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An $M^{[X]}/G(a,b)/1$ queue with unreliable server, re-service on server's decision, balking and Bernoulli vacation schedule under multiple vacation policy

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Abstract. This paper deals with a non-Markovian batch arrival bulk service queue with unreliable server, re-service on server's decision, Bernoulli vacation schedule under multiple vacation and balking. We consider that the server is unreliable and may stop working due to failure. When this happens, the service is interrupted and restarted after repair. The service time, vacation time, re-service time and repair time assume to follow a general (arbitrary) distribution. In the proposed model, we derived the probability distribution of queue size at a random and departure epoch using supplementary variable techniques. Finally, some performance measures, particular cases and numerical results are obtained.

Keywords: General bulk service, Non-Markovian queue, Breakdown and repair, Bernoulli vacation, Multiple vacation, Balking, re-service.
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1 Introduction

Batch arrival, bulk service vacation queues have wide applications in call centres, manufacturing system, complex communication networks and production systems, etc. At the point of service completion when no clients

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are available in the system, the server takes off for a long period of time, which is known as vacation. Levy and Yechiali [11] introduced a model known as the multiple vacation policy under the assumption that, at the end of a vacation the server takes sequence of vacations until he finds at least one item is waiting in the system. One of the most important results that concerns with such models is the Stochastic decomposition result.

A common model related to multiple vacation model is the Bernoulli vacation. "After each service completion, the serve have an option to go for a short vacation with probability '$\theta(0 \leqslant \theta \leqslant 1)$' or starts a fresh service with the probability '$(1-\theta)$' if the queue length is at least 'a' ". This vacation model was introduced by Keilson and Servi [9] in $GI/G/1$ queuing model. Choudhury [6] studied a non-Markovian queue with a unreliable server, two phase services, repair after delay and Bernoulli schedule multiple vacation policy. Jeyakumar et al. [8] analysed bulk service queueing system with server breakdown and multiple working vacations.

The study of queuing models with server breakdowns can be dated back to the 1950s. In queueing theory, we termed these types of models as unreliable server queueing systems. In real life, we often come across a situation where service stations fail and are repaired. Senthilnathan and Arumuganathan [15] discussed an batch arrival non-Markovian retrial queue with unreliable server, two phase service and two phase repair. Rajadurai [14] studied a non-Markovian retrial G-queue with vacation interruption, unreliable server with different working vacation policy. Chang et al. [4] investigated a retrial queue with unreliable-server and customers feedback. Impatient clients are a very common happening in the queuing system. Customers arrive at the system, but may leave without joining the queue in the system (balking) or clients may join the queue for some time, but may quit the system without getting service (reneging). Choudhury and Medhi [5] discussed Balking and Reneging in multi server Markovian queuing system. Ayyappan and Shymala [3] analyzed the transient behavior of batch arrival non-Markovian queuing model with balking, Bernoulli vacation and state dependent arrival. Kailash C. Madan [13] explored a batch arrival queue with service in three fluctuating modes, balking, random breakdown and standby during breakdown periods. Recently, Binay Kumar [10] studied bulk queueing model with optional service, balking, unreliable server under Bernoulli vacation.

The concept of re-service in queueing model was introduced by Madan [12]. He considered a single server non-Markovian queuing model, in which the essential service was given to all clients in the system. "As soon as the first service is completed, the clients have an option to take second service with probability 'θ' or vacate the system with probability '$(1-\theta)$' ". Haridass et al. [7] analyzed a queuing system with setup, bulk service, multiple vacations and a new concept of admitting re-service on server's choice. Ayyappan and Sathiya [1] discussed the transient behavior of batch arrival queues with optional re-service and two stage regular service.

In queueing literature, there are very few papers discussed individually service interruptions, Bernoulli schedule multiple vacation, re-service and balking. Thus, in this work, we intend to study a queueing model that combine these attributes together with a bulk service queueing system. Moreover, another important characteristic is rendering re-service based on server decision that makes the system more realistic. We also take into consideration elapsed vacation time, the elapsed service time, elapsed re-service time and elapsed repair time as supplementary variables.

The above queueing model can be effectively used as a model in production system. Consider a production system, where the system does not initiate the production until some predefined amount of raw materials are accumulated. There are certain items that may require reprocessing service (re-service) apart from regular service. If the reprocessing service is admitted, then the raw materials waiting for production were increased. So, in many cases the reprocessing service may be provided (or rejected) based on certain constraints which does not affects the production system. At the completion of the processing of raw materials, the process may be stopped for maintenance of the system. This maintenance is considered as a vacation (phase I vacation) in our model. We consider that the raw material arrives in batches of variable size. Moreover, the production process may be disrupted due to breakdown and it is repaired immediately. On completion of each service, the system checks the amount of raw material, whether or not to start the major production. If at that instant, no considerable amount of raw material is found, then the system will perform an another optional job (phase II vacation) for a random length of time until it finds required availability of raw materials. Hence, our model can be estimated by this production system.

We organized the paper as follows. Figure 1 illustrates the schematic representation of the proposed model. The Mathematical analysis of the model is briefly described in Section 2. Section 3 focuses on the queue size distribution at a random epoch. PGF of queue size is obtained in Section 4. Stability condition and some particular cases are examined in Section 5 and Section 6. In Section 7, we derive the mean queue length and expected waiting time of the clients. Section 8 concerns with distribution of queue size at departure epoch. Numerical illustration, conclusions and further work are presented in Section 9 and Section 10 respectively.

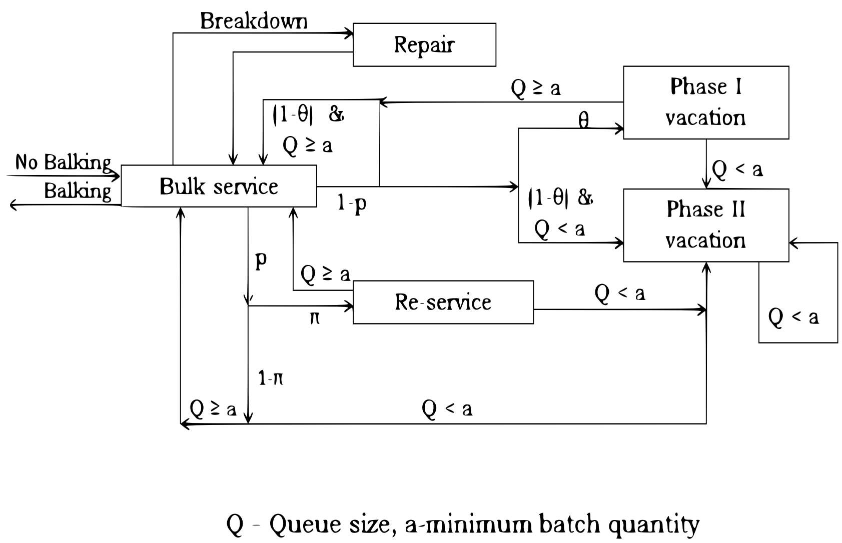

Figure 1: Schematic representation.

2 The Mathematical analysis of our model

We examine a batch arrival bulk service queueing system, where the customers reach the system according to a compound Poisson process, with arrival rate λ. Let $X_1, X_2, ...$ be the size of consecutive arriving batches which are identically independent random variables (i.i.d) with probability mass function (p.m.f) $m_j = Pr\{X = j\}; j \geqslant 1$, and probability generating function M(z). Service follows the "General Bulk Service Rule". The service times $\{B_{e,j}, j \geqslant 1\}$ of the clients are i.i.d. random variable with distribution function (DF) $B_1(g)$ and Laplace-Stieltjes Transform (LST) $\bar{B}_1(s)$. After service completion, the dispatched batch of clients can request for re-service with probability 'p' and it is not essential to accept it; the server concedes the request with probability 'π'. On each service completion, the server has a choice to go for a vacation with probability 'θ' and start fresh service with probability '$1-\theta$'. The re-service time $\{B_{f,j}, j \geqslant 1\}$ of the clients is i.i.d. random variable with DF $B_2(g)$ and LST $\bar{B}_2(s)$. While the server is doing the regular service (or re-service), it may downfall at any moment with a rate 'η' and the service channel will stop working for a short period of time and it is repaired immediately. The repair times $\{R_{e,j}, j \geqslant 1\}$ and $\{R_{f,j}, j \geqslant 1\}$, of the server are i.i.d random variable with distribution functions $R_1(y)$, $R_2(y)$ and LST $\bar{R}_1(s)$, $\bar{R}_2(s)$. After, the repair is completed, the server starts its remaining service (or re-service) to the batch of clients whose service was interrupted due to breakdown. After each service completion, the server takes phase-I vacation with probability 'θ' or may carry on his service to the next unit, if any, with probability '$1-\theta$' (i.e) the server acquire Bernoulli vacation. The phase-I vacation time random variable $\{Q_{e,j}, j \geqslant 1\}$ with DF $Q_1(g)$ and LST $\bar{Q}_1(s)$. Now, for further development, we introduce the approach of multiple vacation policy, where after each service or re-service or phase-I vacation completion, the server takes vacation sequentially until he finds at least 'a' clients in the queue called phase-II vacation (i.e) we have established multiple vacation policy after phase-I vacation. The phase-II vacation time random variable $\{Q_{f,j}, j \geqslant 1\}$ of the server expect to follow a general (arbitrary) distribution with DF $Q_2(g)$ and LST $\bar{Q}_2(s)$. The decision of taking phase-II vacation after each service or re-service or phase I vacation are independent.

3 Queue size distribution

In this section, we obtain the PGF of server's state and number of customers in the queue (the number in the queue excluding the batch being served, if any) at a random epoch.

Let S(t) be the queue length (size), $B_1^0(t)$, $B_2^0(t)$ be the elapsed service and re-service time, $Q_1^0(t)$, $Q_2^0(t)$ be the elapsed phase-I and phase-II vacation time and ${R_1}^0(t)$, ${R_2}^0(t)$ be the elapsed repair time during breakdown while rendering regular and re-service in the system at time 't'.

We define

$$\Omega(t) = \begin{cases} 1, & \text{the server is doing regular service at time 't',} \\ 2, & \text{server is rendering re-service at time 't',} \\ 3, & \text{server is on phase-I vacation at time 't',} \\ 4, & \text{server is on phase-II vacation at time 't',} \\ 5, & \text{server is breakdown and under repair while} \\ & \text{rendering the regular service at time 't',} \\ 6, & \text{server is breakdown and under repair while} \\ & \text{rendering the re-service at time 't'.} \end{cases}$$

Thus, the supplementary variables $B_1^0(t)$, $B_2^0(t)$, $Q_1^0(t)$, $Q_2^0(t)$, ${R_1}^0(t)$ and ${R_2}^0(t)$ are imported in order to attain a bivariate Markov process $\{S(t), D(t)\}$

where

$$D(t) = B_1^0(t) \text{ if } \Omega(t) = 1, \quad D(t) = B_2^0(t) \text{ if } \Omega(t) = 2,$$
$$D(t) = Q_1^0(t) \text{ if } \Omega(t) = 3, \quad D(t) = Q_2^0(t) \text{ if } \Omega(t) = 4,$$
$$D(t) = R_1^0(t) \text{ if } \Omega(t) = 5, \quad D(t) = R_2^0(t) \text{ if } \Omega(t) = 6.$$

Now, we illustrate the following probabilities:

- $B_{e,j}(g,t)$ represents the probability of exactly 'j' clients in the queue at time 't', excluding the batch under service and the server is busy with regular service with elapsed service time is 'g'.
- $B_{f,j}(g,t)$ represents the probability of exactly 'j' clients in the queue at time 't', excluding the batch under re-service and the server is providing re-service with the elapsed re-service time is 'g'.
- $Q_{e,j}(g,t)$ represents the probability of exactly 'j' clients in the queue at time 't' and the server is on phase-I vacation with the elapsed vacation time is 'g'.
- $Q_{f,j}(g,t)$ represents the probability of exactly 'j' clients in the queue at time 't' and the server is on phase-II vacation and with the elapsed vacation time is 'g'.
- $R_{e,j}(g,v,t)$ represents the probability of exactly 'j' clients in the queue at time 't', with elapsed service time is 'g' and the elapsed repair time is 'v'.
- $R_{f,j}(g,v,t)$ represents the probability of exactly 'j' clients in the queue at time 't', with elapsed re-service time of the batch of clients 'g' and the elapsed repair time is 'v'.

For the process,

$$B_{e,j}(g)dg = \lim_{t\to\infty} Pr[S(t) = j,\ D(t) = B_1^0(t);\ g < B_1^0(t) \le g + dg];\ g > 0,\ j \ge 0,$$
$$B_{f,j}(g)dg = \lim_{t\to\infty} Pr[S(t) = j,\ D(t) = B_2^0(t);\ g < B_2^0(t) \le g + dg];\ g > 0,\ j \ge 0,$$
$$Q_{e,j}(g)dg = \lim_{t\to\infty} Pr[S(t) = j,\ D(t) = Q_1^0(t);\ g < Q_1^0(t) \le g + dg];\ g > 0,\ j \ge 0,$$
$$Q_{f,j}(g)dg = \lim_{t\to\infty} Pr[S(t) = j,\ D(t) = Q_2^0(t);\ g < Q_2^0(t) \le g + dg];\ g > 0,\ j \ge 0,$$

and for fixed value of g and $j \ge 0$,

$$R_{e,j}(g,v)dv = \lim_{t\to\infty} Pr[S(t) = j, D(t) = R_1^0(t);$$
$$v < R_1^0(t) \le v + dv/B_1^0(t) = g]; (g,v) > 0,$$
$$R_{f,j}(g,v)dv = \lim_{t\to\infty} Pr[S(t) = j, D(t) = R_2^0(t);$$
$$v < R_2^0(t) \le v + dv/B_2^0(t) = g]; (g,v) > 0.$$

Further, it is assumed that

$$B_1(0) = B_2(0) = Q_1(0) = Q_2(0) = R_1(0) = R_2(0) = 0,$$

and

$$B_1(\infty) = B_2(\infty) = Q_1(\infty) = Q_2(\infty) = R_1(\infty) = R_2(\infty) = 1.$$

$B_1(g)$, $B_2(g)$, $Q_1(g)$, $Q_2(g)$ are continuous at $g = 0$ and $R_1(v)$, $R_2(v)$ are continuous at $v = 0$, so that

$$\phi(g)dg = \frac{dB_1(g)}{1 - B_1(g)}, \ \alpha(g)dg = \frac{dB_2(g)}{1 - B_2(g)}, \quad \nu_1(g)dg = \frac{dQ_1(g)}{1 - Q_1(g)},$$
$$\nu_2(g)dg = \frac{dQ_2(g)}{1 - Q_2(g)}, \ \zeta(v)dv = \frac{dR_1(v)}{1 - R_1(v)} \text{ and } \zeta(v)dv = \frac{dR_2(v)}{1 - R_2(v)}$$

are the hazard rates of $B_1(g)$, $B_2(g)$, $Q_1(g)$, $Q_2(g)$, $R_1(v)$ and $R_2(v)$, respectively.

The Kolmogorov forward equations in transient state:

$$\frac{\partial}{\partial g}B_{e,j}(g,t) + \frac{\partial}{\partial t}B_{e,j}(g,t) + (\lambda + \phi(g) + \eta)B_{e,j}(g,t) = \lambda(1 - \delta_{j,0}) \sum_{k=1}^{j} c_k B_{e,j-k}(g,t) + \int_0^\infty R_{e,j}(g,v,t)\zeta(v)dv, \ j \geq 0, \quad (1)$$

$$\frac{\partial}{\partial g}B_{f,j}(g,t) + \frac{\partial}{\partial t}B_{f,j}(g,t) + (\lambda + \alpha(g) + \eta)B_{f,j}(g,t) = \lambda(1 - \delta_{j,0}) \sum_{k=1}^{j} c_k B_{f,j-k}(g,t) + \int_0^\infty R_{f,j}(g,v,t)\zeta(v)dv, \ j \geq 0, \quad (2)$$

$$\frac{\partial}{\partial g}Q_{e,j}(g,t) + \frac{\partial}{\partial t}Q_{e,j}(g,t) + (\lambda + \nu_1(g))Q_{e,j}(g,t) = \lambda(1 - \delta_{j,0})(1 - \beta)Q_{e,j}(g,t) + \beta\lambda \sum_{k=1}^{j} c_k Q_{e,j-k}(g,t), \ j \geq 0, \quad (3)$$

$$\frac{\partial}{\partial g}Q_{f,j}(g,t) + \frac{\partial}{\partial t}Q_{f,j}(g,t) + (\lambda + \nu_2(g))Q_{f,j}(g,t) = \lambda(1 - \delta_{j,0})(1 - \beta)Q_{f,j}(g,t) + \beta\lambda \sum_{k=1}^{j} c_k Q_{f,j-k}(g,t), \ j \geq 0, \quad (4)$$

$$\frac{\partial}{\partial v}R_{e,j}(g,v,t)+\frac{\partial}{\partial t}R_{e,j}(g,v,t)+(\lambda+\zeta(v))R_{e,j}(g,v,t)$$
$$=\lambda(1-\delta_{j,0})\sum_{k=1}^{j}c_k R_{e,j-k}(g,v,t),\ j\geq 0, \tag{5}$$
$$\frac{\partial}{\partial v}R_{f,j}(g,v,t)+\frac{\partial}{\partial t}R_{f,j}(g,v,t)+(\lambda+\zeta(v))R_{f,j}(g,v,t)$$
$$=\lambda(1-\delta_{j,0})\sum_{k=1}^{j}c_k R_{f,j-k}(g,v,t),\ j\geq 0, \tag{6}$$

where $\delta_{m,j}$ denotes Kronecker's function.

Boundary conditions at $g=0$:

$$B_{e,0}(0,t)=(1-\theta)(1-p)\sum_{r=a}^{b}\int_0^\infty B_{e,r}(g,t)\phi(g)dg+\sum_{r=a}^{b}\int_0^\infty Q_{e,r}(g,t)\nu_1(g)dg$$
$$+(1-\theta)p(1-\pi)\sum_{r=a}^{b}\int_0^\infty B_{e,r}(g,t)\phi(g)dg+\sum_{r=a}^{b}\int_0^\infty Q_{f,r}(g,t)\nu_2(g)dg$$
$$+(1-\theta)\sum_{r=a}^{b}\int_0^\infty B_{f,r}(g,t)\alpha(g)dg, \tag{7}$$
$$B_{e,j}(0,t)=(1-\theta)(1-p)\int_0^\infty B_{e,j+b}(g,t)\phi(g)dg+\int_0^\infty Q_{e,j+b}(g,t)\nu_1(g)dg$$
$$+(1-\theta)p(1-\pi)\int_0^\infty B_{e,j+b}(g,t)\phi(g)dg+\int_0^\infty Q_{f,j+b}(g,t)\nu_2(g)dg$$
$$+(1-\theta)\int_0^\infty B_{f,j+b}(g,t)\alpha(g)dg,\ j\geq 1, \tag{8}$$
$$B_{f,j}(0,t)=p\pi\int_0^\infty B_{e,j}(g,t)\phi(g)dg,\ j\geq 0, \tag{9}$$
$$Q_{e,j}(0,t)=\theta(1-p)\int_0^\infty B_{e,j}(g,t)\phi(g)dg+\theta p(1-\pi)\int_0^\infty B_{e,j}(g,t)\phi(g)dg$$
$$+\theta\int_0^\infty B_{f,j}(g,t)\alpha(g)dg,\ j\geq 0, \tag{10}$$
$$Q_{f,j}(0,t)=(1-\theta)(1-p)\int_0^\infty B_{e,j}(g,t)\phi(g)dg+\int_0^\infty Q_{e,j}(g,t)\nu_1(g)dg$$
$$+(1-\theta)p(1-\pi)\int_0^\infty B_{e,j}(g,t)\phi(g)dg+\int_0^\infty Q_{f,j}(g,t)\nu_2(g)dg$$
$$+(1-\theta)\int_0^\infty B_{f,j}(g,t)\alpha(g)dg,\ j=0,1,2,\ldots,a-1, \tag{11}$$
$$Q_{f,j}(0,t)=0,\ j\geq a, \tag{12}$$

and at $v = 0$ and fixed value of 'g'

$$R_{e,j}(g,0,t) = \eta B_{e,j}(g,t),\ j \geq 0, \tag{13}$$

$$R_{f,j}(g,0,t) = \eta B_{f,j}(g,t),\ j \geq 0. \tag{14}$$

The initial conditions are:

$$B_{e,j}(0) = B_{f,j}(0) = Q_{e,j}(0) = Q_{f,j}(0) = R_{e,j}(0) = R_{f,j}(0) = 0, \quad j = 0,1,2,\ldots. \tag{15}$$

The normalizing condition is:

$$\sum_{j=0}^{\infty}\int_0^{\infty} B_{e,j}(g)dg + \sum_{j=0}^{\infty}\int_0^{\infty} B_{f,j}(g)dg + \sum_{j=0}^{\infty}\int_0^{\infty} Q_{e,j}(g)dg + \sum_{j=0}^{\infty}\int_0^{\infty} Q_{f,j}(g)dg + \sum_{j=0}^{\infty}\int_0^{\infty}\int_0^{\infty} R_{e,j}(g,v)dgdv + \sum_{j=0}^{\infty}\int_0^{\infty}\int_0^{\infty} R_{f,j}(g,v)dgdv = 1. \tag{16}$$

Probability generating functions for $|z| \leq 1$:

$$\begin{aligned}
B_e(g,z,t) &= \sum_{j=0}^{\infty} z^j B_{e,j}(g,t),\ B_e(0,z,t) = \sum_{j=0}^{\infty} z^j B_{e,j}(0,t),\\
B_f(g,z,t) &= \sum_{j=0}^{\infty} z^j B_{f,j}(g,t),\ B_f(0,z,t) = \sum_{j=0}^{\infty} z^j B_{f,j}(0,t),\\
Q_e(g,z,t) &= \sum_{j=0}^{\infty} z^j Q_{e,j}(g,t),\ Q_e(0,z,t) = \sum_{j=0}^{\infty} z^j Q_{e,j}(0,t),\\
Q_f(g,z,t) &= \sum_{j=0}^{\infty} z^j Q_{f,j}(g,t),\ Q_f(0,z,t) = \sum_{j=0}^{\infty} z^j Q_{f,j}(0,t),\\
R_e(g,v.z,t) &= \sum_{j=0}^{\infty} z^j R_{e,j}(g,v,t),\ R_e(g,0.z,t) = \sum_{j=0}^{\infty} z^j R_{e,j}(g,0.t),\\
R_f(g,v.z,t) &= \sum_{j=0}^{\infty} z^j R_{f,j}(g,v,t),\ R_f(g,0.z,t) = \sum_{j=0}^{\infty} z^j R_{f,j}(g,0.t),
\end{aligned} \tag{17}$$

The Laplace transform of $h(t)$ is:

$$\bar{h}(s) = \int_0^{\infty} e^{-st}h(t)dt, \quad \Re(s) > 0. \tag{18}$$

By applying the Laplace transform technique on both sides of the equations (1) to (14) and using equation (15), we get

$$\frac{\partial}{\partial g}\bar{B}_{e,j}(g,s)+(s+\lambda+\phi(g)+\eta)\bar{B}_{e,j}(g,s)=\lambda(1-\delta_{j,0})\sum_{k=1}^{j}c_k\bar{B}_{e,j-k}(g,s)+\int_0^\infty\bar{R}_{e,j}(g,v,s)\zeta(v)dv,\ j\geq 0, \tag{19}$$

$$\frac{\partial}{\partial g}\bar{B}_{f,j}(g,s)+(s+\lambda+\alpha(g)+\eta)\bar{B}_{f,j}(g,s)=\lambda(1-\delta_{j,0})\sum_{k=1}^{j}c_k\bar{B}_{f,j-k}(g,s)+\int_0^\infty\bar{R}_{f,j}(g,v,s)\zeta(v)dv,\ j\geq 0, \tag{20}$$

$$\frac{\partial}{\partial g}\bar{Q}_{e,j}(g,s)+(s+\lambda\beta+\nu_1(g))\bar{Q}_{e,j}(g,s)=\beta\lambda(1-\delta_{j,0})\sum_{k=1}^{j}c_k\bar{Q}_{e,j-k}(g,s),\quad j\geq 0, \tag{21}$$

$$\frac{\partial}{\partial g}\bar{Q}_{f,j}(g,s)+(s+\lambda+\nu_2(g))\bar{Q}_{f,j}(g,s)=\beta\lambda(1-\delta_{jn,0})\sum_{k=1}^{j}c_k\bar{Q}_{f,j-k}(g,s),\quad j\geq 0, \tag{22}$$

$$\frac{\partial}{\partial v}\bar{R}_{e,j}(g,v,s)+(s+\lambda+\zeta(v))\bar{R}_{e,j}(g,v,s)=\lambda(1-\delta_{j,0})\sum_{k=1}^{j}c_k\bar{R}_{e,j-k}(g,v,s),\quad j\geq 0, \tag{23}$$

$$\frac{\partial}{\partial v}\bar{R}_{f,j}(g,v,s)+(s+\lambda+\zeta(v))\bar{R}_{f,j}(g,v,s)=\lambda(1-\delta_{j,0})\sum_{k=1}^{j}c_k\bar{R}_{f,j-k}(g,v,s),\quad j\geq 0, \tag{24}$$

$$\begin{aligned}\bar{B}_{e,0}(0,s)&=(1-\theta)(1-p)\sum_{r=a}^{b}\int_0^\infty\bar{B}_{e,r}(g,s)\phi(g)dg+\sum_{r=a}^{b}\int_0^\infty\bar{Q}_{e,r}(g,s)\nu_1(g)dg\\&+(1-\theta)p(1-\pi)\sum_{r=a}^{b}\int_0^\infty\bar{B}_{e,r}(g,s)\phi(g)dg+\sum_{r=a}^{b}\int_0^\infty\bar{Q}_{f,r}(g,s)\nu_2(g)dg\\&+(1-\theta)\sum_{r=a}^{b}\int_0^\infty\bar{B}_{f,r}(g,s)\alpha(g)dg,\end{aligned} \tag{25}$$

$$\begin{aligned}\bar{B}_{e,j}(0,s)&=(1-\theta)(1-p)\int_0^\infty\bar{B}_{e,j+b}(g,s)\phi(g)dg+\int_0^\infty\bar{Q}_{e,j+b}(g,s)\nu_1(g)dg\\&+(1-\theta)p(1-\pi)\int_0^\infty\bar{B}_{e,j+b}(g,s)\phi(g)dg+\int_0^\infty\bar{Q}_{f,j+b}(g,s)\nu_2(g)dg\\&+(1-\theta)\int_0^\infty\bar{B}_{f,j+b}(g,s)\alpha(g)dg,\ j\geq 1,\end{aligned} \tag{26}$$

$$\bar{B}_{f,j}(0,s) = p\pi \int_0^\infty \bar{B}_{e,j}(g,s)\phi(g)dg,\ \ j \geq 0, \tag{27}$$

$$\begin{aligned}\bar{Q}_{e,j}(0,s) = \theta(1-p) &\int_0^\infty \bar{B}_{e,j}(g,s)\phi(g)dg \\ &+ \theta p(1-\pi)\int_0^\infty \bar{B}_{e,j}(g,s)\phi(g)dg \\ &+ \theta \int_0^\infty \bar{B}_{f,j}(g,s)\alpha(g)dg,\ \ j \geq 0,\end{aligned} \tag{28}$$

$$\begin{aligned}\bar{Q}_{f,j}(0,s) = (1-\theta)(1-p) &\int_0^\infty \bar{B}_{e,j}(g,s)\phi(g)dg + \int_0^\infty \bar{Q}_{e,j}(g,s)\nu_1(g)dg \\ &+ (1-\theta)p(1-\pi)\int_0^\infty \bar{B}_{e,j}(g,s)\phi(g)dg + \int_0^\infty \bar{Q}_{f,j}(g,s)\nu_2(g)dg \\ &+ (1-\theta)\int_0^\infty \bar{B}_{f,j}(g,s)\alpha(g)dg,\ \ j = 0,1,2,\ldots,a-1,\end{aligned} \tag{29}$$

$$\bar{Q}_{f,j}(0,s) = 0,\ \ j \geq a, \tag{30}$$

$$\bar{R}_{e,j}(g,0,s) = \eta \bar{B}_{e,j}(g,s),\ \ j \geq 0, \tag{31}$$

$$\bar{R}_{f,j}(g,0,s) = \eta \bar{B}_{f,j}(g,s),\ \ j \geq 0. \tag{32}$$

Multiplying equations (19) to (24) by z^j, and then taking summation over all possible values of j, $(j = 0,1,2,\ldots)$ and using (17), we get

$$\frac{\partial}{\partial g}\bar{B}_e(g,z,s) + \Big[s + \lambda(1-M(z)) + \phi(g) + \eta\Big]\bar{B}_e(g,z,s) = \int_0^\infty \bar{R}_e(g,v,z,s)\zeta(v)dv, \tag{33}$$

$$\frac{\partial}{\partial g}\bar{B}_f(g,z,s) + \Big[s + \lambda(1-M(z)) + \alpha(g) + \eta\Big]\bar{B}_f(g,z,s) = \int_0^\infty \bar{R}_f(g,v,z,s)\zeta(v)dv, \tag{34}$$

$$\frac{\partial}{\partial g}\bar{Q}_e(g,z,s) + \Big[s + \lambda\beta(1-M(z)) + \nu_1(g)\Big]\bar{Q}_e(g,z,s) = 0, \tag{35}$$

$$\frac{\partial}{\partial g}\bar{Q}_f(g,z,s) + \Big[s + \lambda\beta(1-M(z)) + \nu_2(g)\Big]\bar{Q}_f(g,z,s) = 0, \tag{36}$$

$$\frac{\partial}{\partial v}\bar{R}_e(g,v,z,s) + \Big[s + \lambda(1-M(z)) + \zeta(v)\Big]\bar{R}_e(g,v,z,s) = 0, \tag{37}$$

$$\frac{\partial}{\partial v}\bar{R}_f(g,v,z,s) + \Big[s + \lambda(1-M(z)) + \zeta(v)\Big]\bar{R}_f(g,v,z,s) = 0. \tag{38}$$

Multiplying equations (25) to (32) by appropriate powers of 'z' and take summation over all possible values of 'j' and using PGF, we get

$$\begin{aligned}z^b\bar{B}_e(0,z,s) &= \Big[\int_0^\infty \bar{B}_e(g,z,s)\phi(g)dg + \sum_{r=a}^{b-1}\int_0^\infty \bar{B}_{e,r}(g,s)\phi(g)(z^b - z^r)dg,\\ &- \sum_{r=0}^{a-1}\int_0^\infty \bar{B}_{e,r}(g,s)\phi(g)z^r dg\Big](1-\theta)(1-p\pi) + \int_0^\infty \bar{Q}_e(g,z,s)\nu_1(g)dg\\ &+ (1-\theta)\Big[\int_0^\infty \bar{B}_f(g,z,s)\alpha(g)dg - \sum_{r=0}^{a-1}\int_0^\infty \bar{B}_{f,r}(g,s)\alpha(g)z^r dg\\ &+ \sum_{r=a}^{b-1}\int_0^\infty \bar{B}_{f,r}(g,s)\alpha(g)(z^b - z^r)dg\Big] + \int_0^\infty \bar{Q}_f(g,z,s)\nu_2(g)dg\\ &- \sum_{r=0}^{a-1}\Big[\int_0^\infty \bar{Q}_{e,r}(g,s)\nu_1(g)z^r dg + \int_0^\infty \bar{Q}_{f,r}(g,s)\nu_2(g)z^r dg\Big]\\ &+ \sum_{r=a}^{b-1}(z^b - z^r)\Big[\int_0^\infty \bar{Q}_{f,r}(g,s)\nu_2(g)dg + \int_0^\infty \bar{Q}_{e,r}(g,s)\nu_1(g)dg\Big] \end{aligned} \tag{39}$$

$$\bar{B}_f(0,z,s) = p\pi\int_0^\infty \bar{B}_e(g,z,s)\phi(g)dg, \tag{40}$$

$$\bar{Q}_e(0,z,s) = \theta(1-p\pi)\int_0^\infty \bar{B}_e(g,z,s)\phi(g)dg + \theta\int_0^\infty \bar{B}_{f,r}(g,s)\alpha(g)dg, \tag{41}$$

$$\begin{aligned}\bar{Q}_f(0,z,s) &= (1-\theta)(1-p\pi)\sum_{r=0}^{a-1}\int_0^\infty \bar{B}_{e,r}(g,s)\phi(g)z^r dg\\ &+ \sum_{r=0}^{a-1}\int_0^\infty \bar{Q}_{e,r}(g,s)\nu_1(g)z^r dg + \sum_{r=0}^{a-1}\int_0^\infty \bar{Q}_{f,r}(g,s)\nu_2(g)z^r dg\\ &+ (1-\theta)\sum_{r=0}^{a-1}\int_0^\infty \bar{B}_{f,r}(g,s)\alpha(g)z^r dg,\end{aligned} \tag{42}$$

$$\bar{R}_e(g,0,z,s) = \eta\bar{B}_e(g,z,s), \tag{43}$$

$$\bar{R}_f(g,0,z,s) = \eta\bar{B}_f(g,z,s). \tag{44}$$

Integrating equations (35) to (38), we get

$$\bar{Q}_e(g,z,s) = \bar{Q}_e(0,z,s)(1-\bar{Q}_1(g))e^{-\{H_1(z)\}g}, \tag{45}$$

$$\bar{Q}_f(g,z,s) = \bar{Q}_f(0,z,s)(1-\bar{Q}_2(g))e^{-\{H_1(z)\}g}, \tag{46}$$

$$\bar{R}_e(g,v,z,s) = \bar{R}_e(g,0,z,s)(1-\bar{R}_1(v))e^{-\{H_2(z)\}v}, \tag{47}$$

$$\bar{R}_f(g,v,z,s) = \bar{R}_f(g,0,z,s)(1-\bar{R}_2(v))e^{-\{H_2(z)\}v}. \tag{48}$$

where $H_1(z)$ and $H_2(z)$ are given in Appendix.

Multiplying both sides of the equations (45) and (46) by $\nu_1(g)$ and $\nu_2(g)$ and further integrating over 'g', we get

$$\int_0^\infty \bar{Q}_e(g,z,s)\nu_1(g)dg = \bar{Q}_e(0,z,s)\bar{Q}_1[H_1(z)], \tag{49}$$

$$\int_0^\infty \bar{Q}_f(g,z,s)\nu_2(g)dg = \bar{Q}_f(0,z,s)\bar{Q}_2[H_1(z)]. \tag{50}$$

Multiplying both sides of the equations (47) and (48) by $\zeta(v)$ and then integrating over 'v', we obtain

$$\int_0^\infty \bar{R}_e(g,v,z,s)\zeta(v)dv = \bar{R}_e(g,0,z,s)\bar{R}_1[H_2(z)], \tag{51}$$

$$\int_0^\infty \bar{R}_f(g,v,z,s)\zeta(v)dv = \bar{R}_f(g,0,z,s)\bar{R}_2[H_2(z)]. \tag{52}$$

Again integrating equations (45) to (48), we obtain

$$\int_0^\infty \bar{Q}_e(g,z,s)dg = \bar{Q}_e(z,s) = \bar{Q}_e(0,z,s)\left[\frac{1-\bar{Q}_1(H_1(z))}{H_1(z)}\right], \tag{53}$$

$$\int_0^\infty \bar{Q}_f(g,z,s)dg = \bar{Q}_f(z,s) = \bar{Q}_f(0,z,s)\left[\frac{1-\bar{Q}_2(H_1(z))}{H_1(z)}\right], \tag{54}$$

$$\int_0^\infty \bar{R}_e(g,v,z,s)dv = \bar{R}_e(g,z,s) = \bar{R}_e(g,0,z,s)\left[\frac{1-\bar{R}_1(H_2(z))}{H_2(z)}\right], \tag{55}$$

$$\int_0^\infty \bar{R}_f(g,v,z,s)dv = \bar{R}_f(g,z,s) = \bar{R}_f(g,0,z,s)\left[\frac{1-\bar{R}_2(H_2(z))}{H_2(z)}\right]. \tag{56}$$

Utilizing equations (43) and (51) in equation (33) on simplification, we obtain

$$\frac{\partial}{\partial g}\bar{B}_e(g,z,s) + (s+\lambda(1-M(z))+\phi(g)+\eta)\bar{B}_e(g,z,s) = \eta\bar{R}_1(H_2(z))\bar{B}_e(g,z,s).$$

On integrating the above equation, we get

$$\bar{B}_e(g,z,s) = \bar{B}_e(0,z,s)(1-\bar{B}_1(g))e^{-\{\psi_1(z)\}g}, \tag{57}$$

where $\psi_1(z)$ is given in Appendix.

Multiplying both sides of (57) by $\phi(g)$ and integrating over 'g', we get

$$\int_0^\infty \bar{B}_e(g,z,s)\phi(g)dg = \bar{B}_e(0,z,s)\bar{B}_1[\psi_1(z)]. \tag{58}$$

Again integrating equation (57), we get

$$\int_0^\infty \bar{B}_e(g,z,s)dg = \bar{B}_e(z,s) = \bar{B}_e(0,z,s)\left[\frac{1-\bar{B}_1(\psi_1(z))}{\psi_1(z)}\right]. \tag{59}$$

Utilizing equations (44) and (52) in equation (34) on simplification, we obtain

$$\frac{\partial}{\partial g}\bar{B}_f(g,z,s) + (s+\lambda(1-M(z))+\alpha(g)+\eta)\bar{B}_f(g,z,s) = \eta\bar{R}_2(H_2(z))\bar{B}_f(g,z,s).$$

On integrating the above equation, we get

$$\bar{B}_f(g,z,s) = \bar{B}_f(0,z,s)(1-\bar{B}_2(g))e^{-\{\psi_2(z)\}g}. \tag{60}$$

Multiplying both sides of (60) by $\alpha(g)$ and integrating over 'g', we get

$$\int\limits_0^\infty \bar{B}_f(g,z,s)\alpha(g)dg = \bar{B}_f(0,z,s)\bar{B}_2[\psi_2(z)]. \tag{61}$$

Again integrating equation (60), we get

$$\int_0^\infty \bar{B}_f(g,z,s)dg = \bar{B}_f(z,s) = \bar{B}_f(0,z,s)\left[\frac{1-\bar{B}_2(\psi_2(z))}{\psi_2(z)}\right]. \tag{62}$$

Utilizing equations (49), (58) and (61) in equation (39), we obtain

$$\bar{B}_e(0,z,s) = \frac{\left[\sum\limits_{r=0}^{a-1} q_r z^r(\bar{Q}_2(H_3(z))-1) + \sum\limits_{r=a}^{b-1}\omega_r(z^b-z^r)\right]}{[z^b-\bar{B}_1(\psi_3(z))Y(z)]}, \tag{63}$$

where $Y(z), q_r$ and ω_r are given in Appendix.

From equation (55),

$$\bar{R}_e(z,s) = \int_0^\infty \bar{R}_e(g,z,s)dg = \int_0^\infty \bar{R}_e(g,0,z,s)\left[\frac{1-\bar{R}_1(H_2(z))}{H_2(z)}\right]dg.$$

Using equations (43) and (59) in the above equation, we get

$$\bar{R}_e(z,s) = \eta\bar{B}_e(0,z,s)\left[\frac{1-\bar{B}_1(\psi_1(z))}{\psi_1(z)}\right]\left[\frac{1-\bar{R}_1(H_2(z))}{H_2(z)}\right]. \tag{64}$$

From equation (56),

$$\bar{R}_f(z,s) = \int_0^\infty \bar{R}_f(g,z,s)dg = \int_0^\infty \bar{R}_f(g,0,z,s)\left[\frac{1-\bar{R}_2(H_2(z))}{H_2(z)}\right]dg.$$

Similarly, by using equations (40), (44) and (62) in the above equation, we get

$$\bar{R}_f(z,s) = \eta p\pi\bar{B}_1(\psi_1(z))\bar{B}_e(0,z,s)\left[\frac{1-\bar{B}_2(\psi_2(z))}{\psi_2(z)}\right]\left[\frac{1-\bar{R}_2(H_2(z))}{H_2(z)}\right]. \tag{65}$$

Thus $\bar{B}_e(z,s), \bar{B}_f(z,s), \bar{Q}_e(z,s), \bar{Q}_f(z,s), \bar{R}_e(z,s)$ and $\bar{R}_f(z,s)$ are completely determined from equations (59), (62), (53), (54), (64) and (65) where $\bar{B}_e(0,z,s)$ is given in equation (63).

4 Probability generating function of queue size

By applying Tauberian property, the PGF of the queue size at an arbitrary time epoch in steady state is given as

$$\lim_{s\to 0} s\bar{h}(s) = \lim_{t\to\infty} h(t). \tag{66}$$

Therefore, in steady state, the PGF of queue size when the server is busy, on vacation and repair are given below:

$$B_e(z) = \frac{\left[\begin{array}{l}[1-\bar{B}_1(\psi_3(z))]\times \\ \Big[\sum_{r=0}^{a-1} q_r z^r[\bar{Q}_2(H_3(z))-1] + \sum_{r=a}^{b-1}\omega_r(z^b - z^r)\Big]\end{array}\right]}{\psi_3(z)[z^b-(\bar{B}_1(\psi_3(z))Y(z)]}, \tag{67}$$

$$B_f(z) = \frac{\left[\begin{array}{l}p\pi\bar{B}_1(\psi_3(z))[1-\bar{B}_2(\psi_4(z))]\times \\ \Big[\sum_{r=0}^{a-1} q_r z^r[\bar{Q}_2(H_3(z))-1] + \sum_{r=a}^{b-1}\omega_r(z^b - z^r)\Big]\end{array}\right]}{H_4(z)\psi_3(z)[z^b-(\bar{B}_1(\psi_3(z))Y(z)]}, \tag{68}$$

$$Q_e(z) = \frac{\left[\begin{array}{l}\theta\bar{B}_1(\psi_3(z))(1-p\pi(1-\bar{B}_2(\psi_4(z))[1-\bar{B}_2(\psi_4(z))]\psi_4(z)\times \\ \Big[\sum_{r=0}^{a-1} q_r z^r[\bar{Q}_2(H_3(z))-1] + \sum_{r=a}^{b-1}\omega_r(z^b - z^r)\Big]\end{array}\right]}{H_4(z)\psi_3(z)[z^b-(\bar{B}_1(\psi_3(z))Y(z)]}, \tag{69}$$

$$Q_f(z) = \frac{\Big[\sum_{r=0}^{a-1} q_r z^r[1-\bar{Q}_2(H_3(z)]\Big]}{H_3(z)}, \tag{70}$$

$$R_e(z) = \frac{\left[\begin{array}{l}\eta[1-\bar{R}_1(H_4(z))][1-\bar{B}_1(\psi_3(z))]\times \\ \Big[\sum_{r=0}^{a-1} q_r z^r[\bar{Q}_2(H_3(z))-1] + \sum_{r=a}^{b-1}\omega_r(z^b - z^r)\Big]\end{array}\right]}{H_4(z)\psi_3(z)[z^b-(\bar{B}_1(\psi_3(z))Y(z)]}, \tag{71}$$

$$R_f(z) = \frac{\left[\begin{array}{l}\eta p\pi\bar{B}_1(\psi_3(z))[1-\bar{R}_2(H_4(z))][1-\bar{B}_2(\psi_4(z))]\times \\ \Big[\sum_{r=0}^{a-1} q_r z^r[\bar{Q}_2(H_3(z))-1] + \sum_{r=a}^{b-1}\omega_r(z^b - z^r)\Big]\end{array}\right]}{H_4(z)\psi_4(z)[z^b-(\bar{B}_1(\psi_3(z))Y(z)]}, \tag{72}$$

Finally, the PGF of queue size is

$$P(z) = B_e(z) + Q_e(z) + Q_f(z) + B_f(z) + R_e(z) + R_f(z). \tag{73}$$

Substituting equations (67) to (72) in equation (73), we get

$$P(z)=\frac{\begin{bmatrix}\Big[\sum_{r=0}^{a-1}q_r z^r(\bar{Q}_2(H_3(z))-1)+\sum_{r=a}^{b-1}\omega_r(z^b-z^r)\Big]\times\Bigg[[1-\bar{B}_1(\psi_3(z))]\\ \psi_4(z)H_3(z)[H_4(z)+\eta(1-\bar{R}_1(H_4(z))]+p\pi\bar{B}_1(\psi_3(z))\psi_3(z)H_3(z)\\ (1-\bar{B}_2(\psi_4(z)))\Big[H_4(z)+\eta(1-\bar{R}_2(H_4(z)))\Big]+(1-\bar{Q}_1(H_3(z)))\\ \times\theta\bar{B}_1(\psi_3(z))(1-p\pi(1-\bar{B}_2(\psi_4(z))))\psi_3(z)\psi_4(z)H_4(z)\Bigg]\\ +\sum_{r=0}^{a-1}q_r z^r[z^b-\bar{B}_1(\psi_3(z))(1-p\pi(1-\bar{B}_2(\psi_4(z)))\\ \times(1-\theta(1-\bar{Q}_1(H_3(z)))]\psi_3(z)\psi_4(z)H_4(z)(1-\bar{Q}_2(H_3(z)))\end{bmatrix}}{H_3(z)H_4(z)\psi_3(z)\psi_4(z)\Big[z^b-\bar{B}_1(\psi_3(z))Y(z)\Big]} \tag{74}$$

where ω_r , q_r, $\psi_3(z)$, $\psi_4(z)$, $H_3(z)$, $H_4(z)$ are given in Appendix .

Equation (74) gives the PGF of the queue size involving 'b' unknowns. By Rouche's theorem of complex variables, "$[z^b-\bar{B}_1(\psi_1(z))Y(z)]$ has 'b' zeroes out of which '$b-1$' zeroes inside and one on the unit circle $|z|=1$. Since P(z) is analytic, the numerator must vanish at these points and gives 'b' equations with 'b' unknowns". These equations can be solved by using MATLAB.

5 Stability condition

The PGF should satisfy $P(1)=1$. Now, apply L'Hopital rule and equating the expression to 1 results in

$$\begin{aligned}&[\lambda\beta E(X)E(Q_2)\sum_{r=0}^{a-1}q_r+\sum_{r=a}^{b-1}\omega_r(b-r)][E(B_1)(1+\eta E(R_1))+p\pi E(B_2)\\ &\qquad\times(1+\eta E(R_2))+E(Q_1)\theta]+E(Q_2)\sum_{r=0}^{a-1}q_r[b-\lambda E(X)(E(B_1))\\ &\qquad\times(1+\eta E(R_1))+p\pi E(B_2)(1+\eta E(R_2)+\theta\beta E(Q_1)]\\ &=\Big[b-\lambda E(X)[E(B_1)(1+\eta E(R_1))+p\pi E(B_2)(1+\eta E(R_2))+\theta\beta E(Q_1)]\Big],\end{aligned}$$

since ω_r, q_r are probabilities of 'r' clients in the queue, it follows that $P(1)=1$ is satisfied if $[z^b-\bar{B}_1(\psi_3(z))Y(z)]>0$. If

$$\rho=\lambda E(X)[E(B_1)(1+\eta E(R_1))+p\pi E(B_2)(1+\eta E(R_2))+\theta\beta E(Q_1)]/b,$$

then $\rho<1$ is the condition to be convinced for the existence of steady state for the model under consideration.

6 Particular cases

Case (i): When there is no Bernoulli vacation, no server breakdown the equation (74) reduces to

$$P(z) = \frac{\left[\begin{array}{l}\displaystyle\sum_{r=a}^{b-1}\omega_r(z^b - z^r)\Big[(\bar{B}_1(H_4(z)) - 1) + (\bar{B}_2(H_4(z)) - 1)p\pi \\ (\bar{B}_1(H_4(z))\Big] + \displaystyle\sum_{r=0}^{a-1} q_r z^r \Big(\bar{Q}(H_4(z)) - 1\Big)(z^b - 1)\end{array}\right]}{(-H_4(z))[z^b - \bar{B}_1(H_4(z))]}.$$

which is the PGF of Haridass and Arumuganathan [7] for a bulk service queueing system with server choice of admitting re-service under multiple vacation, without server breakdown and setup.

Case (ii): If no bulk service, no Bernoulli vacation and no breakdown and balking, the equation (74) reduces to

$$P(z) = \frac{\Big[q_0(z-1)\Big(\bar{Q}(\lambda - \lambda M(z)) - 1\Big)\Big]}{(-\lambda + \lambda M(z))[z - \bar{B}(\lambda - \lambda M(z))]}.$$

which exactly coincides with PGF of Ayyappan and Sathiya [2] for a batch arrival non-Markovian queueing model with multiple vacation, single type of service, without breakdown and no restricted admissibility.

7 Performance measures

7.1 Expected queue size

The average queue length L_q at an arbitrary time epoch is obtained by differentiating $P(z)$ at $z = 1$ and is given by

$$L_q = \left[\frac{D^{(V)}(1)\ N^{(VI)}(1) - N^{(V)}(1)\ D^{(VI)}(1)}{6(D^{(V)}(1))^2}\right], \tag{75}$$

where

$$\begin{aligned} D^{(VI)}(1) &= 360X_1^3\beta[2X_2R_{11}R_{12}(b - P_{13}) + X_1R_{11}R_{12}A \\ &\quad +(b - P_{13})(R_{12}P_{11} + R_{11}P_{12})], \\ N^{(V)}(1) &= 120X_1^4\beta R_{11}R_{12}\Big[P_2(B_1R_{11} + p\pi B_2R_{12} + Q_1\theta) \\ &\quad +Q_2\sum_{r=0}^{a-1} q_r(b - P_{13})\Big], \\ D^{(V)}(1) &= 120X_1^4\beta R_{11}R_{12}[b - P_{13}], \end{aligned}$$

and

$$
\begin{aligned}
N^{(VI)}(1) \;=\; & 360X_1^3\beta\Bigg[R_{11}[R_{12}P_1(X_1\beta R_{11}B_1 + P_2(R_{12}P_3 + R_{12}B_1P_4 \\
& +B_1R_{11}P_5)] + p\pi R_{12}[X_1\beta R_{11}R_{12}B_2P_1 + P_2(R_{11}P_6 + B_2R_{11}P_7 \\
& +R_{12}B_2P_8)] + \theta[Q_1X_1\beta R_{11}R_{12}P_1 + P_2(2X_1^2R_{11}R_{12}Q_1P_9 \\
& +R_{11}R_{12}P_{10} + Q_1(R_{12}P_{11} + R_{11}P_{12}))] + R_{11}R_{12}P_1(b - P_{13}) \\
& +X_1R_{11}R_{12}Q_2\sum_{r=0}^{a-1} q_r\Big[b.(b-1) - 2\theta\beta X_1^2Q_1(B_1R_{11} + p\pi R_{12}B_2) \\
& -2X_1^2p\pi B_1B_2R_{11}R_{12} - X_1^2B_3R_{11}^2 - B_1(X_2R_{11} + \eta X_1^2R_3) \\
& -X_1^2B_4p\pi R_{12}^2 - p\pi B_2(X_2R_{12} + \eta X_1^2R_4) - \theta\beta(X_1^2\beta Q_3 + X_2Q_1)\Big] \\
& +Q_2\sum_{r=0}^{a-1} q_r(b - P_{13})(X_2R_{11}R_{12} + R_{11}P_{12} + R_{12}P_{11})\Bigg],
\end{aligned}
$$

where all the notations used above are given explicitly in Appendix A.

7.2 Expected waiting time in the queue

By Little's formula, the mean waiting time of the batch of clients (customers) in the queue is given as, $W_q = \frac{L_q}{\lambda M'(1)}$, where L_q is given in equation (75).

7.3 System state probabilities

Pr[the server is engaged in regular service and re-service] is

$$P(B) = \frac{P_2(B_1 + p\pi B_2)}{b - P_{13}},$$

Pr[the server is on vacation(phase I and phase II)] is

$$P(Q) = \frac{\theta Q_1P_2 + Q_2\sum_{r=0}^{a-1} q_r(b - P_{13})}{b - P_{13}},$$

Pr[the server is under repair while rending regular service and re-service] is

$$P(R) = \frac{\eta P_2(R_{11}B_1 + p\pi B_2R_{12})}{b - P_{13}}.$$

8 Queue size distribution at departure epoch

In this section, we derive the PGF of the limiting distribution of queue size at departure point. Let P_i^+ be the steady state probability that 'i' clients are left back in the system at a departure point. Thus, we have

$$P_i^+ = E\int_0^\infty P_i(g)\phi(g)dg, \qquad i \geq 0,$$

where E is the normalization constant. Using this relationship, the PGF of P_i^+ can be attained:

$$P^+(z) = E\int_0^\infty \sum_{r=0}^{\infty} P_j(g)\phi(g)z^j dg = EB_e(z,0)\bar{B}(\lambda - \lambda M(z)).$$

From $P^+(1) = 1$, we have

$$E = \frac{B_1(b - P_{13})}{\begin{bmatrix}[(b - P_{13}) - \theta Q_1 P_2 - Q_2 \sum_{r=0}^{a-1} q_r(b - P_{13}) \\ - p\pi B_2 P_2 - \eta R_{11} B_1 P_2 - \eta p\pi B_2 R_{12} P_2]\end{bmatrix}}.$$

From the relation

$$P(z) = \frac{B_e(0,z)[1 - \bar{B}_1(\lambda - \lambda M(z))]}{\lambda - \lambda M(z)},$$

we get

$$P^+(z) = \frac{B_1(b - P_{13})H_4(z)\bar{B}_1(H_4(z))B_e(z)}{\begin{bmatrix}[(b - P_{13}) - \theta Q_1 P_2 - Q_2 \sum_{r=0}^{a-1} q_r(b - P_{13}) \\ - p\pi B_2 P_2 - \eta R_{11} B_1 P_2 - \eta p\pi B_2 R_{12} P_2][1 - \bar{B}(H_4(z))]\end{bmatrix}},$$

9 Numerical illustration

The unknown probabilities of the PGF of the queue size are deliberated using numerical techniques. Using MATLAB, the roots of the function $[z^b - \bar{B}_1(\psi_3(z))Y(z)]$ are obtained and simultaneous equations are solved.

A numerical example is evaluated with the following assumptions and notations:

1. Service time follows k-Erlang distribution with $k = 2$.
2. Re-service time follows k-Erlang distribution with $k = 2$.
3. Batch size distribution of the arrival is considered as Geometric with mean 2.
4. Phase- I and phase-II vacation time are assumed to follow Exponential distribution with parameter ν_1 and ν_2 .
5. Repair time are assumed to follow Exponential distribution with parameter ζ.
6. Minimum service capacity $a = 3$ and maximum service capacity $b = 6$.

The performance measures are fixed such that it satisfies the stability condition. The utilization factor ρ, mean queue length L_q, expected mean time in the queue W_q are calculated for various service and arrival rate and the outcomes are tabulated.

From Tables 1, 2 and Figure 2, 3 we observe that:
1. The mean queue length, mean waiting time and ρ increases , when the arrival rate λ increases.
2. The mean queue length, mean waiting time and ρ decreases, when the service rate ϕ increases.
3. Figure 3, shows that when the service rate ϕ increases, the mean queue length (Exponential(L_{q1}) and Erlang-2 distribution (L_{q2})) decreases.

From Tables 3, 4, and Figures 4, 5 we observe that:
1. As arrival rate λ increases, the probability of busy period P(B), probability of repair period P(R) increases but the probability of vacation period P(Q) decreases.
2. As the probability of admitting re-service 'π' increases, the probability of busy period P(B), probability of repair period P(R) increases but the probability of vacation period P(Q) decreases.

Table 1: Arrival rate vs performance measure.
$a = 3$, $b = 6$, $\phi = 5$, $\zeta = 3$, $\nu_1 = 3$, $\nu_2 = 4$, $\eta = 1$, $\alpha = 3$, $\pi = 0.4$, $\theta = 0.5$, $\beta = 0.5$ and $p = 0.6$

λ	ρ	L_q	W_q
2.0	0.3044	0.1007	0.0252
2.1	0.3197	0.3915	0.0932
2.2	0.3349	0.7015	0.1594
2.3	0.3501	1.0328	0.2245
2.4	0.3806	1.7697	0.2892
2.5	0.3653	1.3880	0.3539
2.6	0.3958	2.1811	0.4194
2.7	0.4110	2.6258	0.4863
2.8	0.4262	3.1080	0.5550
2.9	0.4414	3.6323	0.6263
3.0	0.4567	4.2042	0.7007

Table 2: Service rate vs performance measure.
$a = 3, b = 6, \lambda = 2, \zeta = 3, \nu_1 = 3, \nu_2 = 4, \eta = 1, \alpha = 3,$
$\pi = 0.4, \theta = 0.5, \beta = 0.5,$ and $p = 0.6$

ϕ	ρ	Erlang-2		Exponential	
		L_{q_1}	W_{q_1}	L_{q_2}	W_{q_2}
3.0	0.4230	2.1253	0.5313	0.7447	0.1862
3.1	0.4134	1.9280	0.4820	0.6211	0.1553
3.2	0.4044	1.7499	0.4375	0.5085	0.1271
3.3	0.3960	1.5883	0.3971	0.4055	0.1014
3.4	0.3881	1.4408	0.3602	0.3109	0.0777
3.5	0.3806	1.3057	0.3264	0.2236	0.0559
3.6	0.3736	1.1814	0.2954	0.1427	0.0357
3.7	0.3669	1.0667	0.2667	0.0675	0.0169

Table 3: Arrival rate vs Servers state.
$a = 3, b = 6, \zeta = 3, \nu_1 = 3, \nu_2 = 4, \eta = 1, \alpha = 2,$
$\pi = 0.4, \theta = 0.5, \beta = 0.5,$ and $p = 0.6$

λ	$P(B)$	$P(Q)$	$P(R)$
2.0	0.1608	0.7856	0.2144
2.1	0.1695	0.7740	0.2260
2.2	0.1783	0.7623	0.2377
2.3	0.1871	0.7505	0.2495
2.4	0.1961	0.7386	0.2614
2.5	0.2050	0.7266	0.2734
2.6	0.2140	0.7146	0.2854
2.7	0.2231	0.7025	0.2975
2.8	0.2322	0.6904	0.3096
2.9	0.2414	0.6781	0.3219
3.0	0.2507	0.6658	0.3342

10 Conclusion

In this paper, we have studied the transient and steady state behavior of a non-Markovian batch arrival bulk service queue with the following features: unreliable server, server choice of admitting re-service, Bernoulli vacation schedule under multiple vacation and balking. We investigate the queue size distributions at a random epoch as well as at a departure epoch. We have obtained the following

Table 4: Effects of admitting clients vs Servers state
$a = 3, b = 6, \zeta = 3, \nu_1 = 3, \nu_2 = 4, \eta = 1, \alpha = 2,$
$\lambda = 2, \theta = 0.5, \beta = 0.5,$ and $p = 0.6$

π	$P(B)$	$P(Q)$	$P(R)$
0.1	0.1114	0.8515	0.1485
0.2	0.1275	0.8301	0.1699
0.3	0.1439	0.8081	0.1919
0.4	0.1608	0.7856	0.2144
0.5	0.1780	0.7626	0.2374
0.6	0.1956	0.7391	0.2609
0.7	0.2137	0.7151	0.2849
0.8	0.2321	0.6906	0.3094
0.9	0.2509	0.6655	0.3345
1.0	0.2701	0.6399	0.3601

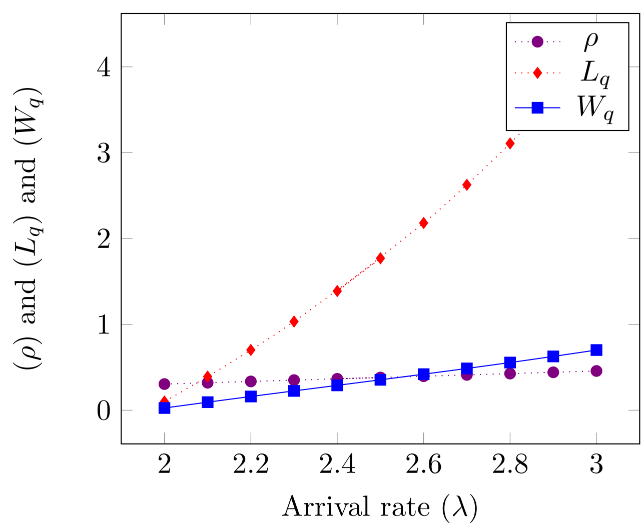

Figure 2: Arrival rate *(vs)* performance measure.

results: expected queue length, expected waiting time of the batch of clients in the queue. Some particular cases, system state probabilities and numerical illustration were found. To obtain these results, we apply supplementary variable technique. The present model can be extended with the concept of random setup with N-policy and closedown.

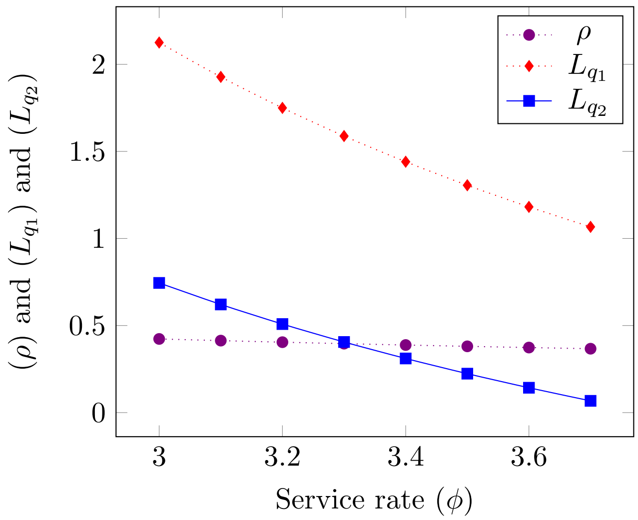

Figure 3: Service rate *(vs)* performance measure.

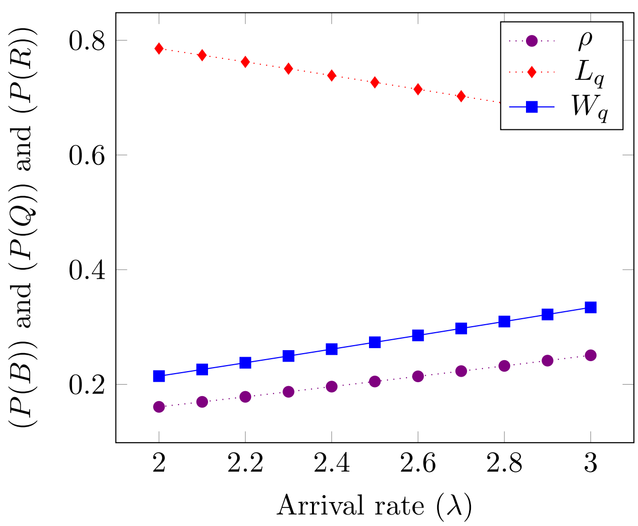

Figure 4: Arrival rate *(vs)* Servers state.

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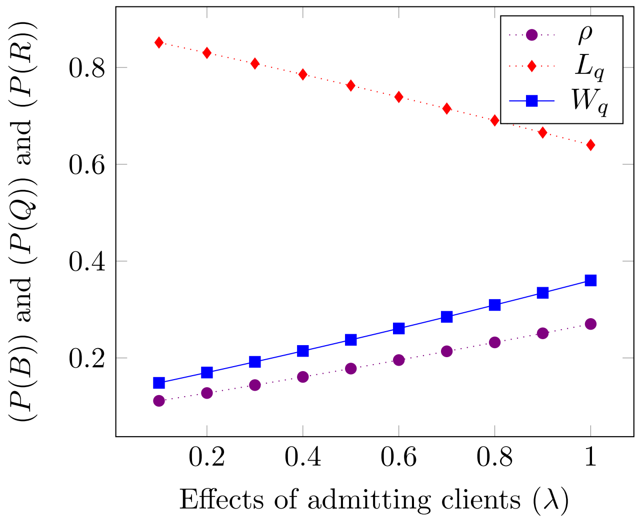

Figure 5: Effects of admitting clients *(vs)* Servers state.

Appendix

The following expressions are used throughout this work

$P_1 = 2X_1Q_2\Sigma_{r=0}^{a-1} rq_r + X_1^2 Q_4 \beta \Sigma_{r=0}^{a-1} q_r + X_2 Q_2 \Sigma_{r=0}^{a-1} q_r,$

$P_2 = X_1 Q_2 \beta \Sigma_{r=0}^{a-1} q_r + \Sigma_{r=a}^{b-1} \omega_r (b-r), \quad P_3 = (X_1)^2 B_3 R_{11}^2 + X_2 B_1 R_{11} + \eta X_1^2 B_1 R_3,$

$P_4 = X_2 R_{11} + \eta X_1^2 R_3, \quad P_5 = R_{12} X_2 + X_2 + \eta X_1^2 R_4 + \eta X_2 R_{12},$

$P_6 = X_1^2 B_4 R_{12}^2 + X_2 B_2 R_{12} + \eta X_1^2 R_4 B_2, \quad P_7 = 2X_1^2 B_1 R_{11} R_{12} + X_2 R_{12} = \eta X_1^2 R_4,$

$P_8 = R_{11} X_2 + X_2 + \eta X_1^2 R_3 + \eta X_2 R_{11}, \quad P_9 = B_1 R_{11} + p\pi B_2 R_{12}, \quad P_{10} = \beta X_1^2 Q_3 + 2X_2 Q_1,$

$P_{11} = X_2 + \eta X_1^2 R_3 + \eta X_2 R_{11}, \quad P_{12} = X_2 + \eta X_1^2 R_4 + \eta X_2 R_{12},$

$P_{13} = X_1(B_1 R_{11} + p\pi B_2 R_{12} + \theta\beta Q_1), \quad R_{11} = 1 + \eta E(R_1), \quad R_{12} = 1 + \eta E(R_2),$

$R_1 = E(R_1), \quad R_2 = E(R_2), \quad R_3 = E(R_1^2), \quad R_4 = E(R_2^2), \quad B_1 = E(B_1), \quad B_2 = E(B_2),$

$B_3 = E(B_1^2), \quad B_4 = E(B_2^2), \quad Q_1 = E(Q_1), \quad Q_2 = E(Q_2), \quad Q_3 = E(Q_1^2), \quad Q_4 = E(Q_2^2),$

$H_1(z) = s + \lambda\beta(1 - M(z), \quad H_2(z) = s + \lambda(1 - M(z)), \quad H_3(z) = \lambda\beta(1 - M(z)),$

$H_4(z) = \lambda(1 - M(z)), \quad \psi_1(z) = [H_2(z) + \eta(1 - \bar{R}_1[H_2(z))],$

$\psi_2(z) = [H_2(z) + \eta(1 - \bar{R}_2[H_2(z))], \quad \psi_3(z) = [H_4(z) + \eta(1 - \bar{R}_1[H_4(z))],$

$\psi_4(z) = [H_4(z) + \eta(1 - \bar{R}_2[H_4(z))], \quad Y(z) = (1 - p\pi(1 - \bar{B}_2(\psi_4(z)))(1 - \theta(1 - \bar{Q}_1(H_3(z))),$

$$q_r = \int_0^\infty [(1-\theta)(1-p\pi)B_{e,r}(g)\phi(g) + (1-\theta)B_{f,r}(g)\alpha(g) + Q_{e,r}(g)\nu_1(g) + Q_{f,r}(g)\nu_2(g)]dg, \quad 0 \leqslant r \leqslant a-1,$$

$$\omega_r = \int_0^\infty [(1-\theta)(1-p\pi)B_{e,r}(g)\phi(g) + (1-\theta)B_{f,r}(g)\alpha(g) + Q_{e,r}(g)\nu_1(g) + Q_{f,r}(g)\nu_2(g)]dg, \quad a \leqslant r \leqslant b-1,$$

$M'(1) = E(X)$ and $M''(1) = E(X^2)$.

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