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Hopf bifurcation analysis of a diffusive predator-prey model with Monod-Haldane response

Muniyagounder Sambath[a*] **and Ramajayam Sahadevan**[b]

[a] *Department of Mathematics, Periyar University, Salem-636011, India*
email: sambathbu2010@gmail.com

[b] *Ramanujan Institute for Advanced Study in Mathematics,*
University of Madras, Chennai–600 005, India
email: ramajayamsaha@yahoo.co.in

Abstract. In this paper, we have studied the diffusive predator-prey model with Monod-Haldane functional response. The stability of the positive equilibrium and the existence of Hopf bifurcation are investigated by analyzing the distribution of eigenvalues without diffusion. We also study the spatially homogeneous and non-homogeneous periodic solutions through all parameters of the system which are spatially homogeneous. In order to verify our theoretical results, some numerical simulations are also presented.

Keywords: stability, prey-predator, Monod-Haldane response, Hopf bifurcation.
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1 Introduction

One of the important themes in both theoretical ecology and applied mathematics is the dynamic relationship between predators and their prey due to its universal existence and importance in population dynamics (see [8, 9]). Predator-prey interactions have shaped all life on earth and this underlying commonality helps to explain the recent development of parallel independent research paths in many diverse fields. For instance, the persistent

*Corresponding author.
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threat of immediate violent death is a hallmark of predator-prey interactions that has profound implications for human psychology, neurobiology, physiology, developmental biology, ecology and evolution. The first model to describe the size (density) dynamics of two populations interacting as a predator-prey system was developed independently by Lotka (1925) and Volterra (1931). Since the classical Lotka-Volterra models suffer from some unavoidable limitations in describing precisely many realistic phenomena in biology, in some cases, they should make way to some more sophisticated models from both mathematical and biological points of view.

Ruan and Xiao [10] proposed a predator-prey model with Monod-Haldane-type functional response of the following form:

$$\begin{aligned} \frac{du}{dt} &= ru(1-\tfrac{u}{K}) - \tfrac{\beta uv}{\alpha+u^2}, \\ \frac{dv}{dt} &= -\gamma v + \tfrac{\mu\beta uv}{\alpha+u^2}, \end{aligned} \tag{1}$$

where u and v are population densities of prey and predator, respectively; r is the birth rate, K is a carrying capacity, β is the maximum uptake rate of the prey and μ is the conversion rate of prey into predator, γ is the death rate of predator and α is half-saturation. The predator consumes the prey with Monod-Haldane response.

Then, after nondimensionalization and reduction of parameters, we take

$$\tilde{u} = \frac{u}{K},\ \tilde{v} = \mu v,\ \tilde{t} = rt,$$

and rescale the parameters via,

$$a = \frac{\alpha}{K^2},\ b = \frac{\beta}{\mu K^2 r},\ d = \frac{\beta\mu}{K\gamma},\ c = \frac{\gamma}{r}.$$

This leads to (after dropping the tildes) model (1) becomes

$$\begin{aligned} \frac{du}{dt} &= u(1-u) - \tfrac{buv}{a+u^2}, \\ \frac{dv}{dt} &= c\left(-v + \tfrac{duv}{a+u^2}\right). \end{aligned} \tag{2}$$

For considering the spatial effect on the population dynamics, we have the spatial version of the model (2) as the following initial boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + u(1-u) - \dfrac{buv}{a+u^2}, & x \in \Omega,\ t > 0, \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v + c\left(-v + \dfrac{duv}{a+u^2}\right), & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{d\nu} = \dfrac{\partial v}{d\nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \geq 0,\ v(x,0) = v_0(x) \geq 0, & x \in \Omega. \end{cases} \tag{3}$$

Here Δ is the Laplacian operator on $\Omega \in R^N$, where d_1 and d_2 denote respectively diffusivity of prey and predator which are kept independent of space and time. The no-flux boundary condition means that the statical environment Ω is isolated and ν is the outward unit normal to $\partial\Omega$. The initial values $u_0(x),\ v_0(x)$ are assumed to be positive and bounded in Ω.

Many researchers have investigated the stability and Hopf bifurcation of predator-prey system with various functional response [2–4, 6, 7, 11–15, 17, 18, 20, 21]. In particular predator-prey model with nonmonotonic functional response was considered [5, 16, 19, 23]. The spatially homogeneous and non-homogeneous Hopf bifurcations occur at the positive steady state of diffusive Holling-Tanner model studied in [7] and diffusive predator-prey model with predator saturation and competition response discussed in [11, 12]. Wang and Wei [18] analyzed Hopf bifurcation and steady state bifurcation of diffusive predator-prey system with Ivlev-type functional response and also showed that the Allee effect has significant impact on the dynamics. In [1], Baek et.al., studied the impulsive predator-prey systems with Monod-Haldane functional response and seasonal effects, they studied the local and global stabilities of prey-free solutions by using the Flquet theory of impulsive differential equations and comparison techniques. Ruan and Xiao [10] discussed the global dynamics of the predator-prey system with this nonmonotonic functional response and proved that the system would undergo a saddle-node bifurcation, Hopf bifurcation and the Bogdanov-Takens bifurcation. Zhang [22] studied the Hopf bifurcation analysis of three-species diffusive predator-prey model with Monod-Haldane with delays. Motivated by the discussions, in this article, we study the stability and Hopf bifurcation of a diffusive predator-prey model with Monod-Haldane response.

The plan of the article is as follows: In Section 2, we analyze the local stability and Hopf bifurcation of system (2). In Section 3 and 4, bifurcations of spatially homogeneous and non-homogeneous periodic solutions are rigorously proved the system (3) with numerical example and some concluding comments are made in Section 5.

2 Local stability and bifurcation analysis

It is straightforward to check that equation (2) admits three distinct equilibrium solutions namely (i) $E_1(0,0)$, (ii) $E_2(1,0)$ and (iii) $E_*(u_*, v_*)$. Note that $E_1 = (0,0)$ represent extinct of both prey and predator. By direct calculation we find that the eigenvalues of the associated Jacobin matrix are -1 and c. Hence the equilibrium point E_1 is a saddle point and it is unstable. The equilibrium point $E_2 = (1,0)$ represents either no the predator or only prey. The eigenvalues of the Jacobin matrix of (2) at E_2 are -1 and $c\left(\frac{d}{1+a} - 1\right)$. Thus we conclude that the equilibrium point E_2 is globally asymptotically stable if $\frac{d}{1+a} < 1$. On the other hand if $\frac{d}{1+a} > 1$ then the eigenvalues are real, distinct but opposite sign indicating that E_2 is a saddle point and it is unstable.

Next we consider the third equilibrium point $E_* = (u_*, v_*)$ where

$$u_* = \frac{1}{2}\left(d + \sqrt{d^2 - 4a}\right) > 0, \quad v_* = \frac{(1-u_*)(a+u_*^2)}{b} > 0.$$

From the biological point of view, it is more interesting to study the dynamical behavior of the positive equilibrium point $E_* = (u_*, v_*)$. The Jacobian matrix of the system (2) at the positive equilibrium point $E_* = (u_*, v_*)$ is

$$L_0(c) = \begin{pmatrix} -u_* + \dfrac{2bu_*^2 v_*}{(a+u_*^2)^2} & -\dfrac{bu_*}{(a+u_*^2)} \\ \dfrac{cdv_*(a-u_*^2)}{(a+u_*^2)^2} & -c\left(\dfrac{(a+u_*^2) - du_*}{(a+u_*^2)}\right) \end{pmatrix}. \quad (4)$$

The characteristic equation of $L_0(c)$ is given by $\lambda^2 - \lambda T + D = 0$, where

$$T = trL_0(c) = \frac{-(c+u_*)(a+u_*^2)^2 + u_*(acd + cdu_*^2 + 2bu_*v_*)}{(a+u_*^2)^2},$$

and

$$D = detL_0(c) = \frac{c[bv_*(a-u_*^2) + (2u_* - 1)(a+u_*^2)(a+u_*(u_*-b))]}{(a+u_*^2)^2}.$$

The roots of the characteristic equation are

$$\lambda_1 = \frac{T}{2} + \frac{1}{2}\sqrt{T^2 - 4D}, \;\; \lambda_2 = \frac{T}{2} - \frac{1}{2}\sqrt{T^2 - 4D}.$$

From the above, it is clear that the eigenvalues are real and distinct (real only) if $T^2 > 4D$ $(= 4D)$.

Let the roots of characteristic equation be complex, that is $\lambda_1 = p(c) + i\omega(c)$ and $\lambda_2 = p(c) - i\omega(c)$ where $p(c) = \frac{T}{2}$ and $\omega(c) = \frac{1}{2}\sqrt{4D - T^2}$. Obviously the equilibrium solution E_* is a stable spiral when $p(c) < 0$ while unstable $p(c) > 0$. If $p(c) = 0$, then we get

$$c_0 = \frac{u_*[2bu_*v_* - (a+u_*^2)^2]}{(a+u_*^2)((a+u_*^2) - du_*)}.$$

In this case E_* is a center. Next

$$\frac{dp}{dc}(c)|_{c=c_0} = \frac{-H}{(a+u_*^2)} < 0,$$

where $(H) = (a+u_*^2) - du_* > 0$. By the Poincare-Andronov-Hopf bifurcation theorem, the system (2) undergoes a Hopf bifurcation at E_* when $c = c_0$.

Thus we have the following.

Theorem 1. *Assume that the condition (H) holds. The equilibrium (u_*, v_*) of the system (2) is locally asymptotically stable when $c > c_0$ and unstable when $c < c_0$; the system (2) undergoes a Hopf bifurcation at the positive equilibrium (u_*, v_*) when $c = c_0$.*

3 Direction of Hopf bifurcation

In this section we wish to study the nature of the Hopf bifurcation which demands further analysis of the normal form of the system. Now we investigate the direction of Hopf bifurcation and stability of bifurcated periodic solutions arising through Hopf bifurcation. We translate the positive equilibrium $E_* = (u_*, v_*)$ to the origin by the translation $\hat{u} = u - u_*,\ \hat{v} = v - v_*$. For convenience, we denote $\hat{u}$ and $\hat{v}$ again by u and v respectively. Thus the local system (2) becomes

$$\begin{cases} \dfrac{du}{dt} = (u+u_*)(1-(u+u_*)) - \dfrac{b(u+u_*)(v+v_*)}{(a+(u+u_*)^2)}, \\ \dfrac{dv}{dt} = c\left(-(v+v_*) + \dfrac{d(u+u_*)(v+v_*)}{(a+(u+u_*)^2)}\right). \end{cases} \tag{5}$$

Rewrite (5) as

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = L_0(c)\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f(u,v,c) \\ g(u,v,c) \end{pmatrix}, \tag{6}$$

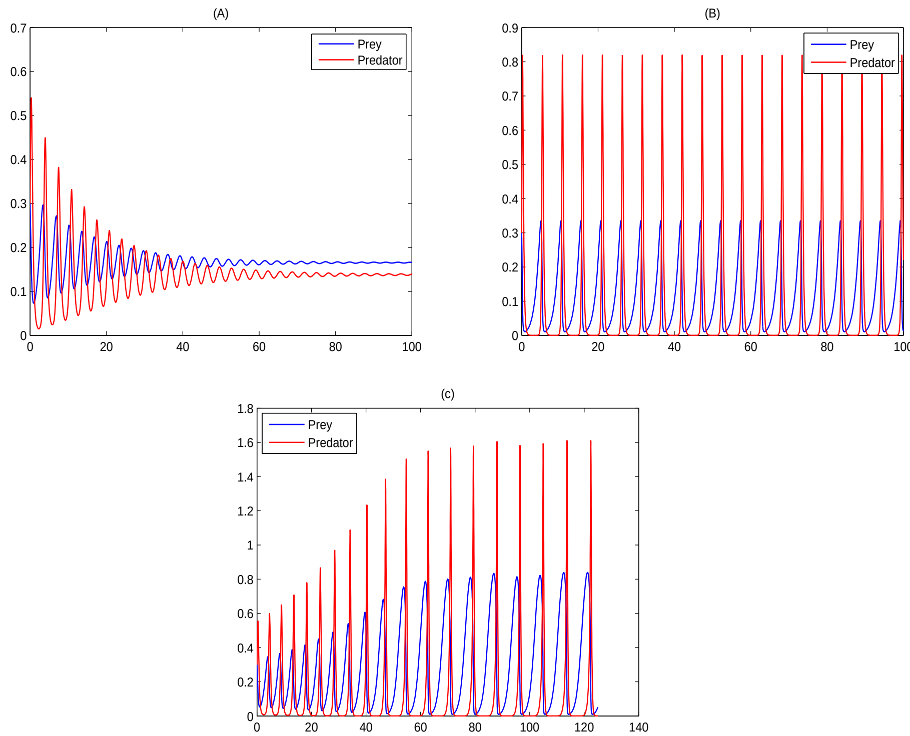

Figure 1: The trajectory graphs of the system (2) with $a = 0.1,\ b = 2$ and $d = 1$. $A)\ c > c_0,\ B)\ c = c_0,\ C)\ c < c_0$.

where $L_0(c)$ is as defined in (4) and

$$\begin{aligned} f(u, v, c) &= (ba_1 - 1)u^2 - ba_2uv + ba_3u^2v - ba_4u^3 + \ldots, \\ g(u, v, c) &= -ca_1u^2 + ca_2uv - ca_3u^2v + ca_4u^3 + \ldots, \end{aligned}$$

with

$$\begin{aligned} a_1 &= \frac{u_*v_*(3a - u_*^2)}{(a + u_*^2)^3}, \\ a_2 &= \frac{b(u_*^2 - a)}{(a + u_*^2)^2}, \\ a_3 &= \frac{u_*(3a - u_*^2)}{(a + u_*^2)^3}, \\ a_4 &= \frac{v_*(a - 7u_*^2)}{(a + u_*^2)^3}. \end{aligned}$$

Therefore the characteristic roots of $L_0(c)$ are $\lambda_{1,2} = p(c) \pm i\omega(c)$, where

$$p(c) = \frac{1}{2}(T), \qquad \omega(c) = \sqrt{(4D - T^2)}.$$

Note that the characteristic roots $\lambda_1,\ \lambda_2$ are a pair of complex conjugates, when $\left(4D - T^2\right) > 0$ and $\lambda_1 = i\omega(c_0)$ and $\lambda_2 = -i\omega(c_0)$ when $c = c_0$.

Set the following matrix for finding normal form of system (2) with suitable M and N

$$B = \begin{pmatrix} 1 & 0 \\ M & N \end{pmatrix},$$

and let

$$\begin{pmatrix} 1 \\ M - iN \end{pmatrix},$$

be the eigenvector corresponding to $\lambda = p(c) + i\omega(c)$ with

$$M = \frac{a + u_*^2}{bu_*}\left(-u_* + \frac{2bu_*^2 v_*}{(a + u_*^2)^2} - p(c)\right), \quad N = \frac{a + u_*^2}{bu_*}\omega(c).$$

Clearly

$$B^{-1} = \begin{pmatrix} 1 & 0 \\ \frac{-M}{N} & \frac{1}{N} \end{pmatrix}.$$

Using the transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = B \begin{pmatrix} x \\ y \end{pmatrix},$$

the system (5) becomes

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = L_0(c) \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} F(x, y, c) \\ G(x, y, c) \end{pmatrix}, \tag{7}$$

where

$$L_0(c) = \begin{pmatrix} p(c) & -\omega(c) \\ \omega(c) & p(c) \end{pmatrix},$$

with

$$\begin{aligned} F(x,y,c) &= (ba_1 - ba_2M - 1)x^2 - (ba_2N)xy + (ba_3N)x^2y \\ &\quad + b(a_3M - a_4)x^3 \ldots, \\ G(x,y,c) &= \frac{-M}{N}F(x,y,c) + \frac{1}{N}g_1(x,y,c), \end{aligned}$$

and

$$\begin{aligned} g_1(x,y,c) &= -c(a_1 - a_2M)x^2 + (ca_2N)xy - (ca_3N)x^2y \\ &\quad + c(a_4 - a_3M)x^3 \ldots . \end{aligned}$$

Rewrite (7) in the polar coordinates as

$$\begin{cases} \dot{r} = p(c)r + a(c)r^3 + \ldots, \\ \dot{\theta} = \omega(c) + c(c)r^2 + \ldots \ . \end{cases} \tag{8}$$

Then the Taylor expansion of (8) at $c = c_0$ yields

$$\begin{cases} \dot{r} = p'(c_0)(c - c_0)r + a(c_0)r^3 + \ldots, \\ \dot{\theta} = \omega(c_0) + \omega'(c_0)(c - c_0) + c(c_0)r^2 + \ldots \ . \end{cases} \tag{9}$$

To determine the stability of Hopf bifurcation periodic solution, we need to calculate the sign of the coefficient $\mathcal{A}(c_0)$ given by

$$\begin{aligned} \mathcal{A}(c_0) &= \frac{1}{16}\left[F_{xxx} + F_{xyy} + G_{xxy} + G_{yyy}\right]|_{(0,0,c_0)} \\ &\quad + \frac{1}{16\omega(c_0)}\left[F_{xy}(F_{xx} + F_{yy}) - G_{xy}(G_{xx} + G_{yy})\right. \\ &\quad \left. - F_{xx}G_{xx} + F_{yy}G_{yy}\right]|_{(0,0,c_0)}, \end{aligned}$$

where

$$\begin{aligned} &F_{xxx} = 6(ba_3M - ba_4), \quad F_{xyy} = G_{yyy} = F_{yy} = G_{yy} = 0, \\ &G_{xxy} = -(2ba_3M + 2ca_3), \quad F_{xy} = -ba_2N, \\ &F_{xx} = 2(ba_1 - ba_2M - 1), \quad G_{xy} = ba_2M + ca_2, \\ &G_{xx} = -2\frac{M}{N}(ba_1 - ba_2M - 1) + 2\frac{1}{N}c(a_2M - a_1), \end{aligned}$$

subscripts denote partial derivative. Thus we obtain

$$\Lambda = -\frac{\mathcal{A}(c_0)}{p'(c_0)}.$$

Now, from the Poincare-Andronov Hopf bifurcation theorem, $p'(c)|_{c=c_0} = -\frac{1}{2} < 0$ and, from the above calculations of $\mathcal{A}(c_0)$, we have the following conclusion.

Theorem 2. *Assume that the condition (H) holds.*

(i) If $\mathcal{A}(c_0) < 0$, the bifurcated periodic solutions are stable and the direction of Hopf bifurcation is supercritical.

(ii) If $\mathcal{A}(c_0) > 0$, the bifurcated periodic solutions are unstable and the direction of Hopf bifurcation is subcritical.

4 Stability and direction of spatial Hopf bifurcation

In this section, we mainly focus on the existence of spatially homogeneous and non-homogeneous periodic solutions bifurcating from the Hopf bifurcation of the reaction-diffusion system

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 u_{xx} + u\,(1-u) - \dfrac{buv}{a+u^2}, & x \in (0, l\pi),\ t > 0,\\ \dfrac{\partial u}{\partial t} = d_2 v_{xx} + c\left(-v + \dfrac{uv}{a+u^2}\right), & x \in (0, l\pi),\ t > 0,\\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in 0, l\pi,\ t > 0,\\ u(x,0) = u_0(x) \geq 0,\ v(x,0) = v_0(x) \geq 0, & x \in (0, l\pi). \end{cases} \tag{10}$$

To cast our discussion into the framework of the Hopf bifurcation theorem, we translate (10) into the following system by the transition $\hat{u} = u - u_*,\ \hat{v} = v - v_*$. For the sake of convenience, we still denote $\hat{u}$ and $\hat{v}$ by u and v respectively. Thus the reaction-diffusion system (10) becomes

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1 u_{xx} = \mathcal{F}(c, u, v), & x \in (0, l\pi),\ t > 0,\\ \dfrac{\partial u}{\partial t} - d_2 v_{xx} = \mathcal{G}(c, u, v), & x \in (0, l\pi),\ t > 0,\\ u_x(0,t) = u_x(l\pi, t) = 0, \quad v_x(0,t) = v_x(l\pi, t) = 0, & t > 0,\\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in (0, l\pi). \end{cases} \tag{11}$$

Define

$$\begin{cases} \mathcal{F}(\alpha, u, v) = (u+u_*)(1-(u+u_*)) - \dfrac{b(u+u_*)(v+v_*)}{(a+(u+u_*)^2)},\\ \mathcal{G}(\alpha, u, v) = c\left(-(v+v_*) + \dfrac{d(u+u_*)(v+v_*)}{(a+(u+u_*)^2)}\right), \end{cases}$$

where $\mathcal{F}, \mathcal{G} : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are C^∞ smooth with $\mathcal{F}(c, 0, 0) = \mathcal{G}(c, 0, 0) = 0$.

Now we define the real-valued Sobolev space

$$X = \left\{ (u, v) \in [H^2(0, l\pi)]^2 : (u_x, v_x)|_{x=0,l\pi} = 0 \right\},$$

and the complexification of X :

$$X_{\mathbb{C}} := X \oplus iX = \{u_1 + iu_2 : u_1, u_2 \in X\}.$$

The linearized operator of the system (10) evaluated at (u_*, v_*) is

$$L(c) = \begin{pmatrix} d_1 \frac{\partial^2}{\partial x^2} + A(c) & B(c) \\ C(c) & d_2 \frac{\partial^2}{\partial x^2} + D(c) \end{pmatrix},$$

with the domain $D_{L(c)} = X_{\mathbb{C}}$ where

$$A(c) = \mathcal{F}_u(c, 0, 0) = -u_* + \frac{2bu_*^2 v_*}{(a + u_*^2)^2}, \; C(c) = \mathcal{G}_u(c, 0, 0) = \frac{cdv_*(a - u_*^2)}{(a + u_*^2)^2},$$

$$B(c) = \mathcal{F}_v(c, 0, 0) = -\frac{bu_*}{(a + u_*^2)}, \; D(c) = \mathcal{G}_v(c, 0, 0) = -c\left(\frac{(a + u_*^2) - du_*}{(a + u_*^2)}\right),$$

with (u_*, v_*) as defined in Section 2.

The following condition is essential to guarantee that the Hopf bifurcation occurs:

$(H1)$ There exists a number $c^H \in \mathbb{R}$ and a neighborhood O of c^H such that for $c \in O$, $L(c)$ has a pair of complex, simple, conjugate eigenvalues $p(c) \pm i\omega(c)$, continuously differentiable in c, with $p(c^H) = 0$, $\omega_0 = \omega(c^H) > 0$ and $p'(c^H) \neq 0$; all other eigenvalues of $L(c)$ have non-zero real parts for $c \in O$.

Now we recall the Hopf bifurcation result appearing in [21] and apply it to the analysis of our model. It is well known that the eigenvalue problem

$$-\varphi'' = \mu\varphi, \quad x \in (0, l\pi); \quad \varphi'(0) = \varphi'(l\pi) = 0,$$

has eigenvalues $\mu_n = \frac{n^2}{l^2}$ $(n = 0, 1, 2, \ldots)$, with corresponding eigenfunctions $\varphi_n(x) = \cos(\frac{nx}{l})$. Let

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \sum_{n=0}^{\infty} \begin{pmatrix} a_n \\ b_n \end{pmatrix} \cos\frac{nx}{l},$$

be an eigenfunction of $L(c)$ corresponding to an eigenvalue $\rho(c)$, that is, $L(c)(\phi,\psi)^T = \rho(c)(\phi,\psi)^T$. Then, from a straightforward analysis, we obtain the following relation:

$$L_n(c)\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \rho(c)\begin{pmatrix} a_n \\ b_n \end{pmatrix}, \quad n = 0,1,2,\dots,$$

where

$$L_n(c) = \begin{pmatrix} -d_1\frac{n^2}{l^2} + A(c) & B(c) \\ C(c) & -d_2\frac{n^2}{l^2} + D(c) \end{pmatrix}.$$

It follows that eigenvalues of $L(c)$ are given by the eigenvalues of $L_n(c)$ for $n = 0,1,2,\dots$. The characteristic equation of $L_n(c)$ is

$$\rho^2 - T_n(c)\rho + D_n(c) = 0, \quad n = 0,1,2,\dots,$$

where

$$\begin{cases} T_n(c) = -u_* + \dfrac{2bu_*^2 v_*}{(a+u_*^2)^2} - c\left(\dfrac{(a+u_*^2) - du_*}{(a+u_*^2)}\right) - \dfrac{(d_1+d_2)n^2}{l^2}, \\ D_n(c) = (A(c)D(c) - B(c)D(c)) + d_1\dfrac{n^2}{l^2}D(c) \\ \qquad\qquad +d_2\dfrac{n^2}{l^2}\left(d_1\dfrac{n^2}{l^2} + u_* - \dfrac{2bu_*^2 v_*}{(a+u_*^2)^2}\right). \end{cases} \tag{12}$$

Therefore the eigenvalues are determined by

$$\rho(c) = \frac{T_n(c) \pm \sqrt{T_n^2(c) - 4D_n(c)}}{2}, \quad n = 0,1,2,\dots .$$

If the condition $(H1)$ holds, we see that, at $c = c^H$, $L(c)$ has a pair of simple purely imaginary eigenvalues $\pm i\omega_0$ if and only if there exists a unique $n \in \mathbb{N}\cup\{0\}$ such that $\pm i\omega_0$ are the purely imaginary eigenvalues of $L_n(c)$. In such a case, denote the associated eigenvector by $q = q_n = (a_n, b_n)^T \cos\frac{n\pi}{l}$, with $a_n, b_n \in \mathbb{C}$, such that $L_n(c)(a_n,b_n)^T = i\omega_0(a_n,b_n)^T$ or $L(c^H)q = i\omega_0 q$.

We identify the Hopf bifurcation value c^H which satisfies the condition $(H1)$ taking the following form, if there exists $n \in \mathbb{N}\cup\{0\}$ such that

$$T_n(c^H) = 0,\ D_n(c^H) = 0 \text{ and } T_j(c^H) \neq 0,\ D_j(c^H) \neq 0 \text{ for any } j \neq n, \tag{13}$$

and for the unique pair of complex eigenvalues $p(c) \pm i\omega(c)$ near the imaginary axis $p'(c^H) \neq 0$. It is easy to derive from (12) that $T_n(c) < 0$ and

$D_n(c) > 0$, if $2bu_*v_* \leq (a+u_*^2)^2$, which implies that $(0,0)$ is a locally asymptotically stable steady state of system (10).
If $2bu_*v_* > (a+u_*^2)^2$, we define

$$c_0^* = \frac{u_*[(a+u_*^2)^2 - 2bu_*v_*]}{(a+u_*^2)((a+u_*^2)-du_*)} > 0. \tag{14}$$

Hence the potential Hopf bifurcation point lives in the interval $(0, c_0^*]$. For any Hopf bifurcation c^H in $(0, c_0^*]$, $p(c^H) \pm i\omega(c^H)$ are the eigenvalues of $L(c^H)$ where

$$p(c^H) = \frac{T_n(c^H)}{2}, \quad \omega(c^H) = \sqrt{D_n(c^H) - p^2(c^H)},$$

and

$$p'(c^H) = \frac{1}{2}T_n^{'}(c^H) < 0. \tag{15}$$

From the above discussion, the determination of Hopf bifurcation point reduces to describe the set

$$\Lambda_1 = \left\{c^H \in (0, c_0^*] : \text{for some } n \in \mathbb{N}\cup\{0\}, (13) \text{ is satisfied}\right\},$$

when a set of parameters (d_1, d_2, a, b) are given. In the following, we fix $(d_1, d_2, a, b) > 0$ and choose l appropriately. First $c_0^H = c_0^*$ is always an element of Λ_1 for any $l > 0$ since $T_0(c_0^H) = 0$, $T_j(c_0^H) < 0$ for any $(j \geq 1)$, $D_m(c_0^H) > 0$ for any $m \in \mathbb{N}\cup\{0\}$. This corresponds to the Hopf bifurcation of spatially homogeneous periodic solution. Apparently c_0^H is also the unique value for the Hopf bifurcation of the spatially homogeneous periodic solution for any $l > 0$. Hence in the following we look for spatially non-homogeneous Hopf bifurcation points.
Note that, when $c < c_0^*$, it is easy to show that $T_n(c) = 0$ is equivalent to

$$c = c_0^* - \frac{(a+u_*^2)n^2(d_1+d_2)}{l^2((a+u_*^2)-du_*)}.$$

Substituting it into the second equation of (12), we have

$$\begin{aligned} D_n(c) &= -d_1^2\frac{n^4}{l^4} - \frac{n^2}{l^2}\left(d_2\left(-u_* + \frac{2bu_*^2v_*}{(a+u_*^2)^2}\right) - d_1c_0^*\left(\frac{du_* - (a+u_*^2)}{(a+u_*^2)}\right)\right. \\ &\quad \left. - \frac{\Theta(a+u_*^2)(d_1+d_2)}{(a+u_*^2)-du_*}\right) + \Theta c_0^*, \\ \text{where } \Theta &= \frac{[bv_*(a-u_*^2) + (2u_*-1)(a+u_*^2)(a+u_*(u_*-b))]}{(a+u_*^2)^2}. \end{aligned}$$

Let

$$B_0 = \left(d_2\left(-u_* + \frac{2bu_*^2v_*}{(a+u_*^2)^2}\right) - d_1c_0^*\left(\frac{du_* - (a+u_*^2)}{(a+u_*^2)}\right) - \frac{\Theta(a+u_*^2)(d_1+d_2)}{(a+u_*^2)-du_*}\right),$$

then $D_n(c) > 0$ if and only if

$$\frac{n^2}{l^2} < \frac{-B_0 + \sqrt{B_0^2 + 4d_1^2\Theta c_0^*}}{2d_1^2}.$$

So all the potential Hopf bifurcation points can be labeled as $\Lambda_1 = \{c_n^H\}_{n=0}^N$ for some $N \in \mathbb{N} \cup \{0\}$ where

$$c_n^H = c_0^* - \frac{(a+u_*^2)n^2(d_1+d_2)}{l^2((a+u_*^2)-du_*)}. \tag{16}$$

That is,

$$0 < c_n^H < c_{N-1}^H < \cdots < c_1^H < c_0^H = c_0^*, \tag{17}$$

satisfying

$$0 \le \frac{\frac{(a+u_*^2)-du_*}{(a+u_*^2)}(c_0^* - c_n^H)}{d_1+d_2} < \frac{-B_0 + \sqrt{B_0^2 + 4d_1^2\theta c_0^*}}{2d_1^2}.$$

Now we only need to verify whether $D_i(c_n^H) \neq 0$ for $i \neq n$. Here we derive a condition on the parameters so that $D_i(c_n^H) > 0$ for each $i = 0, 1, 2, \ldots$. Since

$$D_i(c_n^H) = d_1d_2\frac{i^4}{l^4} + \frac{i^2}{l^2}\left(\frac{(d_1c_n^H - d_2c_0^*)((a+u_*^2)-du_*)}{(a+u_*^2)^2}\right) + c_n^H\Theta,$$

we choose the diffusion coefficient d_2 as small as possible so that $d_1c_n^H - d_2c_0^* > 0$, that is, given the fixed N defined by (17), for every $0 < n \le N,\ d_2 < \epsilon(l,a,b,N)$, where

$$\epsilon(l,a,b,N) := \frac{\dfrac{c_0^*((a+u_*^2)-du_*)}{(a+u_*^2)} - \dfrac{N^2}{l^2}}{\dfrac{c_0^*((a+u_*^2)-du_*)}{(a+u_*^2)} + \dfrac{N^2}{l^2}} > 0. \tag{18}$$

Therefore $D_i(c_n^H) > 0$.

Then summarizing our analysis above and using Hopf bifurcation theorem in [21], we have the main result of this section on the existence of both spatially homogeneous and non-homogeneous periodic solutions bifurcating from Hopf bifurcation.

Theorem 3. *[21] Assume that $(a+u_*^2)^2 < 2bu_*v_*$. For any c_n^H, defined by (16), if there exists $\epsilon = \epsilon(l,a,b,N)$ defined by (18) such that $0 < d_2 < \epsilon$, then the system (10) undergoes a Hopf bifurcation at each $c = c_n^H$ $(0 \le n \le N)$. With s sufficiently* *small, for* $c = c(s),\ c(0) = c_n^H$, *there exists a family of* $T(s)-$ *periodic continuously differentiable solutions* $(u(s)(x,t), v(s)(x,t))$ *and the bifurcating periodic solutions can be parametrized in the form*

$$\begin{cases} u(s)(x,t) = s\left(a_n e^{2\pi it/T(s)} + \bar{a}_n e^{-2\pi it/T(s)}\right)\cos\frac{nx}{l} + o(s^2), \\ v(s)(x,t) = s\left(b_n e^{2\pi it/T(s)} + \bar{b}_n e^{-2\pi it/T(s)}\right)\cos\frac{nx}{l} + o(s^2), \end{cases} \tag{19}$$

$$T(s) = \frac{2\pi}{\omega_0^n}(1+\tau_2 s^2) + o(s^4), \quad \tau_2 = -\frac{1}{\omega_0^n}\left(Im(\sigma(c_n^H)) - \frac{Re(\sigma(c_n^H))}{p'(c_n^H)}\omega'(c_n^H)\right),$$

and

$$T''(0) = \frac{4\pi}{w_0^n}\tau_2 = -\frac{4\pi}{(\omega_0^n)^2}\left(Im(\sigma(c_n^H)) - \frac{Re(\sigma(c_n^H))}{p'(c_n^H)}\omega'(c_n^H)\right).$$

If all eigenvalues (except $\pm i\omega_0^n$*) of* $L(c_n^H)$ *have negative real parts, then the bifurcating periodic solutions are stable (resp. unstable) if* $Re(\sigma(c_n^H)) < 0(resp. > 0)$. *The bifurcation is supercritical (resp. subcritical) if*
$-\frac{1}{p'(c_n^H)}Re(\sigma(c_n^H)) < 0(resp. > 0)$. *Moreover*

(i) The bifurcating periodic solutions from c_0^H *are spatially homogeneous which coincide with the periodic solutions of the corresponding ODE system.*

(ii) The bifurcating periodic solutions from $c_n^H,\ n > 0$, *are spatially non-homogeneous.*

Next we follow the methods in [21] to calculate the direction of Hopf bifurcation and the stability of the bifurcating periodic orbits bifurcating from $c = c_0^*$. We have the following result.

Theorem 4. *[21] For the system (2), the bifurcating (spatially homogeneous) periodic solutions bifurcating from* $c = c_0^H$ *are locally asymptotically stable (resp. unstable) if* $Re(\sigma(c_0^H)) < 0$ *(resp.> 0). Furthermore the direction of Hopf bifurcation at* c_0^H *is supercritical (resp. subcritical) if* $Re(\sigma(c_0^H)) < 0$ *(resp. >0).*

Proof. Following the notations and calculation in [21], we set

$$q = (a_0, b_0)^T = \left(1, \frac{c_0^*((a+u_*^2) - du_*) - i\omega_0(a+u_*^2)}{bu_*}\right)^T,$$

$$q^* = (a_0^*, b_0^*)^T = \left(\frac{\omega_0(a+u_*^2) + ic_0^*((a+u_*^2) - du_*)}{2\omega_0\pi l(a+u_*^2)}, \frac{-ibu_*}{2\omega_0\pi l(a+u_*^2)}\right)^T,$$

such that $\langle q^*, q\rangle = 1$ and $\langle q^*, \bar{q}\rangle = 0$. Then, by direct computation, we get

$$c_0 = x_1 + iy_1,\ d_0 = x_2 + iy_2,\ e_0 = x_3,\ f_0 = x_4,\ g_0 = x_5 + iy_5,\ h_0 = x_6 + iy_6,$$

where

$$
\begin{aligned}
&x_1 = 2(a_1 b - 1) - 2a_2 b \frac{c_0^*((a+u_*^2) - du_*)}{bu_*}, \quad y_1 = 2a_2 b \frac{\omega_0(a+u_*^2)}{bu_*},\\
&x_2 = -2ca_1 + 2ca_2 \frac{c_0^*((a+u_*^2) - du_*)}{bu_*}, \quad y_2 = -2ca_2 \frac{\omega_0(a+u_*^2)}{bu_*},\\
&x_3 = 2(a_1 b - 1) - 2a_2 b \frac{c_0^*((a+u_*^2) - du_*)}{bu_*},\\
&x_4 = -2ca_1 + 2ca_2 \frac{c_0^*((a+u_*^2) - du_*)}{bu_*},\\
&x_5 = 6b\left(\frac{a_3 c_0^*((a+u_*^2) - du_*)}{bu_*} - a_4\right), \quad y_5 = \frac{-2ba_3\omega_0(a+u_*^2)}{bu_*},\\
&x_6 = 6c\left(a_4 - \frac{a_3 c_0^*((a+u_*^2) - du_*)}{bu_*}\right), \quad y_6 = \frac{2ca_3\omega_0(a+u_*^2)}{bu_*}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\langle q^*, Q_{qq}\rangle &= \frac{1}{2\omega_0(a+u_*^2)}[(\omega_0(a+u_*^2)x_1 + c_0^*((a+u_*^2) - du_*)y_1 - bu_* y_2)\\
&\quad + i(\omega_0(a+u_*^2)y_1 + bu_* x_2 - c_0^*((a+u_*^2) - du_*)x_1)],\\
\langle q^*, Q_{q\bar{q}}\rangle &= \frac{1}{2\omega_0(a+u_*^2)}[\omega_0(a+u_*^2)x_3 + i(bu_* x_4 - c_0^*((a+u_*^2) - du_*)x_3)],\\
\langle \bar{q}^*, Q_{qq}\rangle &= \frac{1}{2\omega_0(a+u_*^2)}[(\omega_0(a+u_*^2)x_1 - c_0^*((a+u_*^2) - du_*)y_1 + bu_* y_2)\\
&\quad + i(c_0^*((a+u_*^2) - du_*)x_1 + \omega_0(a+u_*^2)y_1 - bu_* x_2)],\\
\langle \bar{q}^*, Q_{q\bar{q}}\rangle &= \frac{1}{2\omega_0(a+u_*^2)}[\omega_0(a+u_*^2)x_3 + i(c_0^*((a+u_*^2) - du_*)x_3 - bu_* x_4)],\\
\langle q^*, C_{qq\bar{q}}\rangle &= \frac{1}{2\omega_0(a+u_*^2)}[(\omega_0(a+u_*^2)x_5 + c_0^*((a+u_*^2) - du_*)y_5 - bu_* y_6)\\
&\quad + i(\omega_0(a+u_*^2)y_5 + bu_* x_6 - c_0^*((a+u_*^2) - du_*)x_5)].
\end{aligned}
$$

Direct computation gives

$$
\begin{aligned}
H_{20} &= \begin{pmatrix} c_0 \\ d_0 \end{pmatrix} - \langle q^*, Q_{qq}\rangle \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} - \langle \bar{q}^*, Q_{qq}\rangle \begin{pmatrix} \bar{a}_0 \\ \bar{b}_0 \end{pmatrix} = 0,\\
H_{11} &= \begin{pmatrix} e_0 \\ f_0 \end{pmatrix} - \langle q^*, Q_{q\bar{q}}\rangle \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} - \langle \bar{q}^*, Q_{q\bar{q}}\rangle \begin{pmatrix} \bar{a}_0 \\ \bar{b}_0 \end{pmatrix} = 0.
\end{aligned}
$$

Then, by Yi et al. [21], it implies that $w_{20} = w_{11} = 0$; hence

$$\langle q^*, Q_{w_{20}\bar{q}}\rangle = \langle q^*, Q_{w_{11}\bar{q}}\rangle = 0.$$

After elementary but lengthy computations, we obtain

$$\begin{aligned}
Re(\sigma(c_0^H)) &= Re\left\{\frac{i}{2\omega_0}\langle q^*, Q_{qq}\rangle . \langle q^*, Q_{q\bar{q}}\rangle + \frac{1}{2}\langle q^*, C_{qq\bar{q}}\rangle\right\} \\
&= \frac{-1}{8\omega_0^3(a+u_*^2)^2}\Big\{(\omega_0(a+u_*^2)x_1 + c_0^*((a+u_*^2)-du_*)y_1 - bu_*y_2) \\
&\quad (bu_*x_4 - c_0^*((a+u_*^2)-du_*)x_3) + \omega_0(a+u_*^2)x_3(\omega_0(a+u_*^2)y_1 \\
&\quad + bu_*x_2 - c_0^*((a+u_*^2)-du_*)x_1)\Big\} + \frac{1}{4\omega_0(a+u_*^2)}\Big\{\omega_0(a+u_*^2)x_5 \\
&\quad + c_0^*((a+u_*^2)-du_*)y_5 - bu_*y_6\Big\}.
\end{aligned}$$

It follows from (15) that $p'(c_0^H) < 0$ and then, by Theorem 2.1 in [21], the periodic solutions bifurcating from $c = c_0^H$ are locally asymptotically stable (resp. unstable) if $Re(\sigma(c_0^H)) < 0$ (resp.> 0). Furthermore the direction of Hopf bifurcation at c_0^H is supercritical (resp. subcritical) if $Re(\sigma(c_0^H)) < 0$ (resp. >0). □

5 Conclusion

In this article we have considered the diffusive predator-prey model with spatial and without spatial effect. By considering the predator-prey system without spatial effect we find that the distribution of the roots of the characteristic equations of the local system (2) at each of the feasible equilibria and stability of positive equilibrium point is investigated. In particular, the system (2) undergoes a Hopf bifurcation at the positive equilibrium (u_*, v_*) when $c = c_0$ (see Figure 1). This paper shown that modest changes in the parameters c namely death rate of predator lead to dramatic changes in the qualitative dynamics of solutions. Also provided suitable remedial measures to maintain the balance of the ecosystem. Moreover when the direction of the Hopf bifurcation is supercritical, the bifurcating periodic solution is stable and when the direction of the Hopf bifurcation is subcritical, the bifurcating periodic solution is unstable.

The positive constant steady state solutions of the system (10) are locally asymptotically stable when $2bu_*v_* \leq (a+u_*^2)^2$. When $2bu_*v_* > (a+u_*^2)^2$, is the unique homogeneous Hopf bifurcation point where spatially homogeneous orbits bifurcate from (u_*, v_*) for any $l > 0$. Further there exists multiple non-homogeneous Hopf bifurcation points c_n^H, with $1 \leq n \leq N$, satisfying $0 < c_N^H < c_{N-1}^H < \cdots < c_1^H < c_0^H = c_0^*$. At these points, spatially non-homogeneous periodic orbits bifurcate from (u_*, v_*) for suitable $l > 0$.

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