Journal of Mathematical Modeling
Vol. 5, No. 2, 2017, pp. 171-197

JMM

An interior-point algorithm for $P_*(\kappa)$-linear complementarity problem based on a new trigonometric kernel function

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Abstract. In this paper, an interior-point algorithm for $P_*(\kappa)$-Linear Complementarity Problem (LCP) based on a new parametric trigonometric kernel function is proposed. By applying strictly feasible starting point condition and using some simple analysis tools, we prove that our algorithm has $O((1+2\kappa)\sqrt{n}\log n \log \frac{n}{\epsilon})$ iteration bound for large-update methods, which coincides with the best known complexity bound. Moreover, numerical results confirm that our new proposed kernel function is doing well in practice in comparison with some existing kernel functions in the literature.

Keywords: kernel function, linear complementarity problem, primal-dual interior point methods, large-update methods.
AMS Subject Classification: 65K05, 90C51.

1 Introduction

Polynomial time Interior Point Method (IPM) idea was first investigated by Karmarkar in [16] for solving Linear Optimization (LO) problems. Later,

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Received: 26 June 2017 / Revised: 5 October 2017 / Accepted: 18 November 2017.

Kojima et al. [22] and Megiddo [27] developed this idea to primal-dual IPMs for LO problems. Kojima et al. in [21] first proposed the existence and uniqueness of the central path for any $P_*(\kappa)$-Linear Complementarity Problem (LCP). Nesterov and Nemirovski in [28] extended this algorithm to general convex optimization problems such as Convex Quadratic (CQ) problems, Second Order Cone Optimization (SOCO) problems, Semidefinite Optimization (SDO) problems and nonlinear complementarity problems.

The goal of this paper is to focus on the linear complementarity problem formulated in the standard from as:

$$\begin{cases} s = Mx + q, & (x,s) \geq 0, \\ xs = 0, \end{cases} \tag{1}$$

where, M is real $n \times n$ matrix, $q \in \mathbb{R}^n$ and xs in the second equation denotes a coordinate-wise (Hadamard) product of vectors x and s. Note that problem (1) is named $P_*(\kappa)$-LCP if M is a $P_*(\kappa)$-matrix.

Definition 1. Suppose that $\kappa \geq 0$. A matrix $M \in \mathbb{R}^{n\times n}$ is called a $P_*(\kappa)$-matrix if, for any real vector $x \in \mathbb{R}^n$, the following inequality holds:

$$(1+4\kappa) \sum_{i \in J_+(x)} x_i(Mx)_i + \sum_{i \in J_-(x)} x_i(Mx)_i \geq 0,$$

where

$$J_+(x) = \{i \in J : x_i(Mx)_i \geq 0\}, \qquad J_-(x) = \{i \in J : x_i(Mx)_i < 0\},$$

and

$$J = \{1, 2, \ldots, n\}.$$

Corollary 1. *Every $P_*(0)$-matrix is positive semidefinite.*

Definition 2. A matrix $M \in \mathbb{R}^{n\times n}$ is called P_* matrix if it is a $P_*(\kappa)$ matrix for some $\kappa \geq 0$, that is:

$$P_* = \bigcup_{\kappa \geq 0} P_*(\kappa).$$

Lemma 1 (Lemma 4.1 in [21])**.** *Suppose that $M \in \mathbb{R}^{n\times n}$ is a $P_*(\kappa)$ matrix, thus the matrix M' is a nonsingular matrix for any positive diagonal matrix $X, S \in \mathbb{R}^{n\times n}$, where*

$$M' = \begin{pmatrix} -M & I \\ S & X, \end{pmatrix},$$

$X = diag(x)$ and $S = diag(s)$.

As a consequence of Lemma 1, one has the following result.

Corollary 2. *Suppose that $M \in \mathbb{R}^{n\times n}$ is a $P_*(\kappa)$ matrix, and two vectors $x, s \in \mathbb{R}^n$. Thus for any vector $a \in \mathbb{R}^n$, the system*

$$\begin{cases} -M\Delta x + \Delta s = 0, \\ S\Delta x + X\Delta s = a, \end{cases}$$

has a unique solution as $(\Delta x, \Delta s)$, where $X = diag(x)$ and $S = diag(s)$.

According to records, there are several optimization problems that can be formulated as a $P_*(\kappa)$-LCP, for example the Karush-Kuhn-Tucker (KKT) optimality conditions for linear optimization and convex quadratic optimization problems, the optimal invariant capital stock problem, the market equilibrium problem and the optimal stopping problem [38]. For more data on the LCP and its applications, we refer to [8, 15].

A close look at the IPM literatures tells us that by using kernel functions we can get the best known complexity bound. Important work in this direction was done by Peng et al. in [29]. Subsequently, they constructed a new variant of the feasible interior-point algorithm for LO problem based on Self-Regular (SR) barrier (proximity) function and showed that their algorithm in the large neighborhood of the central path has $O(\sqrt{n}\log n \log \frac{n}{\epsilon})$ as the worst case iteration bound. Then, they developed their algorithm to general convex optimization problems such as Complementarity Problem (CP), SOCO, and SDO problems. Primal-dual IPMs for LO problems based on the so-called *eligible kernel functions*, which are not necessarily SR-barrier function, were first studied by Bai et al. in [1]. Based on this class of kernel functions, they showed that their algorithm enjoys $O(\sqrt{n}\log n \log \frac{n}{\epsilon})$ as the worst case iteration bound for large-update methods. In recent papers, several interior point algorithms have been constructed. A comparative study on the kernel functions for LCP is provided in [5–7, 23, 24].

The primal-dual IPMs based on trigonometric kernel functions received much more attention in recent works. This kind of barrier functions was first proposed by El Ghami et al. [9]. They derived the worst case iteration complexity as $O(n^{\frac{3}{4}} \log \frac{n}{\varepsilon})$ for large-update methods. Later, Kherifam in [17] suggested an interior point algorithm for solving SDO problems based on a new trigonometric kernel function and derived the same complexity as in [9]. Later on, El Ghami [10] investigated an IPM for solving $P_*(\kappa)$-LCP based on the trigonometric kernel function which was previously introduced for LO in [9]. He derived the worst case iteration bound as $O\left((1+2\kappa)n^{\frac{3}{4}} \log \frac{n}{\epsilon}\right)$ for large-update methods.

To improve iteration bound of primal-dual IPMs based on the trigonometric kernel function in a large neighborhood of the central path, Peyghami et al. in [32] introduced another new trigonometric kernel function and showed that their algorithm enjoys $O\left(n^{\frac{2}{3}} \log \frac{n}{\epsilon}\right)$ iteration complexity bound which improved the obtained results by El Ghami et al. [9, 10] and Khierfam [17], but it is not the best known iteration complexity bound. Then Peyghami and Fathi in [31] proposed another trigonometric kernel function. They derived the worst case iteration complexity bound as $O(\sqrt{n}(\log n)^2 \log \frac{n}{\epsilon})$ for large-update methods. Recently, an interior point algorithm for solving $P_*(\kappa)$-linear complementarity problem based on the trigonometric kernel function was introduced by Fathi et al. [11]. They obtained the best known complexity bound for large-update methods, that is, $O((1+2\kappa)\sqrt{n} \log n \log \frac{n}{\epsilon})$. In recent years, several interior point algorithms based on the trigonometric kernel functions have been proposed [3, 4, 11, 13, 14, 18–20, 25, 26, 31, 32, 34]. Based on the some of them, the best known iteration complexity bound for large update methods is derived [3, 11, 13, 14, 20, 34].

In this paper, we introduce a large-update primal-dual interior-point algorithm for $P_*(\kappa)$-LCP based on a new family of kernel functions with trigonometric barrier term. Using an elegant analysis, we show that our algorithm enjoys $O((1 + 2\kappa)\sqrt{n} \log n \log \frac{n}{\epsilon})$ as the worst case complexity, which coincides with the best known iteration complexity bound. Finally, we gives some numerical results.

The paper is organized as follows: In Section 2, we recall some basic concepts of interior-point methods and the central path for LCP. In Section 3, we first introduce a new kernel function, then we survey some properties of this kernel function. In Section 4, we obtain the step size during an inner iteration. The worst case iteration bound for the primal-dual IPMs based on the new kernel function is presented in Section 5. We illustrate the practical performance of the new proposed kernel function in Section 6. Some numerical results are presented in Section 7. Finally, we end the paper by some concluding remarks in Section 8.

Some of the notations used throughout the paper are as follows: The set of real vectors with length n, the set of nonnegative vectors and the set of positive vectors are showed as $\mathbb{R}^n, \mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$, respectively. For a given vector $x \in \mathbb{R}^n$ diagonal matrix of x is defined by $X = \operatorname{diag}(x)$. The index set is denoted as $J = \{1, 2, \ldots, n\}$. Also $\|\cdot\|$ denotes the 2-norm of a vector. For two given vectors x and s, the vectors xs and $\frac{x}{s}$ show that the coordinate-wise operations on the vectors, i.e., whose components are $x_i s_i$ and $\frac{x_i}{s_i}$, respectively. Furthermore, $v_{\min}$ is minimum components of the real vector v. We say that $f(t) = \Theta(g(t))$, if there exist positive constants ω_1

and ω_2 so that $\omega_1 g(t) \leq f(t) \leq \omega_2 g(t)$ satisfies for all $t \in \mathbb{R}_{++}$. We also say $f(t) = O(g(t))$, if there exists a positive constant ω so that $f(t) \leq \omega g(t)$, for all $t \in \mathbb{R}_{++}$.

2 Central path and $P_*(\kappa)$-matrices

In this section, we focus on the central path for $P_*(\kappa)$-LCP and some properties of primal-dual IPMs and present a generic interior point algorithm for LCP. Throughout this paper, without lose of generality, we assume that the system (1) satisfies the Interior Point Condition (IPC), that is, there exists a point $(x^0, s^0) > 0$, such that

$$Mx^0 + q = s^0,$$

which means that the interior of the feasible region is not empty.

The key idea of path following IPMs for LCP is to replace the last equation in (1), the so called *complementarity condition* with the parameterized equation $xs = \mu\mathbf{e}$, where μ is a real positive parameter and $\mathbf{e}$ denotes the all-one vector of length n. Therefore, this replacement leads us to the following system:

$$\begin{cases} s = Mx + q, \\ xs = \mu\mathbf{e}, \\ (x, s) > 0. \end{cases} \tag{2}$$

Since IPC holds and M is a $P_*(\kappa)$ matrix, the parameterized system (2) has a unique solution for any real positive parameter μ [29]. Let us, represent the solution of system (2) by $(x(\mu), s(\mu))$. The set of all μ-centers, with $\mu > 0$, i.e., $\{(x(\mu), s(\mu)) : \mu > 0\}$ gives the homotopy path and it is called the central path of the LCP [36]. It is shown that, if the parameter μ goes to zero, the limit of the central path exists and satisfies the complementarity condition and belong to the solution set of (1).
In what follows, we discuss how the algorithm computes the step length in Newton method. Therefore, the system (2) can be converted to the following system:

$$\begin{cases} -M\Delta x + \Delta s = 0, \\ S\Delta x + X\Delta x = \mu\mathbf{e} - xs. \end{cases} \tag{3}$$

The second equation in (3) is so-called the *centering equation*. By using Corollary 2, the system (3) has a unique solution for any two positive vectors x and s. System (3) can be easily rewritten as below:

$$\begin{cases} \bar{M}d_x + d_s = 0, \\ d_x + d_s = v^{-1} - v, \end{cases} \tag{4}$$

where

$$d := \sqrt{\frac{x}{s}}, \qquad v := \sqrt{\frac{xs}{\mu}}, \tag{5}$$

$$d_x := \frac{v\Delta x}{x}, \qquad d_s := \frac{v\Delta s}{s}, \tag{6}$$

and $\bar{M} = DMD$, with $D = \mathrm{diag}(d)$.
One can easily see that the right hand-side of the last equation in (4), which so-called the *scaled centering* equation is equal to negative gradient of the following scaled barrier function:

$$\Psi_c(v) := \sum_{i=1}^{n} \psi_c(v_i) = \sum_{i=1}^{n} \left(\frac{v_i^2 - 1}{2} - \log v_i \right),$$

where, v_i denotes the i-th component of the variance positive vector v.
The scaled barrier function $\Psi_c(v)$ has the following properties:

(*i*) $\Psi_c(v)$ is strictly convex for $v \in \mathbb{R}^n_{++}$.

(*ii*) $\Psi_c(v)$ attains its minimal value at $v = \mathbf{e}$, i.e., $\Psi_c(\mathbf{e}) = 0$.

We call $\psi_c(t)$ a kernel function of the classical logarithmic barrier function. In the next proposition, some properties of the kernel function are characterized.

Proposition 1. *A function $\psi_c : \mathbb{R}_{++} \to \mathbb{R}_+$ is called a kernel function if ψ_c satisfies the following conditions [1]:*

(i) $\psi_c(1) = \psi_c'(1) = 0$,

(ii) $\lim_{t \to 0^+} \psi_c(t) = \lim_{t \to \infty} \psi_c(t) = +\infty$,

(ii) $\psi_c(t)$ *is a strictly convex function for all* $t > 0$.

From the above discussion, one may easily write:

$$\Psi_c(v) = 0 \Leftrightarrow v = \mathbf{e} \Leftrightarrow x = x(\mu), \; s = s(\mu).$$

Recently, Peng et al. in [29] used a new proximity function $\Psi(v)$, such that $\Psi(v) = \sum_{i=1}^{n} \psi(v_i)$ with $\psi(1) = \psi'(1) = 0$ and the function $\psi(t)$ is a strictly convex function for all $t > 0$. Therefore, the scaled centering equation in (4) can be rewritten as below:

$$d_x + d_s = -\nabla \Psi(v). \tag{7}$$

Now, using equation (7), the system (4) may be rewritten as follows:

$$\begin{cases} -M\Delta x + \Delta s = 0, \\ S\Delta x + X\Delta s = -\mu v \nabla\Psi(v). \end{cases} \tag{8}$$

From Corollary 2, the system (8) has a unique solution $(\Delta x, \Delta s)$ for any two vectors $(x, s) > 0$. Moreover, one has:

$$\Delta x = 0,\ \Delta s = 0 \Leftrightarrow v = \mathbf{e}.$$

For the moment, we describe one step of the Algorithm 1. We start the algorithm by proximity parameter τ, barrier parameter update θ, for any $\theta \in [0, 1]$, and a strictly feasible point (x^0, s^0). Note that, at the begining of the algorithm, the point (x^0, s^0) is in a τ neighborhood of the given μ-center. The algorithm consists of the inner while loop and outer while loop, which are called inner and outer iterations, respectively. Note that, any outer iteration consists of update barrier parameter μ by $(1-\theta)\mu$, that is, the parameter μ decreasing to $(1-\theta)\mu$ for some $\theta \in [0, 1]$ and a sequence of one or more inner iterations. Then, we solve the Newton system to derive the unique search direction. The generic algorithm is as follows [29].

The choice of the barrier update parameter θ plays an important role in the theory and practice of IPMs. For a constant θ, let $\theta = \frac{1}{2}$, the algorithm is called the large-update methods, while for the case when the θ is depended on n, $\theta = \frac{1}{\sqrt{n}}$, the algorithm is called the small-update methods. Note that, iteration complexity bound of the algorithm for small-update methods has the best known complexity, that is $O(\sqrt{n}\log\frac{n}{\epsilon})$ in theory, while the large update methods are practically more efficient [35].

3 The new kernel function

In this section, first a new class of kernel functions with trigonometric barrier term is defined; then some properties of these function are studied. The new kernel function is given by:

$$\psi(t) = \frac{t^2-1}{2} - (\sqrt{3}-1)^p \int_1^t \frac{dx}{(\tan(h(x))-1)^p}, \qquad p \geq 2, \tag{9}$$

where

$$h(x) = \frac{1+x}{4+2x}\pi. \tag{10}$$

Note that when $t \to 0^+$, the function $h(t)$ converges to $\frac{\pi}{4}$. It follows that $\lim_{t\to o^+}\psi(t) = +\infty$. Moreover, we can easily see that $\lim_{t\to\infty}\psi(t) = +\infty$. These imply that the function $\psi(t)$ given by (9) is a kernel function [1].

Algorithm 1. Generic Primal-dual IPMs for LCP

Input
  a proximity function $\Psi(v)$
  a threshold parameter $\tau > 0$
  an accuracy parameter $\varepsilon > 0$
  a barrier update parameter θ, $0 < \theta < 1$
begin
  $x := \mathbf{e}; s := \mathbf{e}; \mu := 1; v := \mathbf{e};$
  while $n\mu > \varepsilon$ do
  begin
    $\mu := (1-\theta)\mu$
    while $\Psi(v) > \tau$ do
    begin
      $x := x + \alpha\Delta x$
      $s := s + \alpha\Delta s$
      $v := \sqrt{\frac{xs}{\mu}}$
    end
  end
end

To analyze Algorithm 1, we need the first three derivatives of the function (9), as:

$$\psi'(t) = t - \frac{(\sqrt{3}-1)^p}{(\tan(h(t))-1)^p} \tag{11}$$

$$\psi''(t) = 1 + \frac{2(\sqrt{3}-1)^p \pi p M(t)}{(4+2t)^2(\tan(h(t))-1)^{p+1}} \tag{12}$$

$$\psi'''(t) = \frac{4(\sqrt{3}-1)^p \pi p M(t)}{(4+2t)^3(\tan(h(t))-1)^{p+2}} K(t). \tag{13}$$

where:

$$\begin{aligned} M(t) &= 1 + \tan^2(h(t)) \\ K(t) &= -2(\tan(h(t))-1) + \frac{2\pi}{4+2t}\tan(h(t))\left(\tan(h(t))-1\right) \\ &\quad -\frac{(p+1)\pi M(t)}{4+2t}. \end{aligned} \tag{14}$$

One can easily see that $\psi(1) = \psi'(1) = 0$. Therefore, we can denote the function $\psi(t)$ given by (9) as below:

$$\psi(t) = \int_1^t \int_1^\xi \psi''(\zeta) d\zeta d\xi. \tag{15}$$

We define a norm-based proximity measure $\delta(v)$ as:

$$\delta(v) := \frac{1}{2}\|\nabla\Psi(v)\| = \frac{1}{2}\sqrt{\sum_{i=1}^{n}(\psi'(v_i))^2}, \qquad v \in \mathbb{R}^n_{++}. \tag{16}$$

Note that the function $\psi(t)$ defined by (9), is a decreasing function in $(0, 1]$, and an increasing function in the interval $[1, +\infty)$. In the next lemma, we derive some important properties of the new kernel function.

Lemma 2. *For the function $h(t)$, defined by (10), let $t > 0$. Then, we have:*

$$\tan(h(t)) \geq \frac{t+2}{2\pi}.$$

Proof. First, we define a function $g(t)$ as:

$$g(t) := \tan(h(t)) - \frac{t+2}{2\pi}.$$

Therefore, one has:

$$\begin{aligned} g'(t) &= \frac{2\pi}{(4+2t)^2}(1+\tan^2(h(t))) - \frac{1}{2\pi} \\ &= \frac{2\pi}{(4+2t)^2\cos^2(h(t))} - \frac{1}{2\pi} = \frac{1}{\cos^2(h(t))}\left[\frac{2\pi}{(4+2t)^2} - \frac{1}{2\pi}\cos^2(h(t))\right]. \end{aligned}$$

For all $x \in [0, \pi]$, we have:

$$\begin{aligned} \sin(\frac{\pi}{2} - x) &= \cos(x), \\ \sin(x) &\leq x, \end{aligned}$$

it follows that

$$\begin{aligned} g'(t) &= \frac{1}{\cos^2(h(t))}\left[\frac{2\pi}{(4+2t)^2} - \frac{1}{2\pi}\sin^2(\frac{\pi}{2} - h(t))\right] \\ &\geq \frac{1}{\cos^2(h(t))}\left[\frac{2\pi}{(4+2t)^2} - \frac{2\pi}{(4+2t)^2}\right] = 0. \end{aligned}$$

The lemma follows from the fact that $g(0) > 0$. □

Note that for all $t > 0$, $\tan(h(t)) > 1$. Next lemma shows some properties of the new kernel function.

Lemma 3. *For the function $\psi(t)$, defined by (9), we have:*

i) $\psi''(t) > 1, \qquad \forall\, t > 0,$

ii) $t\psi''(t) - \psi'(t) > 0, \qquad \forall\, t > 1,$

iii) $t\psi''(t) + \psi'(t) > 0, \qquad \forall\, 0 < t < 1,$

iv) $\psi'''(t) < 0, \qquad \forall\, t > 0.$

Proof. The first item easily follows from,

$$\psi''(t) = 1 + \frac{2(\sqrt{3}-1)^p \pi p M(t)}{(4+2t)^2(\tan(h(t))-1)^{p+1}} \geq 1.$$

Since,

$$t\psi''(t) - \psi'(t) = \frac{(\sqrt{3}-1)^p}{(\tan(h(t))-1)^{p+1}}\left[\frac{2\pi p t M(t)}{(4+2t)^2} + \tan(h(t)) - 1\right] > 0,$$

the second item is concluded. To prove the third item, we have

$$t\psi''(t) + \psi'(t) = \frac{(\sqrt{3}-1)^p}{(\tan(h(t))-1)^{p+1}}\left[\frac{2\pi p t M(t)}{(4+2t)^2} - \tan(h(t)) + 1\right].$$

We consider the expression in the bracket as function $k(t)$ such that:

$$k(t) := \frac{2p\pi t}{(2t+4)^2}\left(1 + \tan^2(h(t))\right) - \tan(h(t)) + 1.$$

The first derivative of $k(t)$ is as follows

$$\begin{aligned} k'(t) &= M(t)\frac{2\pi}{(4+2t)^3}\left[p(2t+4) - 4pt + \frac{4\pi p t}{4+2t}\tan(h(t)) - (2t+4)\right] \\ &\geq M(t)\frac{2\pi}{(4+2t)^3}\left[-pt + 4p - 2t - 4\right] \\ &\geq M(t)\frac{2\pi}{(4+2t)^3}\left[3p - 2t - 4\right] \geq 0, \end{aligned}$$

where, the first inequality is due to Lemma 2 and the last inequality follows from the fact that $p \geq 2$ and $t \in (0,1]$. This implies that $k(t)$ is an increasing function for all $t \in (0, 1]$. From $k(0) = 0$, we conclude that for all $t \in (0, 1]$,

$$t\psi''(t) + \psi'(t) > 0.$$

Using the fact that for all $t > 0$, $M(t) > \tan(h(t))(\tan(h(t)) - 1)$ and $p \geq 2$, we conclude that $K(t) < 0$ for all $t > 0$. Therefore, one has $\psi'''(t) < 0$ for all $t > 0$. It completes the proof. □

In what follows, we construct the exponential convexity (e-convexity) property of the new kernel function. This property plays an important role in the analysis of the primal-dual algorithm.

Lemma 4. *(Lemma 2.1.2 in [29]) Suppose that function $\psi(t)$ is a twice differentiable function, for all $t > 0$. Therefore, the following properties are equivalent:*

i) $\psi(\sqrt{t_1 t_2}) \leq \frac{1}{2}(\psi(t_1) + \psi(t_2)), \qquad \forall \quad t_1, t_2 > 0.$

ii) $\psi'(t) + t\psi''(t) \geq 0, \qquad \forall \quad t > 0.$

iii) $\psi(e^{\xi})$ *is a convex function.*

As a consequence of Lemmas 3, and 4, the kernel function $\psi(t)$ defined by (9) has the e-convexity property.

Lemma 5. *For the new kernel function $\psi(t)$ defined by (9), one has:*

i) $\frac{1}{2}(t-1)^2 \leq \psi(t) \leq \frac{1}{2}\psi'(t)^2$, *for all $t > 0$.*

ii) $\Psi(v) \leq 2\delta(v)^2$.

iii) $\|v\| \leq \sqrt{n} + \sqrt{2\Psi(v)} \leq \sqrt{n} + 2\delta(v)$.

Proof. The proof is a straightforward result of (15), Lemma 3 and Lemma 3.4 in [30]. □

As a consequence of the second part of Lemma 5, one has the following corollary.

Corollary 3. *Suppose that $\Psi(v) \geq 1$, then one has*

$$\delta(v) \geq \frac{1}{\sqrt{2}}.$$

For the moment, we focus on the growth behavior of the proximity function $\Psi(v)$ during an iteration of Algorithm 1. Suppose that at the start of each outer iteration, just before the μ-update we have $\Psi(v) \leq \tau$. Due to the update of μ, the vector v is divided by the factor $\sqrt{1-\theta}$, with $0 \leq \theta < 1$, which in general leads to an increase in the value of $\Psi(v)$. Then, during the subsequent inner iteration, $\Psi(v)$ decreases until it passes the threshold τ again. In what follows, we present two lemmas, which are important in deriving the iteration complexity bound.

Lemma 6. *Suppose that $\psi(t)$ is given by (9) and $\beta \geq 1$. One has*

$$\psi(\beta t) \leq \psi(t) + \frac{\beta^2 - 1}{2} t^2.$$

Proof. Let

$$\psi(t) = \frac{t^2 - 1}{2} + p(t),$$

where the function $p(t)$ is defined as follows

$$p(t) = -(\sqrt{3} - 1)^p \int_1^t \frac{dx}{(\tan(h(x)) - 1)^p}.$$

Then, we have

$$\psi(\beta t) - \psi(t) = \frac{\beta^2 - 1}{2} t^2 + p(\beta t) - p(t).$$

Since $\beta \geq 1$, to complete the proof, it suffices to show that the function $p(t)$ is a decreasing function. It immediately follows from,

$$p'(t) = -\frac{(\sqrt{3} - 1)^p}{(\tan(h(x)) - 1)^p} < 0.$$

This completes the proof. □ □

Lemma 7. *Suppose that $0 < \theta < 1$ and $v_+ = \frac{v}{\sqrt{1-\theta}}$. One has*

$$\Psi(v_+) \leq \Psi(v) + \frac{\theta}{2(1-\theta)}(2\Psi(v) + 2\sqrt{2n\Psi(v)} + n).$$

Proof. Using Lemma 6 with $\beta = \frac{1}{\sqrt{1-\theta}}$, we have

$$\Psi(\beta v) \leq \Psi(v) + \frac{1}{2}\sum_{i=1}^{n} (\beta^2 - 1) v_i^2 = \Psi(v) + \frac{\theta \|v\|^2}{2(1-\theta)}.$$

Now, from the third item of Lemma 5, the statement of the lemma follows.

□

4 An estimation for the step size

In this section, we compute the largest possible value for the step size $= \alpha$ during an inner iteration. After a damped step we have:

$$x_+ = x + \alpha\Delta x, \quad s_+ = s + \alpha\Delta s.$$

Using (6), we obtain:

$$x_+ = \frac{x}{v}(v + \alpha d_x), \quad s_+ = \frac{s}{v}(v + \alpha d_s).$$

This implies that,

$$v_+^2 = \frac{x_+ s_+}{\mu} = (v + \alpha d_x)(v + \alpha d_s).$$

From the fact that M is the $P_*(\kappa)$ matrix and (8), i.e. for $\Delta x \in \mathbb{R}^n$, $M\Delta x = \Delta s$, one has:

$$(1 + 4\kappa) \sum_{i \in J_+(x)} (\Delta x_i)\Delta s_i + \sum_{i \in J_-(x)} \Delta x_i \Delta s_i \geq 0,$$

where, $J_+(x) = \{i \in J : \Delta x_i \Delta s_i \geq 0\}$ and $J_-(x) = J - J_+(x)$. Since

$$d_x d_s = \frac{v^2 \Delta x \Delta s}{xs} = \frac{\Delta x \Delta s}{\mu},$$

and $\mu > 0$, we have

$$(1 + 4\kappa) \sum_{i \in J_+(x)} (d_x)_i (d_s)_i + \sum_{i \in J_-(x)} (d_x)_i (d_s)_i \geq 0. \tag{17}$$

For notational convenience, we define the following notations:

$$\delta := \delta(v), \qquad \sigma_+ := \sum_{i \in J_+(x)} (d_x)_i (d_s)_i, \qquad \sigma_- := -\sum_{i \in J_-(x)} (d_x)_i (d_s)_i.$$

Therefore, the equation (17) can be rewritten as below:

$$(1 + 4\kappa) \sum_{i \in J_+(x)} (d_x)_i (d_s)_i + \sum_{i \in J_-(x)} (d_x)_i (d_s)_i = (1 + 4\kappa)\sigma_+ - \sigma_- \geq 0. \tag{18}$$

To estimate the bound for $\|d_x\|$ and $\|d_s\|$, we need the following technical lemma.

Lemma 8. *One has $\sigma_+ \leq \delta^2$ and $\sigma_- \leq (1 + 4\kappa)\delta^2$.*

Proof. From definition of σ_+, σ_- and δ, one has

$$\sigma_+ = \sum_{i\in J_+}(d_x)_i(d_s)_i \leq \frac{1}{4}\sum_{i\in J_+}((d_x)_i + (d_s)_i)^2 \leq \frac{1}{4}\sum_{i\in J}((d_x)_i + (d_s)_i)^2 = \delta^2.$$

From the fact that M is the $P_*(\kappa)$ matrix and using (18), one can easily see that

$$\sigma_- \leq (1+4\kappa)\sigma_+ \leq (1+4\kappa)\delta^2.$$

This completes the proof. □

Upper bounds for $\|d_x\|$ and $\|d_s\|$ are proved in the following lemma.

Lemma 9. *The inequalities* $\|d_x\| \leq 2\sqrt{1+2\kappa}\delta$ *and* $\|d_s\| \leq 2\sqrt{1+2\kappa}\delta$ *hold.*

Proof. The proof is similar to the Lemma 4.4 in [30]. We just restate it here. From the fact that $\sum_{i\in J}(d_x)_i(d_s)_i = \sigma_+ - \sigma_-$ and definition δ, one has

$$2\delta = \|d_x + d_s\| = \sqrt{\sum_{i=1}^{n}((d_x)_i + (d_s)_i)^2} = \sqrt{\sum_{i=1}^{n}((d_x)_i^2 + (d_s)_i^2) + 2(\sigma_+ - \sigma_-)}.$$

Due to (17), one has

$$\begin{aligned} 2\delta &\geq \sqrt{\sum_{i=1}^{n}((d_x)_i^2 + (d_s)_i^2) + 2\left(\frac{\sigma_-}{1+4\kappa} - \sigma_-\right)} \\ &= \sqrt{\sum_{i=1}^{n}((d_x)_i^2 + (d_s)_i^2) - \frac{8\kappa}{1+4\kappa}\sigma_-}. \end{aligned}$$

Hence, we have

$$4\delta^2 + \frac{8\kappa}{1+4\kappa}\sigma_- \geq \sum_{i=1}^{n}((d_x)_i^2 + (d_s)_i^2).$$

From Lemma 8, we obtain

$$4(1+2\kappa)\delta^2 \geq 4\delta^2 + \frac{8\kappa}{1+4\kappa}\sigma_- \geq \sum_{i=1}^{n}((d_x)_i^2 + (d_s)_i^2).$$

This implies that

$$\|d_s\| \leq \sum_{i=1}^{n}((d_x)_i^2 + (d_s)_i^2) \leq 2\sqrt{1+2\kappa}\delta.$$

Similarly, we can prove that $\|d_s\| \leq 2\sqrt{1+2\kappa}\delta$. This completes the proof of the lemma. □

Now, using the e-convexity property of the function $\Psi(v)$, one has

$$\Psi(v_+) = \Psi\left(\sqrt{(v+\alpha d_x)(v+\alpha d_s)}\right) \leq \frac{1}{2}\left[\Psi(v+\alpha d_x) + \Psi(v+\alpha d_s)\right].$$

Let

$$\begin{aligned} f(\alpha) &= \Psi(v_+) - \Psi(v), \\ f_1(\alpha) &= \frac{1}{2}\left[\Psi(v+\alpha d_x) + \Psi(v+\alpha d_s)\right] - \Psi(v). \end{aligned} \tag{19}$$

In what follows, we derive an upper bound on the value of $f(\alpha)$ during an inner iteration. To this end, note that $f_1(\alpha)$ is an upper bound of $f(\alpha)$, that is $f(\alpha) \leq f_1(\alpha)$, and $f(0) = f_1(0) = 0$.
Using the definition of $f_1(\alpha)$, we have

$$f_1'(\alpha) = \frac{1}{2}\sum_{i=1}^{n}\left(\psi'(v_i+\alpha d_{x_i})d_{x_i} + \psi'(v_i+\alpha d_{s_i})d_{s_i}\right).$$

Using (16) and the second equation in (8), we may write:

$$f_1'(0) = \frac{1}{2}\nabla\Psi(v)^T(d_x+d_s) = -\frac{1}{2}\nabla\Psi(v)^T\nabla\Psi(v) = -2\delta(v)^2.$$

Furthermore, we have

$$f_1''(\alpha) = \frac{1}{2}\sum_{i=1}^{n}\left(\psi''(v_i+\alpha d_{x_i})d_{x_i}^2 + \psi''(v_i+\alpha d_{s_i})d_{s_i}^2\right).$$

Lemma 10. *(Lemma 5.5 in [2]) The following inequality holds:*

$$f_1''(\alpha) \leq 2(1+2\kappa)\delta^2\psi''(v_{\min} - 2\alpha\sqrt{1+2\kappa}\delta).$$

Lemma 11. *Suppose that $\rho : [0,\infty) \to (0,1]$ is the inverse of the function $-\frac{1}{2}\psi'(t)$ in the interval $(0,1]$. Thus, the largest possible value for α satisfying $f'(\alpha) \leq 0$ is given by*

$$\bar{\alpha} = \frac{1}{2\sqrt{1+2\kappa}\delta}\left(\rho(\delta) - \rho(\delta + \frac{\delta}{\sqrt{1+2\kappa}})\right). \tag{20}$$

Proof. From e-convexity property of function $\psi(v)$, $P_*(\kappa)$ property of M, Lemma 8, Lemma 9 and Lemma 5.6 in [29], the proof is completed. □

Lemma 12. *(Lemma 5.8 in [2]) Suppose that $\bar{\alpha}$ is given by (20). One has*

$$\bar{\alpha} \geq \frac{1}{(1+2\kappa)\psi''(\rho(\delta + \frac{\delta}{\sqrt{1+2\kappa}}))}. \tag{21}$$

In what follows, we use the notation

$$\tilde{\alpha} = \frac{1}{(1+2\kappa)\psi''(\rho(\delta + \frac{\delta}{\sqrt{1+2\kappa}}))}, \tag{22}$$

and we will use $\tilde{\alpha}$ as the default step size. Lemma 12 implies that $\tilde{\alpha} \leq \overline{\alpha}$.

Lemma 13. *(Lemma 3.7 in [32]) Suppose that $\bar{\alpha}$ is given by (21). Then, for α satisfying $\alpha \leq \bar{\alpha}$, we have:*

$$f(\alpha) \leq -\alpha\delta^2.$$

In the next lemma, we compute the amount of decrease in the proximity function during an inner iteration.

Lemma 14. *Suppose that $\Psi(v) \geq 1$ and $\rho : [0, \infty) \to (0, 1]$ is the inverse of the function $-\frac{1}{2}\psi'(t)$ in the interval $(0, 1]$, and $\tilde{\alpha}$ is defined as in (22). Then, one has*

$$f(\tilde{\alpha}) \leq -\Theta\left(\frac{\delta^{\frac{p-1}{p}}}{(1+2\kappa)p}\right). \tag{23}$$

Proof. From Lemma 13 and the fact that $\tilde{\alpha} \leq \overline{\alpha}$, we have $f(\tilde{\alpha}) \leq -\tilde{\alpha}\delta^2$. Now, we compute the inverse function $-\frac{1}{2}\psi'(t)$, for all $t \in (0, 1]$. By solving the equation $-\frac{1}{2}\psi'(t) = s$ for t, we conclude that:

$$-\left[t - \left(\frac{\sqrt{3}-1}{\tan(h(t))-1}\right)^p\right] = 2s.$$

This implies that

$$\left(\frac{\sqrt{3}-1}{\tan(h(t))-1}\right)^p \leq 2s + 1,$$

where, the last inequality is obtained from the fact that $t \leq 1$. Now, putting $t = \rho(2\delta)$, we get $4\delta = -\psi'(t)$. Thus, we have

$$\left(\frac{\sqrt{3}-1}{\tan(h(t))-1}\right)^p \leq 4\delta + 1.$$

We note that for all $0 < t \leq 1$, $\tan(h(t)) \in (1, \sqrt{3}]$. Therefore, $\frac{\sqrt{3}-1}{\tan(h(t))-1} \geq 1$. This implies that

$$\frac{\sqrt{3}-1}{\tan(h(t))-1} \quad \leq \quad (4\delta+1)^{\frac{1}{p}}. \tag{24}$$

It follows that

$$\frac{1}{\tan(h(t))-1} \quad \leq \quad \frac{(4\delta+1)^{\frac{1}{p}}}{\sqrt{3}-1} \leq 2(4\delta+1)^{\frac{1}{p}}. \tag{25}$$

Thus, we have

$$\begin{aligned}\psi''(t) &= 1 + \frac{2\pi p \left(1+\tan^2(h(t))\right)}{(4+2t)^2(\tan(h(t))-1)} \left(\frac{\sqrt{3}-1}{\tan(h(t))-1}\right)^p \\ &\leq 1 + \frac{4\pi p}{(4+2t)^2}(1+\tan^2(h(t)))(4\delta+1)^{\frac{p+1}{p}}.\end{aligned}$$

Since $(4+2t)^2 \geq 16$ for all $0 < t \leq 1$ and $1+\tan^2(h(t)) \leq 4$, it follows that:

$$\psi''(t) \leq 1 + p\pi(4\delta+1)^{\frac{p+1}{p}}. \tag{26}$$

Using (26), we obtain a lower bound for $\tilde{\alpha}$, with $t \in (0, 1]$

$$\begin{aligned}\tilde{\alpha} &= \frac{1}{(1+2\kappa)\psi''(t)} \geq \frac{1}{(1+2\kappa)\left(1+p\pi(4\delta+1)^{\frac{p+1}{p}}\right)} \\ &= \Theta\left(\frac{1}{(1+2\kappa)p\delta^{\frac{p+1}{p}}}\right),\end{aligned}$$

which implies that

$$f(\tilde{\alpha}) \leq -\frac{\delta^2}{(1+2\kappa)\psi''(\rho(2\delta))} \leq -\Theta\left(\frac{\delta^{\frac{p-1}{p}}}{(1+2\kappa)p}\right).$$

This proves the lemma. □

A direct consequence of applying the second part of Lemma 5 to (23) is as follows

$$f(\tilde{\alpha}) \leq -\Theta\left(\frac{\delta^{\frac{p-1}{p}}}{(1+2\kappa)p}\right) \leq -\Theta\left(\frac{\Psi^{\frac{p-1}{2p}}}{(1+2\kappa)p}\right). \tag{27}$$

5 Iteration complexity

In this section, we proceed by computing the worst case total iteration complexity bound for Algorithm 1, based on the new measure function Ψ induced from kernel function consisting of trigonometric function in its barrier term as defined in (9) for large-update methods. To this end, using Lemma 7 and just after reducing μ to $(1-\theta)\mu$ with $\theta \in (0,1)$, one may write

$$\Psi(v_+) \le \Psi(v) + \frac{\theta}{2(1-\theta)}(2\Psi(v) + 2\sqrt{2n\Psi(v)} + n). \tag{28}$$

From the structure of Algorithm 1, and at the begining of an outer iteration and just before updating the parameter μ, we have $\Psi(v) \le \tau$. Therefore, from (28) it is easy to see that the proximity function $\Psi(v)$ exceeds the threshold τ after the μ-update. Therefore, we need to compute the number of inner iterations that are required to return the iterates back to the situation where measure function $\Psi(v) \le \tau$ after the μ-update. In what follows, we represent the value of proximity function $\Psi(v)$ after updating μ by Ψ_0, and the subsequent values by Ψ_j, for all $j = 1, \dots, L-1$, where L is the total number of inner iterations performed in an outer iteration. From (28) and the fact that proximity function $\Psi(v) \le \tau$, it implies that

$$\Psi_0 \le \tau + \frac{\theta}{2(1-\theta)}(2\tau + 2\sqrt{2n\tau} + n). \tag{29}$$

Due to (27), the decreasing of Ψ in any inner iteration is denoted by

$$\Psi_{j+1} \le \Psi_j - \varsigma \Delta\Psi_j, \qquad j = 0, 1, \dots, L-1, \tag{30}$$

where ς is some positive constant and $\Delta\Psi_j$, is denoted by

$$\Delta\Psi_j = \frac{\Psi^{\frac{p-1}{2p}}}{(1+2\kappa)p}. \tag{31}$$

In the sequel, we recall the following technical lemma which helps us to state the inner iteration complexity result in an outer iteration. Furthermore, one can find its proof in [29].

Lemma 15. *Given $\alpha \in [0,1]$ and $t \ge -1$, one has*

$$(1+t)^\alpha \le 1 + \alpha t.$$

The worst case upper bound for the total number of inner iteration in an outer iteration is given by the following theorem.

Theorem 1. *Suppose that $\tau = O(n) \geq 1$. If L represent the total number of inner iterations in an outer iteration of the Algorithm 1, one has*

$$L \leq 1 + \frac{2p^2(1+2\kappa)\Psi_0^{\frac{p+1}{2p}}}{(p-1)\varsigma}. \tag{32}$$

Proof. From definition of L, i.e., $\Psi_{L-1} > \tau$ and $\Psi_L \leq \tau$, and (30), for any $j = 0, 1, \ldots, L-1$, we have

$$\begin{aligned} 0 \quad \leq \quad & \Psi_{j+1}^{\frac{p+1}{2p}} \leq \left(\Psi_j - \varsigma \frac{\Psi^{\frac{p-1}{2p}}}{p(1+2\kappa)}\right)^{\frac{p+1}{2p}} = \Psi_j^{\frac{p+1}{2p}} \left(1 - \varsigma \frac{\Psi^{-\frac{p-1}{2p}}}{p(1+2\kappa)}\right)^{\frac{p+1}{2p}} \\ \leq \quad & \Psi_j^{\frac{p+1}{2p}} \left(1 - \frac{\varsigma(p+1)\Psi^{\frac{p-1}{2p}}}{2p^2(1+2\kappa)}\right) = \Psi_j^{\frac{p+1}{2p}} - \frac{\varsigma(p+1)}{2p^2(1+2\kappa)}, \end{aligned} \tag{33}$$

where the last inequality is obtained from Lemma 15. Using (33) subsequently, we obtain

$$\Psi_{j+1}^{\frac{p+1}{2p}} \leq \Psi_0^{\frac{p+1}{2p}} - \frac{j(p+1)\varsigma}{2p^2(1+2\kappa)}.$$

For $j = L-1$, we have

$$0 \quad \leq \quad \Psi_L^{\frac{p+1}{2p}} \leq \Psi_0^{\frac{p+1}{2p}} - \frac{(L-1)(p+1)}{2p^2(1+2\kappa)}\varsigma,$$

which implies that

$$L \leq 1 + \frac{2p^2(1+2\kappa)\Psi_0^{\frac{p+1}{2p}}}{(p-1)\varsigma}.$$

This completes the proof. □

Remark 1. *For the large-update method, we have $\tau = O(n)$ and $\Theta = \Theta(1)$.*

As a consequence of Remark 1 and (29), we conclude that $\Psi_0 = O(n)$. Therefore, Theorem 1 implies the following upper bound for the total number of inner iterations for an outer iteration as

$$L \leq \left\lceil \Theta\left(1 + p(1+2\kappa)\Psi_0^{\frac{p+1}{2p}}\right)\right\rceil = O\left\lceil p(1+2\kappa)n^{\frac{p+1}{2p}}\right\rceil. \tag{34}$$

Theorem 2. *Suppose that $P_*(\kappa)$-LCP defined by (1). Then for large-update methods the total number of iterations to get an ϵ-solution, i.e., a solution that satisfies $x^T s = n\mu \le \epsilon$, is bounded by*

$$O\left((1+2\kappa)pn^{\frac{p+1}{2p}} \log \frac{n}{\epsilon}\right).$$

Proof. Lemma I.36 in [35] implies that the total number of outer iterations for getting $n\mu \le \epsilon$ are bounded above by $O\left(\frac{1}{\theta} \log \frac{n}{\epsilon}\right)$. Moreover, the total number of iterations for Algorithm 1 is obtained from multiplying the total number of inner and outer iterations. Hence, we may derive the following total number of iterations to get an ϵ-solution, i.e., a solution that satisfies $x^T s = n\mu \le \epsilon$, as follows:

$$O\left((1+2\kappa)pn^{\frac{p+1}{2p}} \log \frac{n}{\epsilon}\right).$$

This follows the result. □

So far, this bound significantly improves the iteration bound of large update primal-dual interior point methods based on the trigonometric kernel functions obtained in [9]. The iteration complexity for the small-update methods is straightforward and we left it for the interested readers.

6 Numerical results

Although our main focus in this paper is finding the worst case complexity of Algorithm 1 based on the new considered kernel function with trigonometric barrier term for linear complementary problems, here we present some numerical results of performing Algorithm 1 with the following eight kernel functions introduced in the literature and new proposed kernel function with $p = \{2, 5, 10\}$. The numerical results are obtained using MATLAB 7.6.0 (R2008a) on a PC with CPU 2.0 GHz and 2 G RAM memory. Without loss of generality, we take the accuracy parameter $\epsilon = 10^{-8}$, the threshold parameter $\tau = 3$, the parameter $\mu = 1$ and $p = \{2, 5, 10\}$ in all experiments. In Table 1, we present eight kernel functions presented in the literature. We consider the following problems:

Problem 1. (Lee's example in [23]) For this test problem, we have:

$$M = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}, \qquad q = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Table 1: Eight kernel functions.

Kernel function $\psi_i(t)$	Ref
$\psi_1(t) = \frac{t^2-1}{2} + t^{-1} - 1$	[24]
$\psi_2(t) = \frac{t^2-1}{2} + \frac{6}{\pi}\tan(\frac{1-t}{2+4t}\pi)$	[10]
$\psi_3(t) = \frac{t^2-1}{2} - \log(t) + \frac{1}{8}\tan^2(\frac{1-t}{2+4t}\pi)$	[32]
$\psi_4(t) = \frac{t^2-1}{2} + \frac{4}{\pi}\cot(\frac{\pi t}{1+t})$	[17]
$\psi_5(t) = \frac{t^2-1}{2} + (\frac{1}{t} - 1)\frac{e^{\frac{1}{t}-1}}{e}$	[24]
$\psi_6(t) = \frac{t^2-1}{2} - \int_1^t e^{\tan(\frac{\pi}{2+2x})-1}dx$	[31]
$\psi_7(t) = \frac{t^2-1}{2} + \frac{e^{(e^{4(\frac{1}{t}-1)}-1)}-1}{4}$	[23]
$\psi_8(t) = \frac{t^2-1}{2} + \frac{e^{\frac{1}{t}-1}}{e}$	[24]

Table 2: Number of iterations for Problem 1.

θ	ψ_1	ψ_2	ψ_3	ψ_4	ψ_5	ψ_6	ψ_7	ψ_8	ψ_9	ψ_{10}	ψ_{11}
0.1	27	25	22	126	18	22	18	20	20	19	17
0.2	27	24	21	99	18	21	18	18	19	19	17
0.3	26	23	21	99	18	20	17	18	19	18	17
0.4	26	22	21	75	17	20	17	19	18	18	16
0.5	25	21	20	77	17	19	17	18	18	17	16
0.6	24	20	20	58	18	19	16	17	17	16	16
0.7	23	19	19	58	17	18	16	18	17	15	15

Algorithm 1 starts with initial point $(x^0; s^0) = (0.4\ 0.45\ 2.45\ 2.2)$. An optimal solution of Problem 1 is given by:

$$(x^*; s^*) = (0\ 0\ 2.0500\ 3.1000).$$

Numerical results of applying Algorithm 1 based on the kernel functions given in Table 1 and new proposed kernel function with different values of θ for test Problem 1 are given in Table 2. In Tables 2-5, the functions ψ_9, ψ_{10} and ψ_{11} are new proposed kernel function with $p = \{2, 5, 10\}$, respectively. Moreover, in Tables 2-5, for moment we present kernel function $\psi_i(t)$ for $i \in \{1, 2 \cdots, 11\}$ as ψ_i.

Problem 2. Consider a randomly generated $P_*(0)$-LCP test problem. We select matrix M as $M = AA^T$, where $A = rand(n, n)$ and use the barrier

Table 3: Number of iterations for Problem 2.

n	ψ_1	ψ_2	ψ_3	ψ_4	ψ_5	ψ_6	ψ_7	ψ_8	ψ_9	ψ_{10}	ψ_{11}
2	9	9	9	11	12	9	10	8	9	8	8
5	17	22	17	30	20	24	19	14	16	13	13
10	20	40	18	72	24	30	22	16	19	17	15
20	38	25	20	89	49	37	29	32	35	32	27
50	42	59	26	175	59	45	38	90	42	41	39
100	47	69	40	425	138	48	46	181	46	44	41
200	56	84	53	568	355	57	75	235	63	58	51

Table 4: Number of iterations for Problem 3.

n	ψ_1	ψ_2	ψ_3	ψ_4	ψ_5	ψ_6	ψ_7	ψ_8	ψ_9	ψ_{10}	ψ_{11}
2	30	31	38	38	132	35	32	32	34	32	29
5	219	206	211	286	375	555	245	610	231	211	191
10	1436	1494	1041	1865	831	705	749	1651	737	715	701

parameter $\theta = 0.5$. In this case, the starting point is selected as $x^0 = s^0 = \mathbf{e}$ and $q = s - Mx$.

Problem 3. (Murty's example in [40]) In this case, the parameter θ and starting point are chosen as $\theta = 0.5$ and $x^0 = s^0 = \mathbf{e}$. Also,

$$M = \begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \qquad q = \begin{pmatrix} -1 \\ -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}$$

Problem 4. (Fathi's example in [12]) In this problem, the algorithm starts with initial point as $x^0 = s^0 = \mathbf{e}$ and

$$M = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 5 & 6 \\ 2 & 6 & 9 \end{pmatrix}, \qquad q = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

Table 5: Number of iterations for Problem 4.

θ	ψ_1	ψ_2	ψ_3	ψ_4	ψ_5	ψ_6	ψ_7	ψ_8	ψ_9	ψ_{10}	ψ_{11}
0.1	87	35	48	92	35	47	75	35	38	36	34
0.2	99	35	41	95	43	50	107	36	40	38	35
0.3	100	34	42	136	43	66	59	44	37	33	31
0.4	116	33	36	88	34	49	33	34	34	33	31
0.5	47	35	37	62	48	49	44	35	37	35	33
0.6	63	29	33	74	68	35	28	29	33	31	28
0.7	63	39	35	57	61	47	75	39	38	35	34

Remark 2. *For all test problems, the step size is computed as:*

$$\tilde{\alpha} \quad = \quad \frac{1}{(1+2\kappa)\psi''(\rho(2\delta))}. \tag{35}$$

Based on the obtained results in this section, we conclude that Algorithm 1 with the new proposed kernel function has better results than the others kernel functions.

7 Concluding remarks

In this paper, we proposed a primal-dual interior point algorithm for $P_*(\kappa)$-LCP based on a new family of kernel functions consisting of a trigonometric function in its barrier term in large neighborhood of the central path. Using the feasibility condition to initial point, by a simple analysis we proved that our algorithm has the worst case iteration complexity bound for large-update method as $O\left((1+2\kappa)pn^{\frac{p+1}{2p}}\log\frac{n}{\epsilon}\right)$ for $p \geq 2$. Finally, with $p = O(\log n)$ we obtain complexity bound of the algorithm as $O\left((1+2\kappa)\sqrt{n}\log n\log\frac{n}{\epsilon}\right)$.

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