, \mu h_M \|a\| \ge 2\varepsilon \text{ and } h_M \|\hat{b}\| \ge 2\mu\alpha, \\ \left[L_{\text{cen}}^{N,M}U\right]_i^j, & \text{if } 3N/4 < i < N \text{ and } \mu h_R \|a\| < 2\varepsilon, \\ \left[L_{\text{mp}}^{N,M}U\right]_i^j, & \text{if } 3N/4 < i < N \text{ and } \mu h_R \|a\| \ge 2\varepsilon. \end{cases} \tag{8}$$

If the left transition point $x_i = \tau_1 = 0.25$, the left layer region will be $[0,\ 0.25]$. The hybrid scheme at $x_i = \tau_1$ is

$$[L^{N,M}U]_i^j = \begin{cases} \left[L_{\text{cen}}^{N,M}U\right]_i^j, & \text{if } \tau_1 = 0.25, \\ \left[L_{\text{mp}}^{N,M}U\right]_i^j, & \text{if } \tau_1 < 0.25 \text{ and } h_M \|\hat{b}\| < 2\mu\alpha, \\ \left[L_{\text{up}}^{N,M}U\right]_i^j, & \text{otherwise.} \end{cases} \tag{9}$$

If the right transition point $x_i = \tau_2$ such that $1-\tau_2 = 0.75$, the right layer region will be $[1-\tau_2, 1] = [0.75, 1]$. Then, at $x_i = 1-\tau_2 = 0.75$ we have the following hybrid scheme

$$[L^{N,M}U]_i^j = \begin{cases} \left[L_{\text{cen}}^{N,M}U\right]_i^j, & \text{if } 1-\tau_2 = 0.75, \text{ and } \mu h_R \|a\| < 2\varepsilon, \\ \left[L_{\text{mp}}^{N,M}U\right]_i^j, & \text{if } 1-\tau_2 = 0.75, \text{ and } \mu h_R \|a\| \ge 2\varepsilon, \\ \left[L_{\text{mp}}^{N,M}U\right]_i^j, & \text{if } 1-\tau_2 > 0.75, \text{ and } h_R \|\hat{b}\| < 2\mu\alpha, \\ \left[L_{\text{up}}^{N,M}U\right]_i^j, & \text{otherwise,} \end{cases} \tag{10}$$

and

$$F_i^j = \begin{cases} \dfrac{\Delta t}{2}\left[-c_i^{j+1}U_i^{j-m+1} + f_i^{j+1} + -c_i^j U_i^{j-m} + f_i^j\right], & \text{if } L^{N,M} = L_{\text{cen}}^{N,M} \text{ or } L_{up}^{N,M}, \\ \dfrac{\Delta t}{2}\left[\overline{-c_i^{j+1}U_i^{j-m+1}} + \overline{f_i^{j+1}} + \overline{-c_i^j U_i^{j-m}} + \overline{f_i^j}\right], & \text{if } L^{N,M} = L_{\text{mp}}^{N,M}. \end{cases} \tag{11}$$

with

$$-c_i^j U_i^{j-m} = \Big\{ \underbrace{-c_i^j \Phi_i^{j-m}}_{j\in[0,\ m]} \ \ \textbf{or} \ \ \underbrace{-c_i^j U_i^{j-m}}_{j\in[m+1,\ M]},$$

$$\overline{-c_i^j U_i^{j-m}} = \Big\{ \underbrace{\overline{-c_i^j \Phi_i^{j-m}}}_{j\in[0,\ m]} \ \ \textbf{or} \ \ \underbrace{\overline{-c_i^j U_i^{j-m}}}_{j\in[m+1,\ M]},$$

and the elements in the system of the matrix $L_{\varepsilon,\mu}^{N,M}$ of (7) are as follows

$$
\begin{aligned}
&\overset{+}{\sigma}_i = \frac{\triangle t\varepsilon}{2h_{i+1}\hbar_i} + \frac{\triangle t\mu a_i^j}{4\hbar_i}, \quad \overset{c}{\sigma}_i = \frac{-\triangle t\varepsilon}{2h_{i+1}\hbar_i} - \frac{\triangle t\varepsilon}{2h_i\hbar_i} - \frac{\triangle t}{2}b_i^j + 1, \quad \overset{-}{\sigma}_i = \frac{\triangle t\varepsilon}{2h_{i+1}\hbar_i} - \frac{\triangle t\mu a_i^j}{4\hbar_i}, \quad \text{if } L^{N,M} \equiv L_{cd}^{N,M},\\
&\overset{+}{\sigma}_i = \frac{\triangle t\varepsilon}{2h_{i+1}\hbar_i} + \frac{\triangle t\mu a_i^j}{2h_{i+1}}, \quad \overset{c}{\sigma}_i = \frac{-\triangle t\varepsilon}{2h_{i+1}\hbar_i} - \frac{\triangle t\varepsilon}{2h_i\hbar_i} - \frac{\triangle t\mu a_i^j}{2h_{i+1}} - 0.5\triangle t b_i^j + 1, \quad \overset{-}{\sigma}_i = \frac{-\triangle t\varepsilon}{h_i\hbar_i} \quad \text{if } L^{N,M} \equiv L_{up}^{N,M},\\
&\overset{+}{\sigma}_i = \frac{\triangle t\varepsilon}{h_{i+1}\hbar_i} + \frac{\triangle t\mu a_{i+1/2}^j}{h_{i+1}} - 0.5\triangle t b_{i+1/2}^j + 1, \quad \overset{c}{\sigma}_i = \frac{-\triangle t\varepsilon}{h_{i+1}\hbar_i} - \frac{\triangle t\varepsilon}{h_i\hbar_i} - \frac{\triangle t\mu a_{i+1/2}^j}{h_{i+1}} - 0.5\triangle t b_{i+1/2}^j + 1,\\
&\overset{-}{\sigma}_i = \frac{\triangle t\varepsilon}{h_i\hbar_i} \quad \text{if } L^{N,M} \equiv L_{mp}^{N,M},\\
&\overset{++}{\sigma}_i = \frac{-\triangle t\varepsilon}{2h_{i+1}\hbar_i} + \frac{-\triangle t\mu a_i^{j+1}}{4\hbar_i}, \overset{cc}{\sigma}_i = \frac{\triangle t\varepsilon}{2h_{i+1}\hbar_i} + \frac{\triangle t\varepsilon}{2h_i\hbar_i} + \frac{\triangle t}{2}b_i^{j+1} + 1, \overset{--}{\sigma}_i = \frac{-\triangle t\varepsilon}{2h_{i+1}\hbar_i} + \frac{\triangle t\mu a_i^{j+1}}{24\hbar_i}, \text{ if } L^{N,M} \equiv L_{cd}^{N,M},\\
&\overset{++}{\sigma}_i = \frac{-\triangle t\varepsilon}{2h_{i+1}\hbar_i} - \frac{\triangle t\mu a_i^{j+1}}{2h_{i+1}}, \overset{cc}{\sigma}_i = \frac{\triangle t\varepsilon}{2h_{i+1}\hbar_i} + \frac{\triangle t\varepsilon}{2h_i\hbar_i} + \frac{\triangle t\mu a_i^{j+1}}{2h_{i+1}} + \frac{\triangle t}{2}b_i^{j+1} + 1, \overset{--}{\sigma}_i = \frac{\triangle t\varepsilon}{h_i\hbar_i} \quad \text{if } L^{N,M} \equiv L_{up}^{N,M},\\
&\overset{++}{\sigma}_i = \frac{-\triangle t\varepsilon}{h_{i+1}\hbar_i} - \frac{\triangle t\mu a_{i+1/2}^{j+1}}{h_{i+1}} + \triangle t b_{i+1/2}^{j+1} + 1, \overset{cc}{\sigma}_i = \frac{\triangle t\varepsilon}{h_{i+1}\hbar_i} + \frac{\triangle t\varepsilon}{h_i\hbar_i} + \frac{\triangle t\mu a_{i+1/2}^{j+1}}{h_{i+1}} + 0.5\triangle t b_{i+1/2}^j + 1,\\
&\overset{--}{\sigma}_i = \frac{-\triangle t\varepsilon}{h_i\hbar_i} \quad \text{if } L^{N,M} \equiv L_{mp}^{N,M}.
\end{aligned}
$$

4 Boundedness of discrete solution and error analysis

Lemma 3 (**Discrete minimum principle**). *The discretized form in (7) satisfies discrete minimum principle, that is, if for any mesh function* φ *with* $\varphi_i^j\Big|_{\partial\Lambda^{N,M}} \geq 0$ *and* $\left[L^{N,M}\varphi\right]_i^j\Big|_{\Lambda^{N,M}} \leq 0$*, then* $\varphi_i^j\Big|_{\bar{\Lambda}^{N,M}} \geq 0$.

Proof. It follows from the idea in [45]. □

Before proceeding with the error analysis, we express the decomposition of discrete solution U as follows

$$U = V + W_L + W_R, \tag{12}$$

where

$$\begin{cases} \left[L^{N,M}V\right]_i^j = -c_i^j V_i^{j-m} + f_i^j, & (x_i,t_j) \in \Lambda^{N,M},\\ V_i^j\Big|_{\partial\Lambda^{N,M}} = v(x_i,t_j), \end{cases} \tag{13}$$

$$\begin{cases} \left[L^{N,M}W_L\right]_i^j = -c_i^j (W_L)_i^{j-m}, & (x_i,t_j) \in \Lambda^{N,M},\\ (W_L)_i^j\Big|_{\partial\Lambda^{N,M}} = w_L(x_i,t_j), \end{cases} \tag{14}$$

$$\begin{cases} \left[L^{N,M}W_R\right]_i^j = -c_i^j (W_L)_i^{j-m}, & (x_i,t_j) \in \Lambda^{N,M},\\ (W_R)_i^j\Big|_{\partial\Lambda^{N,M}} = w_R(x_i,t_j). \end{cases} \tag{15}$$

Then, for the numerical solution U and the analytical solution u, we can decompose the error as

$$[U-u]_i^j = [V-v]_i^j + [W_L-w_L]_i^j + [W_R-w_R]_i^j \text{ for all } (x_i,t_j) \in \bar{\Lambda}^{N,M}. \tag{16}$$

Lemma 4 ([45]). *For $s = v, w_L, w_R$ defined on $\bar{\Lambda}$, $S = V, W_L, W_R$ defined on $\bar{\Lambda}^{N,M}$ and for every $(x_i, t_j) \in \bar{\Lambda}^{N,M}$,*
(a) The local truncation error of the developed scheme is defined by

$$\left[L^{N,M}(S-s)\right]_i^j = -c_i^j [S-s]_i^{j-m} + \left[\left(L_{\varepsilon,\mu} - \frac{\partial}{\partial t}\right)s - L^{N,M}s\right]_i^j,$$

on arbitrary mesh with step sizes h_i given by

$$\left|\left[L_{cen}^{N,M}(S-s)\right]_i^j\right| \le \left|[S-s]_i^{j-m}\right| + C\left[\varepsilon h_i \|s_{xxx}\| + \mu h_i \|s_{xx}\| + M^{-2}\|s_{ttt}\|\right],$$
$$\left|\left[L_{up}^{N,M}(S-s)\right]_i^j\right| \le \left|[S-s]_i^{j-m}\right| + C\left[\varepsilon h_i \|s_{xxx}\| + \mu h_{i+1} \|s_{xx}\| + M^{-2}\|s_{ttt}\|\right],$$
$$\left| L_{mp}^{N,M}(S-s)\right]_i^j \left| \le \right|[S-s]_i^{j-m}\left| + C\left[\varepsilon \hbar_i \|s_{xxx}\| + \mu h_{i+1}^2 (\|s_{xxx}\| + \|s_{xx}\|) + M^{-2}\|s_{ttt}\|\right],\right.$$

and on uniform mesh with step size h is given by

$$\left| L_{cen}^{N,M}(S-s)\right]_i^j \left| \le \right|[S-s]_i^{j-m}\left| + C\left[\varepsilon h^2 \|s_{xxxx}\| + \mu h^2 \|s_{xxx}\| + M^{-2}\|s_{ttt}\|\right],\right.$$
$$\left|\left[L_{up}^{N,M}(S-s)\right]_i^j\right| \le \left|[S-s]_i^{j-m}\right| + C\left[\varepsilon h^2 \|s_{xxxx}\| + \mu h \|s_{xx}\| + M^{-2}\|s_{ttt}\|\right],$$
$$\left|\left[L_{mp}^{N,M}(S-s)\right]_i^j\right| \le \left|[S-s]_i^{j-m}\right| + C\left[\varepsilon h \|s_{xxx}\| + \mu h^2 (\|s_{xxx}\| + \|s_{xx}\|) + M^{-2}\|s_{ttt}\|\right].$$

(b) The error bound for the smooth component is

$$\left|[V-v]_i^j\right| \le C\left(M^{-2} + N^{-2}\right).$$

(c) The error bounds on the singular components (the layer components) satisfy

$$\left|(W_L)_i^j\right| \le CN^{-2}, \quad i = N/4, \ldots, N, \; j\Delta t \le \gamma,$$
$$\left|(W_R)_i^j\right| \le CN^{-2}, \quad i = 0, \ldots, 3N/4, \; j\Delta t \le \gamma.$$

By using the idea in parts b and c of Lemma 4, we have the following main result.

Theorem 2 (**Main result**). *Let U_i^{j+1} and $u(x_i, t_{j+1})$ be the numerical solution and the continuous solution of problem (1) - (2). Then*

$$\max_{0\le i\le N, 0\le j\le M} \left|u(x_i, t_{j+1}) - U_i^{j+1}\right|_{\bar{\Lambda}}^{N,M} \le \begin{cases} C\left(M^{-2} + N^{-2}(\ln N)^2\right), & \text{when } \alpha\mu^2 \le \eta\varepsilon, \\ C\left(M^{-2}\ln N + N^{-2}(\ln N)^2\right), & \text{when } \alpha\mu^2 \ge \eta\varepsilon. \end{cases}$$

5 Numerical results and discussion

To confirm the efficiency of the proposed method, we consider test problems with smooth and non-smooth data. We present the numerical results on a uniform mesh as well as on the Shishkin mesh [31]. The maximum point-wise errors and the numerical rate of convergence are computed using the double mesh principle [14] due to the unavailability of the analytical solution. Results are compared with some methods in the literature.

The maximum absolute error is given by

$$E_{rr}^{N_h,M_{\triangle t}} = \max_{0\le i,j\le N_h,M_{\triangle t}} \left| U^{N_h,M_{\triangle t}}(x_i,\ t_j) - U^{2N_h,2M_{\triangle t}}(x_{2i},\ t_{2j}) \right|,$$

and also the rate of convergence is given by the formula

$$Roc^{N_h.M_{\triangle t}} = \log_2\left(\frac{E_{rr}^{N_h.M_{\triangle t}}}{E_{rr}^{2N_h,2M_{\triangle t}}}\right).$$

We consider the following two test examples with smooth data to investigate the applicability of our scheme (7).

Example 1.
$$\begin{cases} \varepsilon u_{xx}(x,t) + \mu(1+x)u_x(x,t) - u(x,t) - u_t(x,t) & \\ \qquad = -u(x,t-\tau) + 16x^2(1-x)^2, & (x,t)\in(0,1)\times(0,2], \\ u(x,t) = 0, & (x,t)\in[0,1]\times[-\tau,0], \\ u(0,t) = 0, u(1,t) = 0, & t\in[0,2]. \end{cases}$$

Example 2.
$$\begin{cases} \varepsilon u_{xx}(x,t) + \mu\left(1+x(1-x)+t^2\right)u_x(x,t) - (1+5xt)u(x,t) - u_t(x,t) & \\ \qquad = -u(x,t-\tau) + x(1-x)\left(e^t-1\right), & (x,t)\in(0,1)\times(0,2], \\ u(x,t) = 0, & (x,t)\in[0,1]\times[-\tau,0], \\ u(0,t) = 0, u(1,t) = 0, & t\in(0,2]. \end{cases}$$

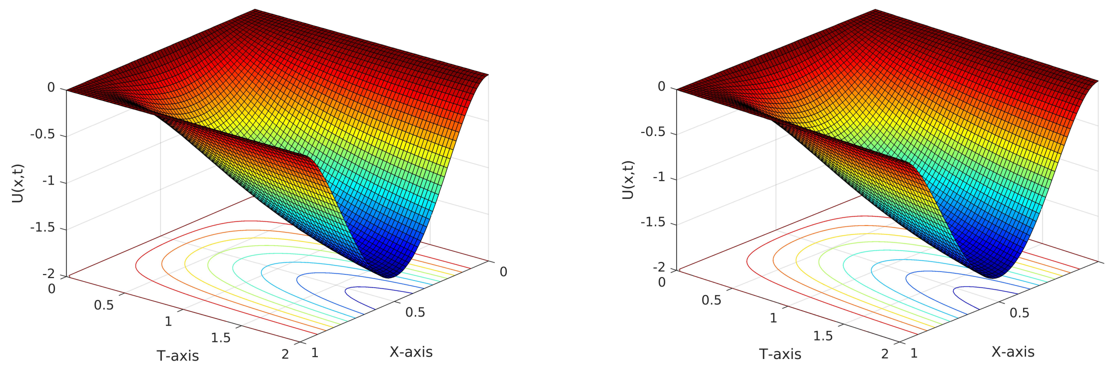
For our third and fourth examples, we consider the following two-parameter singularly perturbed time delay parabolic problem with non-smooth convection coefficient term $a(x,t)$ and source term $f(x,t)$ [5].

Example 3.

$$a(x,t) = \begin{cases} -\left(1+e^{-xt}\right), & 0\le x\le 0.5, \\ 2+x+t, & 0.5 < x \le 1, \end{cases}$$
$$f(x,t) = \begin{cases} \left(e^{t^2}-1\right)(1+xt), & 0\le x\le 0.5, \\ -(2+x)t^2, & 0.5 < x \le 1, \end{cases}$$
$$b(x,t) = 2+xt, \qquad c(x,t) = 1, \qquad y(0,t) = y(1,t) = y(x,0) = 0.$$

Example 4.

$$a(x,t) = \begin{cases} 1+x(1-x)+t, & 0\le x\le 0.5, \\ -(1+3xt), & 0.5 < x \le 1, \end{cases}$$
$$f(x,t) = \begin{cases} (1+x)\left(e^t-1\right), & 0\le x\le 0.5, \\ (-2+x)t, & 0.5 < x \le 1, \end{cases}$$
$$b(x,t) = 1+x+t, \qquad c(x,t) = 1, \qquad y(0,t) = y(1,t) = y(x,0) = 0.$$



(a) (b)

Figure 1: Surface plot for Example 1 using scheme in (7) with $N = M = 64$ and (a) $\varepsilon = 10^{-8}, \mu = 10^{-4}$, (b) $\varepsilon = 10^{-4}, \mu = 10^{-8}$.

The analytical solutions of these test examples do not have closed form. So we have used scheme (7) to obtain the numerical solutions of the given problems. The results in Tables 1 and 2 show the numerical implementation of Example 1 and Tables 3 and 4 show the experimental results of Example 2. The numerical results of both examples validate that the proposed scheme (7) converges uniformly with order two in both variables independently of the perturbation parameters.

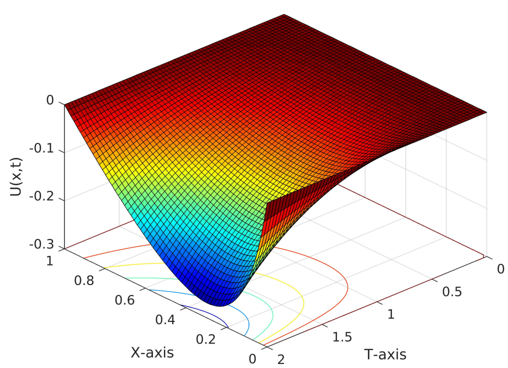
Tables 6 and 7 show the numerical result of the test problem in Example 3, while Tables 8 and 9 summarize the numerical results of Example 4. In both examples, the data given in the convection term and the source term are non-smooth. Consequently, the solutions of these problems show both interior layers and boundary layers as can be seen in Figures 3 and 4. Again, our scheme (7) converges uniformly with order two with respect to both variable as shown from the numerical results of Examples 3 and 4. The comparison of the proposed scheme with existing scheme in [45] is given in Table 5. As can be seen, the maximum point-wise error in our scheme is less than the maximum point-wise error of the scheme in [45]. This shows our scheme is more accurate than the scheme in [45]. In all of our test examples, errors do not depend on the size of perturbation parameters. This is indeed the confirmation that the present method is parameter uniform. This uniformly convergence is also supported by Log-Log plots in Figures 3 and 4. In Examples 1 and 2, the data on the convection coefficient term and source term are smooth. The solution of such problems show boundary layer at the boundary regions. Where as in Examples 3 and 4, the data on the convection coefficient term and source term are non smooth (has discontinuity). Because of this non-smoothness, the solution of the given problems exhibits both interior layer and boundary layer, and we have shown this effect graphically in Figures 3 and 4. The theoretical behaviors of the numerical solutions are supported by Figures 1-4.

6 Conclusions

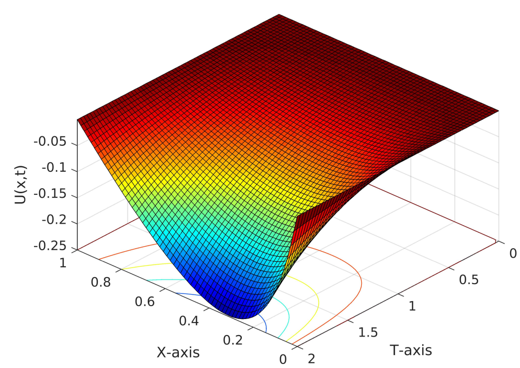
This article introduced a hybrid fitted finite difference scheme to solve two-parameter singularly perturbed time delay parabolic differential equations for both smooth and non-smooth data. In this work,

Table 1: Maximum errors, $Err^{N,M}_{\varepsilon,\mu}$ and rates of convergence, $Roc^{N,M}_{\varepsilon,\mu}$ using scheme (7) for Example 1 with $\mu = 10^{-3}$ and different values of ε.

$\mu = 10^{-3}$ / $\varepsilon \downarrow$	N $\rightarrow$ / M $\rightarrow$	32 / 8	64 / 16	128 / 32	256 / 64	512 / 128
10^0	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	7.1784e-04	1.8213e-04	4.5704e-05	1.1437e-05	2.8598e-06
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	1.9787	1.9946	1.9986	1.9997	-
10^{-2}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	5.6025e-03	1.4060e-03	3.5184e-04	8.7980e-05	2.1996e-05
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	1.9945	1.9986	1.9997	1.9999	-
10^{-4}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	5.8291e-03	1.4625e-03	3.6596e-04	9.1510e-05	2.2879e-05
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	1.9948	1.9987	1.9997	1.9999	-
10^{-6}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	5.8314e-03	1.4631e-03	3.6611e-04	9.1547e-05	2.2888e-05
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	1.9948	1.9987	1.9997	1.9999	-
10^{-8}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	5.8314e-03	1.4631e-03	3.6611e-04	9.1547e-05	2.2888e-05
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	1.9948	1.9987	1.9997	1.9999	-
10^{-10}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	5.8314e-03	1.4631e-03	3.6611e-04	9.1547e-05	2.2888e-05
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	1.9948	1.9987	1.9997	1.9999	-
10^{-12}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	5.8314e-03	1.4631e-03	3.6611e-04	9.1547e-05	2.2888e-05
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	1.9948	1.9987	1.9997	1.9999	-
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-40}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	5.8314e-03	1.4631e-03	3.6611e-04	9.1547e-05	2.2888e-05
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	1.9948	1.9987	1.9997	1.9999	-
$Err^{N,M}_{\varepsilon,\mu}$	$\rightarrow$	5.8314e-03	1.4631e-03	3.6611e-04	9.1547e-05	2.2888e-05
$Roc^{N,M}_{\varepsilon,\mu}$	$\rightarrow$	1.9948	1.9987	1.9997	1.9999	-



(a)



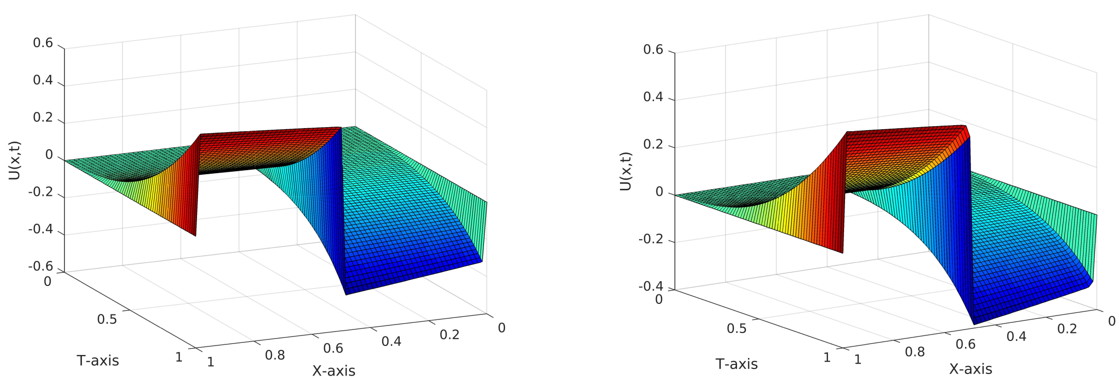
(b)

Figure 2: Surface plot for Example 2 using scheme (7) with $N = M = 64$ and (a) $\varepsilon = 10^{-8}$, $\mu = 10^{-4}$, (b) $\varepsilon = 10^{-4}$, $\mu = 10^{-8}$.

we considered a layer adapted piecewise uniform mesh of Shishkin type in the spatial direction and a uniform mesh in temporal direction. The problem was discretized using the difference operator comprising of Crank-Nicolson method for time and a blend of the central, upwind and midpoint difference operators

Table 2: Maximum errors, $Err_{\varepsilon,\mu}^{N,M}$ and rates of convergence, $Roc_{\varepsilon,\mu}^{N,M}$ using scheme (7) for Example 1 with $\varepsilon = 10^{-3}$ and different values of μ.

$\varepsilon = 10^{-3}$	N $\rightarrow$	32	64	128	256	512
$\mu\downarrow$	M $\rightarrow$	8	16	32	64	128
10^{0}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	6.4634e-03	1.5384e-03	3.7470e-04	9.2432e-05	2.2952e-05
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	2.0709	2.0376	2.0193	2.0098	-
10^{-2}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	5.8135e-03	1.4580e-03	3.6474e-04	9.1193e-05	2.2798e-05
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	1.9954	1.9991	1.9999	2.0000	-
10^{-4}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	5.8075e-03	1.4572e-03	3.6464e-04	9.1181e-05	2.2796e-05
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	1.9947	1.9987	1.9997	2.0000	-
10^{-6}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	5.8074e-03	1.4572e-03	3.6464e-04	9.1180e-05	2.2796e-05
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	1.9947	1.9987	1.9997	1.9999	-
10^{-8}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	5.8074e-03	1.4572e-03	3.6464e-04	9.1180e-05	2.2796e-05
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	1.9947	1.9987	1.9997	1.9999	-
10^{-10}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	5.8074e-03	1.4572e-03	3.6464e-04	9.1180e-05	2.2796e-05
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	1.9947	1.9987	1.9997	1.9999	-
10^{-12}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	5.8074e-03	1.4572e-03	3.6464e-04	9.1180e-05	2.2796e-05
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	1.9947	1.9987	1.9997	1.9999	-
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-40}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	5.8074e-03	1.4572e-03	3.6464e-04	9.1180e-05	2.2796e-05
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	1.9947	1.9987	1.9997	1.9999	-
$Err_{\varepsilon,\mu}^{N,M}$	$\rightarrow$	5.8074e-03	1.4572e-03	3.6464e-04	9.1180e-05	2.2796e-05
$Roc_{\varepsilon,\mu}^{N,M}$	$\rightarrow$	1.9947	1.9987	1.9997	1.9999	-



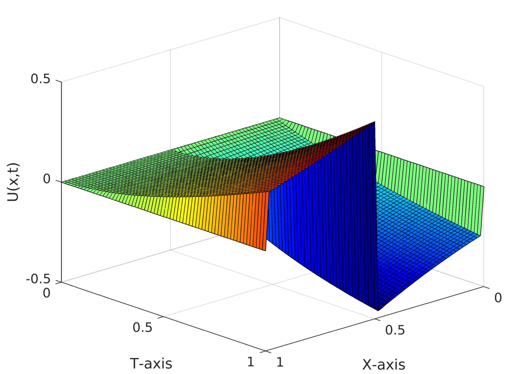
(a) (b)

Figure 3: Surface plot for Example 3 using scheme (7) with $N = M = 64$ and (a) $\varepsilon = 10^{-8}$, $\mu = 10^{-4}$, (b) $\varepsilon = 10^{-4}$, $\mu = 10^{-8}$.

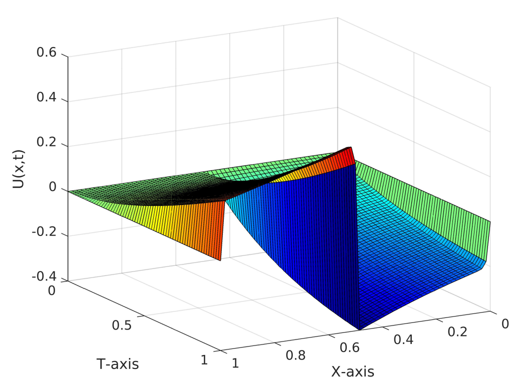
for space. The convergence analysis proved that the method is uniformly convergent of second order in time variable and almost second order in space variable. The method was further implemented using test examples (with smooth and non-smooth data). In both cases, the theoretical results were confirmed.

Table 3: Maximum errors, $Err_{\varepsilon,\mu}^{N,M}$ and rates of convergence, $Roc_{\varepsilon,\mu}^{N,M}$ using scheme (7) for Example 2 with $\mu = 10^{-3}$ and different values of ε.

$\mu = 10^{-3}$	N $\rightarrow$	32	64	128	256
$\varepsilon \downarrow$	M $\rightarrow$	8	16	32	64
10^{0}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	8.9042e-04	2.2425e-04	5.6169e-05	1.4049e-05
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	1.9894	1.9973	1.9993	-
10^{-2}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	3.1319e-03	7.8541e-04	1.9655e-04	4.9150e-05
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	1.9955	1.9985	1.9996	-
10^{-4}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	3.2007e-03	8.0247e-04	2.0080e-04	5.0214e-05
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	1.9959	1.9987	1.9996	-
10^{-6}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	3.2014e-03	8.0264e-04	2.0085e-04	5.0225e-05
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	1.9959	1.9986	1.9996	-
10^{-8}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	3.2014e-03	8.0264e-04	2.0085e-04	5.0225e-05
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	1.9959	1.9986	1.9996	-
10^{-10}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	3.2014e-03	8.0264e-04	2.0085e-04	5.0225e-05
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	1.9959	1.9986	1.9996	-
10^{-12}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	3.2014e-03	8.0264e-04	2.0085e-04	5.0225e-05
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	1.9959	1.9986	1.9996	-
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-40}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	3.2014e-03	8.0264e-04	2.0085e-04	5.0225e-05
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	1.9959	1.9986	1.9996	-
$Err_{\varepsilon,\mu}^{N,M}$	$\rightarrow$	3.2014e-03	8.0264e-04	2.0085e-04	5.0225e-05
$Roc_{\varepsilon,\mu}^{N,M}$	$\rightarrow$	1.9959	1.9986	1.9996	-



(a)



(b)

Figure 4: Surface plot for Example 4 using scheme (7) with $N = M = 64$ and (a) $\varepsilon = 10^{-8}$, $\mu = 10^{-4}$, (b) $\varepsilon = 10^{-4}$, $\mu = 10^{-8}$.

Tabulated results show practically a second order uniform convergence. Furthermore, performance comparison with an existing method in the literature indicates that the proposed method is more accurate.

Table 4: Maximum errors, $Err_{\varepsilon,\mu}^{N,M}$ and rates of convergence, $Roc_{\varepsilon,\mu}^{N,M}$ using scheme (7) for Example 2 with $\varepsilon = 10^{-3}$ and different values of μ.

$\varepsilon = 10^{-3}$ $\mu \downarrow$	N $\rightarrow$ M $\rightarrow$	32 8	64 16	128 32	256 64
10^{0}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	3.5162e-03	8.4508e-04	2.0685e-04	5.0155e-05
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	2.0569	2.0305	2.0441	-
10^{-2}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	3.1970e-03	8.0128e-04	2.0046e-04	5.0122e-05
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	1.9963	1.9990	1.9998	-
10^{-4}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	3.1940e-03	8.0086e-04	2.0041e-04	5.0115e-05
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	1.9957	1.9986	1.9996	-
10^{-6}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	3.1940e-03	8.0085e-04	2.0040e-04	5.0115e-05
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	1.9958	1.9986	1.9996	-
10^{-8}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	3.1940e-03	8.0085e-04	2.0040e-04	5.0115e-05
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	1.9958	1.9986	1.9996	-
10^{-10}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	3.1940e-03	8.0085e-04	2.0040e-04	5.0115e-05
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	1.9958	1.9986	1.9996	-
10^{-12}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	3.1940e-03	8.0085e-04	2.0040e-04	5.0115e-05
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	1.9958	1.9986	1.9996	-
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-40}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	3.1940e-03	8.0085e-04	2.0040e-04	5.0115e-05
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	1.9958	1.9986	1.9996	-
$Err_{\varepsilon,\mu}^{N,M}$	$\rightarrow$	3.1940e-03	8.0085e-04	2.0040e-04	5.0115e-05
$Roc_{\varepsilon,\mu}^{N,M}$	$\rightarrow$	1.9958	1.9986	1.9996	-

Table 5: Comparison of the proposed scheme (7) with an existing scheme in [45] using Example 1.

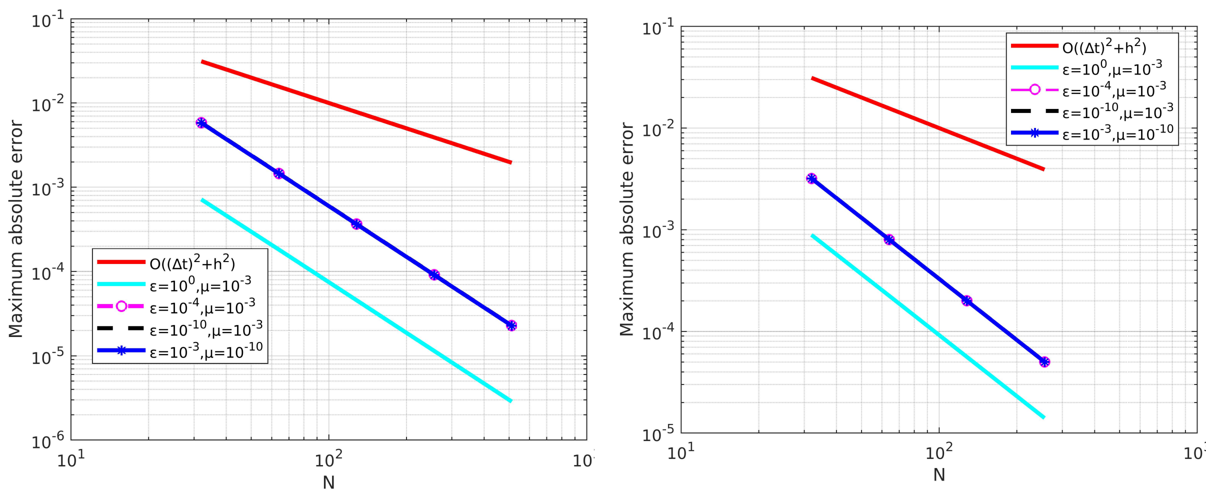
$\mu = 10^{-3}$ $\varepsilon \downarrow$	$N = 32$ $M = 8$	$N = 64$ $M = 16$	$N = 128$ $M = 32$	$N = 256$ $M = 64$	$N = 512$ $M = 128$
Proposed method					
10^{-4}	5.8291e-03	1.4625e-03	3.6596e-04	9.1510e-05	2.2879e-05
10^{-6}	5.8314e-03	1.4631e-03	3.6611e-04	9.1547e-05	2.2888e-05
10^{-8}	5.8314e-03	1.4631e-03	3.6611e-04	9.1547e-05	2.2888e-05
10^{-10}	5.8314e-03	1.4631e-03	3.6611e-04	9.1547e-05	2.2888e-05
scheme in [45]					
10^{-4}	$4.3705e-2$	$1.6704e-2$	$7.3802e-3$	$3.7406e-3$	$1.8967e-3$
10^{-6}	$4.3471e-2$	$1.6596e-2$	$7.3290e-3$	$3.7218e-3$	$1.8873e-3$
10^{-8}	$4.3429e-2$	$1.6573e-2$	$7.3303e-3$	$3.7211e-3$	$1.8870e-3$
10^{-10}	$4.4343e-2$	$1.6572e-2$	$7.3303e-3$	$3.7211e-3$	$1.8870e-3$

Acknowledgements

The authors wish to acknowledge the referees for their constructive suggestions and comments.

Table 6: Maximum errors, $Err^{N,M}_{\varepsilon,\mu}$ and rates of convergence, $Roc^{N,M}_{\varepsilon,\mu}$ using scheme (7) for Example 3 with $\mu = 10^{-3}$ and different values of ε.

$\mu = 10^{-3}$	N $\rightarrow$	32	64	128	256
$\varepsilon \downarrow$	M $\rightarrow$	8	16	32	64
10^{0}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	5.5107e-03	1.4134e-03	3.5905e-04	9.0543e-05
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	1.9631	1.9769	1.9875	-
10^{-2}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	5.7686e-03	1.4501e-03	3.6416e-04	9.1280e-05
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	1.9921	1.9935	1.9962	-
10^{-4}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	5.8369e-03	1.4589e-03	3.6526e-04	9.1417e-05
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	2.0003	1.9979	1.9984	-
10^{-6}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	5.8375e-03	1.4591e-03	3.6529e-04	9.1426e-05
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	2.0003	1.9980	1.9984	-
10^{-8}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	5.8375e-03	1.4591e-03	3.6529e-04	9.1426e-05
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	2.0003	1.9980	1.9984	-
10^{-10}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	5.8375e-03	1.4591e-03	3.6529e-04	9.1426e-05
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	2.0003	1.9980	1.9984	-
10^{-12}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	5.8375e-03	1.4591e-03	3.6529e-04	9.1426e-05
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	2.0003	1.9980	1.9984	-
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-20}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	5.8375e-03	1.4591e-03	3.6529e-04	9.1426e-05
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	2.0003	1.9980	1.9984	-
$Err^{N,M}_{\varepsilon,\mu}$	$\rightarrow$	5.8375e-03	1.4591e-03	3.6529e-04	9.1426e-05
$Roc^{N,M}_{\varepsilon,\mu}$	$\rightarrow$	2.0003	1.9980	1.9984	-



(a) (b)

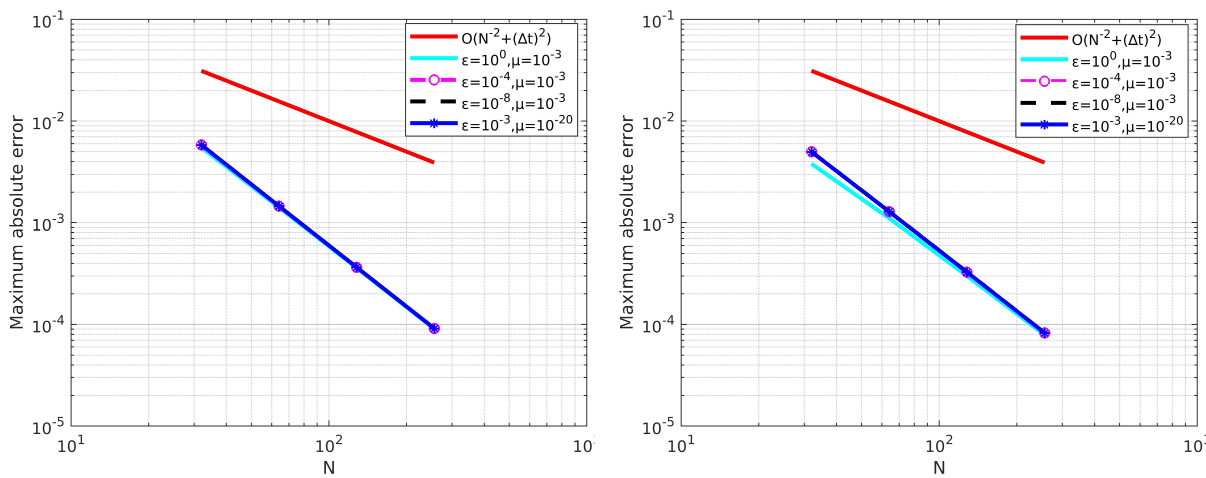
Figure 5: Log-Log plot of N vs maximum absolute error (a) for Example 1 and (b) for Example 2.

References

[1] A. Ansari, S. Bakr, G. Shishkin, *A parameter-robust finite difference method for singularly perturbed delay parabolic partial differential equations*, J. Comput. Appl. Math. **205** (2007) 552–566.

Table 7: Maximum errors, $Err^{N,M}_{\varepsilon,\mu}$ and rates of convergence, $Roc^{N,M}_{\varepsilon,\mu}$ using scheme (7) for Example 3 with $\varepsilon = 10^{-3}$ and different values of μ.

$\varepsilon = 10^{-3}$ $\mu \downarrow$	N $\rightarrow$ M $\rightarrow$	32 8	64 16	128 32	256 64
10^{0}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	6.1608e-03	1.5011e-03	3.7069e-04	9.2110e-05
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	2.0371	2.0177	2.0088	-
10^{-2}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	5.8314e-03	1.4576e-03	3.6493e-04	9.1323e-05
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	2.0002	1.9979	1.9986	-
10^{-4}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	5.8314e-03	1.4576e-03	3.6493e-04	9.1324e-05
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	2.0002	1.9979	1.9986	-
10^{-6}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	5.8314e-03	1.4576e-03	3.6493e-04	9.1324e-05
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	2.0002	1.9979	1.9986	-
10^{-8}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	5.8314e-03	1.4576e-03	3.6493e-04	9.1324e-05
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	2.0002	1.9979	1.9986	-
10^{-10}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	5.8314e-03	1.4576e-03	3.6493e-04	9.1324e-05
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	2.0002	1.9979	1.9986	-
10^{-12}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	5.8314e-03	1.4576e-03	3.6493e-04	9.1324e-05
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	2.0002	1.9979	1.9986	-
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-20}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	5.8314e-03	1.4576e-03	3.6493e-04	9.1324e-05
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	2.0002	1.9979	1.9986	-
$Err^{N,M}_{\varepsilon,\mu}$	$\rightarrow$	5.8314e-03	1.4576e-03	3.6493e-04	9.1324e-05
$Roc^{N,M}_{\varepsilon,\mu}$	$\rightarrow$	2.0002	1.9979	1.9986	-



(a) (b)

Figure 6: Log-Log plot of N vs maximum absolute error (a) for Example 3 and (b) for Example 4.

Table 8: Maximum errors, $Err^{N,M}_{\varepsilon,\mu}$ and rates of convergence, $Roc^{N,M}_{\varepsilon,\mu}$ using scheme (7) for Example 4 with $\mu = 10^{-3}$ and different values of ε.

$\mu = 10^{-3}$ $\varepsilon \downarrow$	N $\rightarrow$ M $\rightarrow$	32 8	64 16	128 32	256 64
10^{0}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	3.8029e-03	1.1072e-03	3.0071e-04	7.8684e-05
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	1.7802	1.8805	1.9342	-
10^{-2}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	4.8349e-03	1.2524e-03	3.2301e-04	8.2004e-05
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	1.9488	1.9550	1.9778	-
10^{-4}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	5.0002e-03	1.2915e-03	3.2799e-04	8.2634e-05
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	1.9529	1.9773	1.9888	-
10^{-6}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	5.0019e-03	1.2919e-03	3.2809e-04	8.2660e-05
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	1.9530	1.9773	1.9888	-
10^{-8}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	5.0019e-03	1.2919e-03	3.2809e-04	8.2660e-05
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	1.9530	1.9773	1.9888	-
10^{-10}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	5.0019e-03	1.2919e-03	3.2809e-04	8.2660e-05
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	1.9530	1.9773	1.9888	-
10^{-12}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	5.0019e-03	1.2919e-03	3.2809e-04	8.2660e-05
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	1.9530	1.9773	1.9888	-
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-20}	$Err^{N,M}_{\varepsilon,\mu} \rightarrow$	5.0019e-03	1.2919e-03	3.2809e-04	8.2660e-05
	$Roc^{N,M}_{\varepsilon,\mu} \rightarrow$	1.9530	1.9773	1.9888	-
$Err^{N,M}_{\varepsilon,\mu}$	$\rightarrow$	5.0019e-03	1.2919e-03	3.2809e-04	8.2660e-05
$Roc^{N,M}_{\varepsilon,\mu}$	$\rightarrow$	1.9530	1.9773	1.9888	-

[2] E.B. Bashier, K.C. Patidar, *A novel fitted operator finite difference method for a singularly perturbed delay parabolic partial differential equation*, Appl. Math. Comput. **217** (2011) 4728–4739.

[3] T.A. Bullo, G.F. Duressa, G.A. Degla, *Robust finite difference method for singularly perturbed two-parameter parabolic convection-diffusion problems*, Int. J. Comput. Methods **18** (2021) 2050034.

[4] Z. Cen, *A second-order finite difference scheme for a class of singularly perturbed delay differential equations*, Int. J. Comput. Math. **87** (2010) 173–185.

[5] M. Chandru, P. Das, H. Ramos, *Numerical treatment of two-parameter singularly perturbed parabolic convection diffusion problems with non-smooth data*, Math. Methods Appl. Sci. **41** (2018) 5359–5387.

[6] M. Chandru, T. Prabha, V. Shanthi, *A parameter robust higher order numerical method for singularly perturbed two parameter problems with non-smooth data*, J. Comput. Appl. Math. **309** (2017) 11–27.

[7] P. Das, *Comparison of a priori and a posteriori meshes for singularly perturbed nonlinear parameterized problems*, J. Comput. Appl. Math. **290** (2015) 16–25.

[8] P. Das, S. Natesan, *Higher-order parameter uniform convergent schemes for robin type reaction-diffusion problems using adaptively generated grid*, Int. J. Comput. Methods **9** (2012) 1250052.

Table 9: Maximum errors, $Err_{\varepsilon,\mu}^{N,M}$ and rates of convergence, $Roc_{\varepsilon,\mu}^{N,M}$ using scheme (7) for Example 4 with $\varepsilon = 10^{-3}$ and different values of μ.

$\varepsilon = 10^{-3}$	N $\rightarrow$	32	64	128	256
$\mu\downarrow$	M $\rightarrow$	8	16	32	64
10^{0}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	5.0529e-03	1.3020e-03	3.2960e-04	8.2863e-05
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	1.9564	1.9819	1.9919	-
10^{-2}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	4.9835e-03	1.2875e-03	3.2703e-04	8.2399e-05
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	1.9526	1.9771	1.9887	-
10^{-4}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	4.9856e-03	1.2878e-03	3.2707e-04	8.2403e-05
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	1.9529	1.9772	1.9888	-
10^{-6}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	4.9856e-03	1.2878e-03	3.2707e-04	8.2403e-05
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	1.9529	1.9772	1.9888	-
10^{-8}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	4.9856e-03	1.2878e-03	3.2707e-04	8.2403e-05
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	1.9529	1.9772	1.9888	-
10^{-10}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	4.9856e-03	1.2878e-03	3.2707e-04	8.2403e-05
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	1.9529	1.9772	1.9888	-
10^{-12}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	4.9856e-03	1.2878e-03	3.2707e-04	8.2403e-05
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	1.9529	1.9772	1.9888	-
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-20}	$Err_{\varepsilon,\mu}^{N,M} \rightarrow$	4.9856e-03	1.2878e-03	3.2707e-04	8.2403e-05
	$Roc_{\varepsilon,\mu}^{N,M} \rightarrow$	1.9529	1.9772	1.9888	-
$Err_{\varepsilon,\mu}^{N,M}$	$\rightarrow$	4.9856e-03	1.2878e-03	3.2707e-04	8.2403e-05
$Roc_{\varepsilon,\mu}^{N,M}$	$\rightarrow$	1.9529	1.9772	1.9888	-

[9] P. Das, S. Natesan, *Numerical solution of a system of singularly perturbed convection diffusion boundary value problems using mesh equidistribution technique*, Aust. J. Math. Anal. Appl. **10** (2013) 1–17.

[10] P. Das, S. Natesan, *Richardson extrapolation method for singularly perturbed convection-diffusion problems on adaptively generated mesh*, CMES Comput. Model. Eng. Sci. **90** (2013) 463–485.

[11] P. Das, S. Natesan, *Adaptive mesh generation for singularly perturbed fourth-order ordinary differential equations*, Int. J. Comput. Math. **92** (2015) 562–578.

[12] M. Dehghan, *Numerical solution of the three-dimensional advection–diffusion equation*, Appl. Math. Comput. **150** (2004) 5–19.

[13] M. Dehghan, *Finite difference procedures for solving a problem arising in modeling and design of certain optoelectronic devices*, Math. Comput. Simul. **71** (2006) 16–30.

[14] E.P. Doolan, J.J. Miller, W.H. Schilders, *Uniform Numerical Methods for Problems with Initial and Boundary Layers*, Boole Press, 1980.

[15] G.F. Duressa, T.B. Mekonnen, *An exponentially fitted method for two parameter singularly perturbed parabolic boundary value problems*, Commun. Korean Math. Soc. **38** (2023) 299–318.

[16] F. Erdogan, Z. Cen, *A uniformly almost second order convergent numerical method for singularly perturbed delay differential equations*, J. Comput. Appl. Math. **333** (2018) 382–394.

[17] K. Friedrichs, W. Wasow, *Singular perturbations of non-linear oscillations*, Duke Math. J. **13** (1946) 367–381.

[18] L. Govindarao, S.R. Sahu, J. Mohapatra, *Uniformly convergent numerical method for singularly perturbed time delay parabolic problem with two small parameters*, Iran. J. Sci. Technol. Trans. A: Sci. **43** (2019) 2373–2383.

[19] J. Gracia, E. O'Riordan, M. Pickett, *A parameter robust second order numerical method for a singularly perturbed two-parameter problem*, Appl. Numer. Math. **56** (2006) 962–980.

[20] B. Gunes, H. Duru, *A computational method for the singularly perturbed delay pseudo-parabolic differential equations on adaptive mesh*, Int. J. Comput. Math. (2023) 1–16.

[21] V. Gupta, M.K. Kadalbajoo, R.K. Dubey, *A parameter-uniform higher order finite difference scheme for singularly perturbed time-dependent parabolic problem with two small parameters*, Int. J. Comput. Math. **96** (2019) 474–499.

[22] M.K. Kadalbajoo, A. Jha, *Exponentially fitted cubic spline for two-parameter singularly perturbed boundary value problems*, Int. J. Comput. Math. **89** (2012) 836–850.

[23] A. Kaushik, *Singular perturbation analysis of bistable differential equation arising in the nerve pulse propagation*, Nonlinear Anal. Real World Appl. **9** (2008) 2106–2127.

[24] A. Kaushik, K. Sharma, M. Sharma, *A parameter uniform difference scheme for parabolic partial differential equation with a retarded argument*, Appl. Math. Model. **34** (2010) 4232–4242.

[25] R.B. Kellogg, A. Tsan, *Analysis of some difference approximations for a singular perturbation problem without turning points*, Math. Comput. **32** (1978) 1025–1039.

[26] N. Kopteva, *Uniform pointwise convergence of difference schemes for convection-diffusion problems on layer-adapted meshes*, Computing **66** (2001) 179–197.

[27] D. Kumar, K. Deswal, *Wavelet-based approximation for two-parameter singularly perturbed problems with robin boundary conditions*, J. Appl. Math. Comput. (2022) 1–25.

[28] M. Kumar, S. Kumar, *A second order uniformly convergent numerical scheme for parameterized singularly perturbed delay differential problems*, Numer. Algorithms **76** (2017) 349–360.

[29] S. Kumar, M. Kumar, *High order parameter-uniform discretization for singularly perturbed parabolic partial differential equations with time delay*, Comput. Math. Appl. **68** (2014) 1355–1367.

[30] T. Linss, *A posteriori error estimation for a singularly perturbed problem with two small parameters*, Int. J. Numer. Anal. Model. **7** (2010) 3.

[31] J.J. Miller, E. ORiordan, G.I. Shishkin, *Fitted Numerical Methods for Singular Perturbation Problems: Error Estimates in the Maximum Norm for Linear Problems in one and two Dimensions*, World scientific, 1996.

[32] J. Miller, E.ORiordan, G. Shishkin, L. Shishkina, *Fitted Mesh Methods for Problems with Parabolic Boundary Layers*, Mathematical Proceedings of the Royal Irish Academy, JSTOR, (1998) 173–190.

[33] J.B. Munyakazi, *A robust finite difference method for two-parameter parabolic convection-diffusion problems*, Appl. Math. Inf. Sci. **9** (2015) 2877.

[34] J.D. Murray, *Mathematical Biology: I. An Introduction*, Springer, New York, 2002.

[35] N.T. Negero, *A parameter-uniform efficient numerical scheme for singularly perturbed time-delay parabolic problems with two small parameters*, Partial Differ. Equ. Appl. Math. **7** (2023) 100518.

[36] R. OMalley, *Two-parameter singular perturbation problems for second-order equations*, J. Math. Mech. **16** (1967) 1143–1164.

[37] R. OMalley, *Boundary value problems for linear systems of ordinary differential equations involving many small parameters*, J. Math. Mech. **18** (1969) 835–855.

[38] E. ORiordan, M.L. Pickett, G.I. Shishkin, *Singularly perturbed problems modeling reaction-convection-diffusion processes*, Comput. Methods Appl. Math. **3** (2003) 424–442.

[39] E. ORiordan, M. Pickett, G. Shishkin, *Parameter-uniform finite difference schemes for singularly perturbed parabolic diffusion-convection-reaction problems*, Math. Comput. **75** (2006) 1135–1154.

[40] K.C. Patidar, *A robust fitted operator finite difference method for a two-parameter singular perturbation problem1*, J. Differ. Equ. Appl. **14** (2008) 1197–1214.

[41] L. Prandtl, *Uber Flussigkeitsbewegung bei sehr kleiner Reibung*, Verhandl. 3rd Int. Math. Kongr. Heidelberg (1904), Leipzig (1905).

[42] H.G. Roos, Z. Uzelac, *The Sdfem for a convection-diffusion problem with two small parameters*, Comput. Methods Appl. Math. **3** (2003) 443–458.

[43] V. Shanthi, N. Ramanujam, S. Natesan, *Fitted mesh method for singularly perturbed reaction-convection-diffusion problems with boundary and interior layers*, J. Appl. Math. Comput. **22** (2006) 49–65.

[44] J. Singh, S. Kumar, M. Kumar, *A domain decomposition method for solving singularly perturbed parabolic reaction-diffusion problems with time delay*, Numer. Methods Partial Differ. Equ. **34** (2018) 1849–1866.

[45] S. Sumit, S. Kumar, Kuldeep, M. Kumar, *A robust numerical method for a two-parameter singularly perturbed time delay parabolic problem*, Comput. Appl. Math. **39** (2020) 1–25.

[46] R. Vulanovic, *A higher-order scheme for quasilinear boundary value problems with two small parameters.*, Computing **67** (2001) 287–303.

[47] S. Yuzbasi, N. Sahin, *Numerical solutions of singularly perturbed one-dimensional parabolic convection–diffusion problems by the bessel collocation method*, App. Math. Comput. **220** (2013) 305–315.

[48] W.K. Zahra, A.M. Mhlawy, *Numerical solution of two-parameter singularly perturbed boundary value problems via exponential spline*, J. King Saud Univ. Sci. **25** (2013) 201–208.