Journal of Mathematical Modeling
Vol. 5, No. 1, 2017, pp. 61-75

JMM

A numerical approach to solve eighth order boundary value problems by Haar wavelet collocation method

Arikera Padmanabha Reddy*, Manjula Harageri and Channaveerapala Sateesha

Department of Mathematics, V. S. K. University, Ballari, India
emails: paddu.padmanabha@gmail.com, manjulaharageri@gmail.com, csatish9980@gmail.com

Abstract. In this paper a robust and accurate algorithm based on Haar wavelet collocation method (HWCM) is proposed for solving eighth order boundary value problems. We used the Haar direct method for calculating multiple integrals of Haar functions. To illustrate the efficiency and accuracy of the concerned method, few examples are considered which arise in the mathematical modeling of fluid dynamics and hydromagnetic stability. Convergence and error bound estimation of the method are discussed. The comparison of results with exact solution and existing numerical methods such as Quintic B-spline collocation method and Galerkin method with Quintic B-splines as basis functions shown that the HWCM is a powerful numerical method for solution of above mentioned problems.

Keywords: Haar wavelet, Eighth order boundary value problems, Collocation method.
AMS Subject Classification: 65T60, 65L10, 65L60.

1 Introduction

The wavelets have many applications in science and engineering since from its inception. Different families of wavelets are Haar, Daubechies, Coiflet, biorthogonal spline, Symlet etc. Alfred Haar [6] introduced the wavelets,

*Corresponding author.
Received: 17 December 2016 / Revised: 15 May 2017 / Accepted: 15 May 2017.

which are yielded from pairs of piecewise constant functions. These functions are symmetric, orthogonal, compactly supported and have explicit expression for scaling functions [14, 20]. Due to these properties Haar wavelets act as mathematical tool in the numerical solution of differential equations, integral equations and signal processing. The operational matrix method was derived by Chen and Hsiao [4, 5] to solve problems of dynamical systems. Lepik [15] improved the method through integration of Haar wavelets, where as this method can be used for solving higher order ordinary differential equations (ODEs), integral equations and integro-differential equations.

Some researchers applied the Haar wavelet collocation method (HWCM) to various boundary value problems(BVPs). Siraj-ul-Islam et al. [11] estimated the numerical solution of second order BVPs. Fazal-i-Haq et al. [7,8] found the solution of the fourth and sixth order BVPs. Reddy et al. [18,19] solved the fifth and seventh order ODEs arising in modeling viscoelastic flow and induction motors. Harpreet Kaur et al. [10] solved nonlinear BVPs having quadratic factor on dependent variable. Further applications of this method has been listed and illustrated by Hariharan and Kannan [9]. These studies motivated us to develop HWCM for eighth order BVPs, which arise in various fields.

The eighth order BVPs arise in many applications viz. hydrodynamics, hydromagnetic stability theory, fluid dynamics, beam and long wave theory, astrophysics, etc. Chandrasekhar [3] explained the infinite horizontal layer of fluid heated from below under the condition of effect of rotation. In dealing the case, when instability sets in as stationary convection, basic equations are modeled by sixth order differential equations which contained Rayleigh number and Taylor number. When instability sets as an over stability, system results into eighth-order ODEs. Shen [21] derived the eighth order differential equation by bending and axial vibrations of an elastic beam with the intelligent constrained layer treatment. Agarwal [1] discussed the existence and uniqueness for the solution of higher order ODEs. Viswanadham et al. [12, 13] devised the solution of eighth order BVPs by Quintic B-spline collocation method (QBCM) and Galerkin method (GM) with Quintic B-splines as basis functions.

The main goal of this work is to construct a simple collocation method combining with Haar family for the numerical solution of linear and nonlinear eighth order BVPs arising in mathematical modeling of various applications. We mainly focus on the following type of boundary value problems to test the simplicity and applicability of the HWCM.

The following form of eighth order BVP is considered

$$y^{(8)}(x) = f(x, y, y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}, y^{(5)}, y^{(6)}, y^{(7)}), \quad x \in (a, b), \tag{1}$$

subject to the following type of boundary conditions:

$$\begin{aligned} &y(a) = \alpha_1, \quad y(b) = \beta_1, \quad y^{(1)}(a) = \alpha_2, \quad y^{(1)}(b) = \beta_2, \\ &y^{(2)}(a) = \alpha_3, \quad y^{(2)}(b) = \beta_3, \quad y^{(3)}(a) = \alpha_4, \quad y^{(3)}(b) = \beta_4, \end{aligned} \tag{2}$$

where α_i,β_i, a and b are real constants for $i = 1, 2, 3, 4$.

The organization of this article is as follows in Section 2, Haar wavelets and their integrals are introduced. In Section 3, a general formulation for the numerical algorithm based on Haar wavelets is presented. Error bound and convergence issues are discussed in Section 4. Few problems are solved in Section 5 to test the effectiveness of the method and conclusion presented in the last section of this paper.

2 Haar wavelets and their integrals

In this section, we use orthogonal basis for the subspaces of $L^2[a, b]$ called Haar wavelet family. The interval $[a, b]$ is divided into 2^{J+1} subintervals of equal length (Δt), where J is called maximal level of resolution. We have coarser resolution values $j = 0, 1, \ldots, J-1$ and translation parameter $k = 0, 1, \ldots, 2^j - 1$ [14]. With these two parameters, i^{th} Haar wavelet in Haar family is defined as

$$h_i(t) = \begin{cases} 1, & \text{for} \quad t \in [\xi_1(i), \xi_2(i)), \\ -1, & \text{for} \quad t \in [\xi_2(i), \xi_3(i)), \\ 0, & \quad\text{otherwise}, \end{cases} \tag{3}$$

where $i = m + k + 1$, $\xi_1(i) = a + 2k\mu\Delta t$, $\xi_2(i) = a + (2k + 1)\mu\Delta t$, $\xi_3(i) = a + 2(k + 1)\mu\Delta t$ and $\mu = 2^{J-j}$. Eq. (3) is valid for $i > 2$, $h_1(t)$ and $h_2(t)$ are called father and mother wavelets in Haar family, which are defined as

$$h_1(t) = \begin{cases} 1, & \text{if} \quad t \in [a, b), \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

$$h_2(t) = \begin{cases} 1, & \text{if} \quad t \in [a, \frac{a+b}{2}), \\ -1, & \text{if} \quad t \in [\frac{a+b}{2}, b), \\ 0, & \quad\text{otherwise}. \end{cases} \tag{5}$$

As a consequence of Haar multiresolution analysis any function which is having finite energy on $[a,b]$ i.e. $f \in L^2[a,b]$ can be decomposed as infinite sum of Haar wavelets:

$$f(x) = \sum_{i=1}^{\infty} a_i h_i(x),$$

where a_i are called Haar coefficients. The above series terminates to finite if $f(x)$ is piecewise constant or approximated by piecewise constant during each subinterval. Since we have an explicit expression for each member of Haar family (3-5), we can integrate as multiple times depending upon the application. The following notations are introduced

$$p_{\gamma,i}(t) = \int_a^t \int_a^t \cdots \int_a^t h_i(x) dx^\gamma \tag{6}$$

and

$$E_{\gamma,i} = \int_a^b p_{\gamma,i}(t) dt. \tag{7}$$

For $i = 1$, Eq.(6) becomes

$$p_{\gamma,1}(t) = \frac{1}{\gamma!}(t-a)^\gamma, \tag{8}$$

and for $i \geq 2$, we get

$$p_{\gamma,i}(t) = \begin{cases} 0, & t \in [a, \xi_1(i)), \\ \frac{1}{\gamma!}(t-\xi_1(i))^\gamma, & t \in [\xi_1(i), \xi_2(i)), \\ \frac{1}{\gamma!}\left\{(t-\xi_1(i))^\gamma - 2(t-\xi_2(i))^\gamma\right\}, & t \in [\xi_2(i), \xi_3(i)), \\ \frac{1}{\gamma!}\left\{(t-\xi_1(i))^\gamma - 2(t-\xi_2(i))^\gamma + (t-\xi_3(i))^\gamma\right\}, & t \in [\xi_3(i), b). \end{cases} \tag{9}$$

3 Method of solution

3.1 Haar wavelet collocation method:

In previous section we have seen that each Haar function in Haar family (3-5) is piecewise constant and not continuous on $[0,1)$. Hence it can not be differentiated at the points of discontinuity. But we can integrate each member many times as noticed from Eq. (6) to (8). Due to this reason we expand the highest derivative in the differential equation (1) into Haar series. Other derivatives are obtained through integration. The proposed method is illustrated with the following four steps [14, 18, 19].

1. For a given resolution J, approximate the highest derivative in Eq.(1) by piecewise constant on each subinterval

$$y^{(8)}(x) = \sum_{i=1}^{2^{J+1}} a_i h_i(x). \tag{10}$$

2. Decompose $y^{(7)}(x), \quad y^{(6)}(x), \ldots, y(x)$ in terms of integrated Haar functions and replace these into the given linear differential equation.

3. Discritize equation obtained in step (2) at collocation points

$$x_l = \frac{(\tilde{x}_l + \tilde{x}_{l-1})}{2}, \quad l = 1, 2, \ldots, 2^{J+1},$$

where $\tilde{x_c}$ is the grid point given by

$$\tilde{x_c} = a + c\frac{(b-a)}{2^{J+1}}, \quad c = 0, 1, 2, \ldots, 2^{J+1}.$$

It results into a$2^{J+1} \times 2^{J+1}$ linear algebraic system of equations.

4. Calculate the wavelet coefficients a_i and obtain the Haar solution for the unknown function y.

The proposed method is further simplified with the help of particular boundary conditions for BVPs with $a = 0$, $b = 1$.

3.2 Constructing the boundary conditions

The following type of boundary conditions is considered

$$\begin{aligned} &y(0) = \alpha_1, \quad y(1) = \beta_1, \quad y^{(1)}(0) = \alpha_2, \quad y^{(1)}(1) = \beta_2, \\ &y^{(2)}(0) = \alpha_3, \quad y^{(2)}(1) = \beta_3, \quad y^{(3)}(0) = \alpha_4, \quad y^{(3)}(1) = \beta_4. \end{aligned} \tag{11}$$

The approximate solution $y(x)$ can be derived as

$$\begin{aligned} y(x) = \alpha_1 + \alpha_2 x + \alpha_3 \tfrac{x^2}{2} + \alpha_4 \tfrac{x^3}{6} + y^{(4)}(0)\tfrac{x^4}{24} + y^{(5)}(0)\tfrac{x^5}{120} \\ + y^{(6)}(0)\tfrac{x^6}{720} + y^{(7)}(0)\tfrac{x^7}{5040} + \sum_{i=1}^{2^{J+1}} a_i p_{8,i}(x). \end{aligned} \tag{12}$$

Using boundary conditions Eq. (11), $y^{(4)}(0)$, $y^{(5)}(0)$, $y^{(6)}(0)$, $y^{(7)}(0)$ are calculated as

$$\begin{aligned} y^{(4)}(0) = & -840\alpha_1 - 480\alpha_2 - 120\alpha_3 - 16\alpha_4 \\ & +840\beta_1 - 360\beta_2 + 60\beta_3 - 4\beta_4 \\ & + \sum_{i=1}^{2^{J+1}} a_i(-840E_{8,i} + 360E_{7,i} - 60E_{6,i} + 4E_{5,i}), \end{aligned} \tag{13}$$

$$\begin{aligned} y^{(5)}(0) = & \ 10080\alpha_1 + 5400\alpha_2 + 1200\alpha_3 + \\ & 120\alpha_4 - 10080\beta_1 + 4680\beta_2 - 840\beta_3 + 60\beta_4 \\ & + \sum_{i=1}^{2^{J+1}} a_i(10080E_{8,i} - 4680E_{7,i} + 840E_{6,i} - 60E_{5,i}), \end{aligned} \tag{14}$$

$$\begin{aligned} y^{(6)}(0) = & -50400\alpha_1 - 25920\alpha_2 - 5400\alpha_3 - 480\alpha_4 \\ & +50400\beta_1 - 24480\beta_2 + 4680\beta_3 - 360\beta_4 \\ & + \sum_{i=1}^{2^{J+1}} a_i(-50400E_{8,i} + 24480E_{7,i} - 4680E_{6,i} + 360E_{5,i}), \end{aligned} \tag{15}$$

$$\begin{aligned} y^{(7)}(0) = & \ 100800\alpha_1 + 50400\alpha_2 + 10080\alpha_3 + 840\alpha_4 \\ & -100800\beta_1 + 50400\beta_2 - 10080\beta_3 + 840\beta_4 \\ & + \sum_{i=1}^{2^{J+1}} a_i(100800E_{8,i} - 50400E_{7,i} + 10080E_{6,i} - 840E_{5,i}), \end{aligned} \tag{16}$$

where

$$\begin{aligned} E_{5,i} &= \int_0^1 p_{5,i}(x)dx, \qquad E_{6,i} = \int_0^1 p_{6,i}(x)dx, \\ E_{7,i} &= \int_0^1 p_{7,i}(x)dx, \qquad E_{8,i} = \int_0^1 p_{8,i}(x)dx. \end{aligned} \tag{17}$$

4 Covergence analysis of Haar wavelet discretization method(HWDM)

The accuracy issues of the HWDM, open from year 1997 were clarified by Majak et al. [16] in 2015. The following results are due to notations introduced by Majak et al. [17]. Consider the following general form of ODE:

$$f(x, y, y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}, y^{(5)}, y^{(6)}, y^{(7)}, y^{(8)}) = 0. \tag{18}$$

Expansion of the eighth order derivative in terms of Haar wavelet series can be written as:

$$\frac{d^8y(x)}{dx^8} = \sum_{i=1}^{\infty} a_i h_i(x) = a_1 h_1(x) + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} a_{2^j+k+1}\ h_{2^j+k+1}(x). \tag{19}$$

In Eq. (19) $2^j + k + 1 = i, \quad k = 0, 1, \ldots, 2^j - 1$. Integrating Eq. (19) eight times from 0 to x we obtain the solution of ODE (18) as

$$y(x) = \frac{a_1}{8!} + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} a_{2^j+k+1}\ p_{8,2^j+k+1}(x) + B(x). \tag{20}$$

Here $p_{8,2^j+k+1}(x)$ is the eighth order integrals of the Haar functions found by using Eq. (9) and $B(x)$ is a boundary term.

Let us assume that $\frac{d^8y}{dx^8} \in L^2(R)$ is continuous and its next derivative is bounded on $[0,1]$, *i.e.* $\exists\, \rho > 0$ such that $\left|\frac{d^9y}{dx^9}\right| \leq \rho$.
Let

$$y_{2^{J+1}}(x) = \frac{a_1}{8!} + \sum_{j=0}^{J}\sum_{k=0}^{2^j-1} a_{2^j+k+1}\; p_{8,2^j+k+1}(x) + B(x),$$

be the approximation to the solution y. The absolute error at the J^{th} resolution is denoted as $\left|E_{2^{J+1}}\right|$ and given by

$$\left|E_{2^{J+1}}\right| = \left|y(x) - y_{2^{J+1}}(x)\right| = \left|\sum_{j=J+1}^{\infty}\sum_{k=0}^{2^j-1} a_{2^j+k+1} \quad p_{8,2^j+k+1}(x)\right|. \tag{21}$$

Norm of the error in Hilbert space $L^2(R)$ [16] is defined as

$$\begin{aligned}\|E_{2^{J+1}}\|_2^2 &= \int_0^1 \Big(\sum_{j=J+1}^{\infty}\sum_{k=0}^{2^j-1} a_{2^j+k+1}\; p_{8,2^j+k+1}(x)\Big)^2 dx \\ &= \sum_{j=J+1}^{\infty}\sum_{k=0}^{2^j-1}\sum_{r=J+1}^{\infty}\sum_{s=0}^{2^r-1} a_{2^j+k+1}a_{2^r+s+1} \\ &\qquad\qquad \times \int_0^1 p_{8,2^j+k+1}(x)\; p_{8,2^r+s+1}(x)dx.\end{aligned} \tag{22}$$

Majak et al. [16] have shown that $|a_{2^j+k+1}| \leq \frac{\rho}{2^{j+1}}$, for $k = 0, 1, \ldots, 2^j - 1$ and $p_{8,i}(x)$ are monotonically increasing on $[0,1)$. Therefore,

$$\begin{aligned}\|E_{2^{J+1}}\|_2^2 \leq \frac{\rho^2}{4} &\sum_{j=J+1}^{\infty}\sum_{k=0}^{2^j-1}\sum_{r=J+1}^{\infty}\sum_{s=0}^{2^r-1} \frac{1}{2^j}\frac{1}{2^r}\left[\frac{1}{720}\left(\frac{1}{2^{j+1}}\right)^2\right. \\ &\left.+\tfrac{1}{288}\left(\tfrac{1}{2^{j+1}}\right)^4 + \tfrac{1}{720}\left(\tfrac{1}{2^{j+1}}\right)^6 + \tfrac{1}{20160}\left(\tfrac{1}{2^{j+1}}\right)^8\right]\left[\tfrac{1}{720}\left(\tfrac{1}{2^{r+1}}\right)^2\right. \\ &\left.+\tfrac{1}{288}\left(\tfrac{1}{2^{r+1}}\right)^4 + \tfrac{1}{720}\left(\tfrac{1}{2^{r+1}}\right)^6 + \tfrac{1}{20160}\left(\tfrac{1}{2^{r+1}}\right)^8\right].\end{aligned} \tag{23}$$

Above inequality can be simplified as

$$\begin{aligned}\|E_{2^{J+1}}\|_2 \leq \tfrac{\eta}{4320}&\left[\left(\tfrac{1}{2^{J+1}}\right)^2 + \tfrac{1}{2}\left(\tfrac{1}{2^{J+1}}\right)^4 + \tfrac{1}{21}\left(\tfrac{1}{2^{J+1}}\right)^6\right. \\ &\left.+\tfrac{1}{2380}\left(\tfrac{1}{2^{J+1}}\right)^8\right].\end{aligned} \tag{24}$$

Therefore,

$$\|E_{2^{J+1}}\|_2 = O\Big[\Big(\frac{1}{2^{J+1}}\Big)^2\Big]. \tag{25}$$

From Eq. (25), we can conclude that the convergence is of order two.

5 Numerical studies

In this section we consider five test problems whose exact solutions are known arising in fluid mechanics, beam and long wave theory, applied mathematics [1, 3, 12, 13, 21]. The Haar solution and exact solution are represented in graphs that are compared with the QBCM and GM.

Example 1. Consider the following linear boundary value problem

$$y^{(8)}(x) = y(x) - 8e^x, \quad x \in (0,1), \tag{26}$$

with boundary conditions:

$$\begin{array}{l} y(0)=1, \;\; y(1)=0, \;\; y^{(1)}(0)=0, \;\; y^{(1)}(1)=-e, \\ y^{(2)}(0)=-1, \;\; y^{(2)}(1)=-2e, \;\; y^{(3)}(0)=-2, \;\; y^{(3)}(1)=-3e. \end{array} \tag{27}$$

Exact solution of this problem is $(1-x)e^x$. In Figure 1 , the comparison of exact and approximate solution for $J=3$ is represented. The comparison of absolute errors of the HWCM for $J=1$, with QBCM [12] is shown in Table 1.

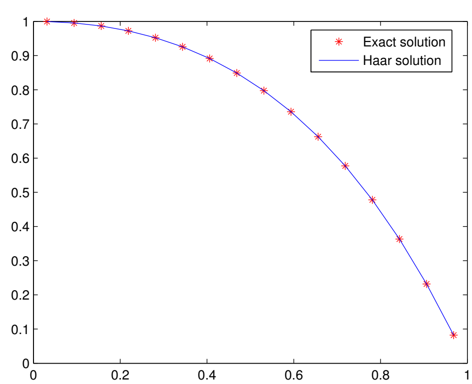

Figure 1: Comparison of exact and approximate solutions of Example 1 for $J=3$.

Table 1: Comparison of QBCM and HWCM results for Example 1.

x	Exact solution	Approx. solution	Abs. error by HWCM	Abs. error by QBCM [12]
0.1	0.9947	0.9947	6.3E-11	5.9E-07
0.2	0.9771	0.9771	6.5E-10	6.5E-07
0.3	0.9449	0.9449	2.0E-09	1.2E-07
0.4	0.8951	0.8951	3.3E-09	9.5E-07
0.5	0.8244	0.8244	3.9E-09	2.0E-06
0.6	0.7288	0.7288	3.4E-09	3.6E-06
0.7	0.6041	0.6041	2.0E-09	5.0E-06
0.8	0.4451	0.4451	6.9E-10	4.3E-06
0.9	0.2460	0.2466	7.6E-11	2.1E-06

Example 2. The following linear boundary value problem is considered

$$y^{(8)}(x) + (48 + 15x + x^3)e^x + xy(x) = 0, \quad x \in (0, 1), \tag{28}$$

with boundary conditions:

$$\begin{array}{l} y(0) = 0, \quad y(1) = 0, \quad y^{(1)}(0) = 1, \quad y^{(1)}(1) = -e, \\ y^{(2)}(0) = 0, \quad y^{(2)}(1) = -4e, \quad y^{(3)}(0) = -3, \quad y^{(3)}(1) = -9e. \end{array} \tag{29}$$

Exact solution of this problem is $x(1-x)e^x$. In Figure 2, the comparison of exact and approximate solution for $J = 4$ is represented. Errors related to this problem obtained by HWCM for $J = 1$ are compared with QBCM [12] and GM [13] are inserted in Table 2.

Example 3. We consider the nonlinear boundary value problem

$$y^{(8)}(x) - e^{-x}y^2(x) = 0, \quad x \in (0, 1), \tag{30}$$

with boundary conditions:

$$\begin{array}{l} y(0) = 1, \quad y(1) = e, \quad y^{(1)}(0) = 1, \quad y^{(1)}(1) = e, \\ y^{(2)}(0) = 1, \quad y^{(2)}(1) = e, \quad y^{(3)}(0) = 1, \quad y^{(3)}(1) = e. \end{array} \tag{31}$$

Exact solution of this problem is e^x. The nonlinear boundary value problem Eq. (30) is converted into a sequence of linear boundary value problems with the aid of quasilinearization technique [2]. The comparison of exact and approximate solution for $J = 5$ is represented in Figure 3. Errors obtained by HWCM for $J = 1$ are compared with QBCM [12] are tabulated in Table 3.

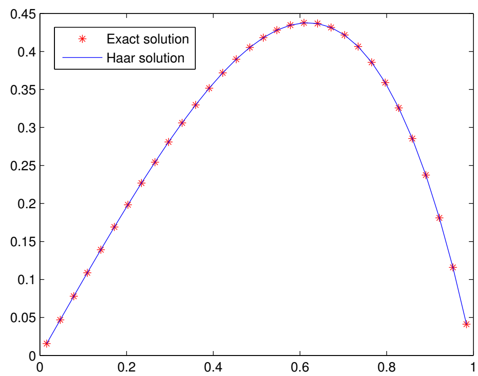

Figure 2: Comparison of exact and approximate solutions of Example 2 for $J = 4$.

Table 2: Comparison of QBCM, GM and HWCM results for Example 2.

x	Exact solution	Approx. solution	Abs. error by HWCM	Abs. error by QBCM [12]	Abs. error GM [13]
0.1	0.0995	0.0995	6.0E-10	2.4E-07	5.2E-08
0.2	0.1954	0.1954	6.2E-09	8.2E-07	2.2E-06
0.3	0.2835	0.2835	1.8E-08	1.9E-06	7.0E-06
0.4	0.3580	0.3580	3.2E-08	4.3E-06	1.1E-05
0.5	0.4122	0.4122	3.7E-08	6.2E-06	1.2E-05
0.6	0.4373	0.4373	3.2E-08	7.2E-06	8.8E-06
0.7	0.4229	0.4229	1.9E-08	7.0E-06	2.5E-06
0.8	0.3561	0.3561	6.6E-09	5.0E-06	1.8E-06
0.9	0.2214	0.2214	7.2E-10	2.4E-06	2.0E-06

Example 4. The following linear boundary value problem is considered

$$y^{(8)}(x) = y(x) - 8(2x\cos(x) + 7\sin(x)), \quad x \in (0,1), \tag{32}$$

subject to the boundary conditions:

$$\begin{aligned} &y(0) = 0, \quad y(1) = 0, \quad y^{(1)}(0) = 0, \quad y^{(1)}(1) = 2sin(1), \\ &y^{(2)}(0) = 0, \quad y^{(2)}(1) = 4\cos(1) + 2\sin(1), \quad y^{(3)}(0) = 7, \\ &y^{(3)}(1) = 6\cos(1) - 6\sin(1). \end{aligned} \tag{33}$$

Analytic solution of this problem is $(x^2 - 1)sin(x)$. In Figure 4, exact and approximate solutions for $J = 3$ is shown. The comparison of exact and

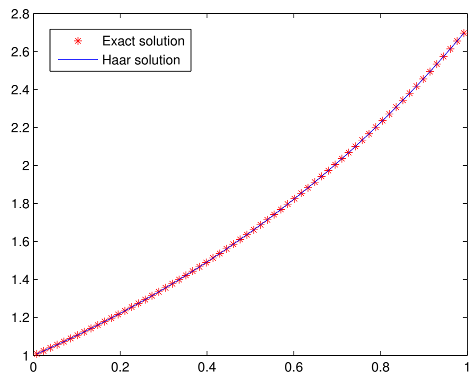

Figure 3: Comparison of exact and approximate solutions of Example 3 for $J = 5$.

Table 3: Comparison of QBCM, GM and HWCM results for Example 3.

x	Exact solution	Approx. solution	Abs. error by HWCM	Abs. error by QBCM [12]
0.1	1.1052	1.1052	6.6E-12	1.2E-07
0.2	1.2214	1.2214	6.9E-11	1.2E-05
0.3	1.3499	1.3499	2.1E-10	4.5E-05
0.4	1.4918	1.4918	3.5E-10	8.2E-05
0.5	1.6487	1.6487	4.1E-10	1.0E-04
0.6	1.8221	1.8221	3.5E-10	6.6E-05
0.7	2.0138	2.0138	2.1E-10	6.6E-05
0.8	2.2255	2.2255	7.2E-11	3.1E-05
0.9	2.4596	2.4596	8.0E-12	1.2E-05

approximate solution for $J = 1$ is shown in Table 4.

Example 5. Consider the nonlinear boundary value problem

$$y^{(8)}(x) + e^{-x}y^2(x) = e^{-x} + e^{-3x}, \quad x \in (0, 1), \tag{34}$$

with the boundary conditions:

$$\begin{aligned} &y(0) = 1, \quad y(1) = \tfrac{1}{e}, \quad y^{(1)}(0) = -1, \quad y^{(1)}(1) = \tfrac{-1}{e}, \quad y^{(2)}(0) = 1, \\ &y^{(2)}(1) = \tfrac{1}{e}, \quad y^{(3)}(0) = -1, \quad y^{(3)}(1) = \tfrac{-1}{e}. \end{aligned} \tag{35}$$

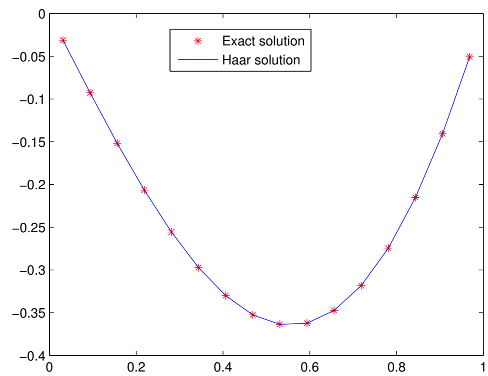

Figure 4: Comparison of exact and approximate solution of Example 4 for $J = 3$.

Table 4: Comparison of numerical results for Example 4.

x	Exact solution	Approx. solution	Abs. error
0.1	-0.0988	-0.0988	2.0E-10
0.2	-0.1907	-0.1907	2.0E-09
0.3	-0.2689	-0.2689	5.9E-09
0.4	-0.3271	-0.3271	1.0E-08
0.5	-0.3596	-0.3596	1.3E-08
0.6	-0.3614	-0.3614	1.1E-08
0.7	-0.3286	-0.3286	6.8E-09
0.8	-0.2582	-0.2582	2.4E-09
0.9	-0.1488	-0.1488	2.3E-10

Exact solution of this problem is e^{-x}. In Figure 5, comparison of exact and approximate solutions for $J = 4$ is represented. The comparison of absolute errors of HWCM for $J = 1$, with GM [13] is inserted in Table 5.

6 Conclusion

In this paper, Haar wavelet collocation method is applied to find the solution for eighth order BVPs. Nonlinear BVPs are solved with the aid of quasilinearization technique. From the convergence analysis we noticed that, the rate of convergence of proposed method is of order two. The exact

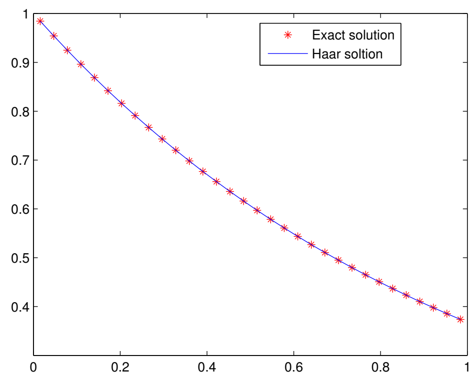

Figure 5: Comparison of exact and approximate solutions of Example 5 for $J = 4$.

Table 5: Comparison of GM and HWCM results for Example 5.

x	Exact solution	Approx. solution	Abs. error by HWCM	Abs. error by GM [13]
0.1	0.9048	0.9048	2.9E-12	3.6E-07
0.2	0.8187	0.8187	2.7E-11	6.3E-06
0.3	0.7408	0.7408	7.6E-11	1.9E-05
0.4	0.6703	0.6703	1.3E-10	3.1E-05
0.5	0.6065	0.6065	1.5E-10	3.6E-05
0.6	0.5488	0.5488	1.3E-10	3.2E-05
0.7	0.4965	0.4965	7.6E-11	1.9E-05
0.8	0.4493	0.4493	2.5E-11	7.2E-06
0.9	0.4065	0.4065	2.4E-12	1.5E-06

and Haar solution in graphs and tables(first three columns of each table) showed that they are almost the same. The comparison of absolute errors obtained by HWCM with QBCM and GM showed that we achieved better accuracy even for lower resolution $J = 1$. These observations ensured that HWCM has given accurate results for small number of grid points. Therefore HWCM is computationally efficient and results are more accurate.

Acknowledgements

Authors are thankful to anonymous reviewer for giving valuable suggestions to improve this article. First author is grateful to Vision Group on Science and Technology, Govt. of Karnataka, India for the financial support under the scheme "Seed Money to Young Scientists for Research (SMYSR-FY-2015-16/GRD:497)".

References

[1] R.P. Agarwal, *Boundary value problems for higher order differential equations*, World scientific, Singapore, 1986.

[2] R.E. Bellman and R.E. Kalaba, *Quasilinearization and nonlinear boundary value problems*, Elsevier, New York, 1965.

[3] S. Chandrasekhar, *Hydrodynamic and hydromagnetic stability*, Clarendon Press, 1961.

[4] C.F. Chen and C.H. Hsiao, *Wavelet approach to optimizing dynamic system*, IEEE Proc. Control Theory Appl. **146** (1999) 213–219.

[5] C.F. Chen and C.H. Hsiao, *Haar wavelet method for solving lumped and distributed parameter systems*, IEEE Proc. Control Theory Appl. **144** (1997) 87–94.

[6] A. Haar, *Zur theoric der orthogonalen Funktionsysteme*, Math. Annal. **69** (1991) 331–371.

[7] Fazal-i-Haq and A. Arshed, *Numerical solution of fourth order boundary value problems using Haar wavelets*, Appl. Math. Sci. **63** (2011) 3131–3146.

[8] Fazal-i-Haq, A. Arshed and H. Iltaf , *Solution of sixth-order boundary value problems by collocation method using Haar wavelets*, Appl. Math. Sci. **43** (2012) 5729–5735.

[9] G. Hariharan and K. Kannan , *An overview of Haar wavelet method for solving differential and integral equations*, Wl. Appl. Sci. J. **12** (2013) 1–14.

[10] K. Harpreet, R. C. Mittal and M. Vinod, *Haar wavelet quasilinearization approach for solving nonlinear boundary value problems*, Amer. J. Comp. Math. **1** (2011) 176–182.

[11] Siraj-ul-Islam, A. Imran and S. Bozidar, *The numerical solution of second order boundary value problems by collocation method with the Haar Wavelets*, Math. Comp. Model. **52** (2010) 1577–1590.

[12] K.N.S. Kasi Viswanadham and Y. Showri Raju, *Quintic B-spline collocation method for eighth order boundary value problems*, Adv. Comp. Math. Appl. **1** (2012) 47-52.

[13] K.N.S. Kasi Viswanadham and B. Sreenivasulu, *Numerical solution of eighth order boundary value problems by Galerkin method with Quintic B-splines*, Int. J. Comp. Appl. **89** (2014) 975–8887.

[14] U. Lepik and H. Hein, *Haar wavelets with applications*, Springer, 2014.

[15] U. Lepik, *Haar wavelet method for solving higher order differential equations*, Int. J. Math. Comput. **1** (2008) 84–94.

[16] J. Majak, B.S. Shvartsman, M. Kirs, M. Pohlak and H. Herranen, *Convergence theorem for the Haar wavelet based discretization method*, Comp. Stru. **126** (2015) 227–232.

[17] J. Majak, B. Shvartsman, K. Karjust, M. Mikola, A. Haavajoe and M. Pohlak, *On the accuracy of the Haar wavelet discretization method*, Comp. part B. **80** (2015) 321–327.

[18] A. Padmanabha Reddy, S.H. Manjula, C. Sateesha and N. M. Bujurke, *Haar wavelet approach for the solution of seventh order ordinary differential equations*, Math. Model. Eng. Prob. **3** (2016) 108–114.

[19] A. Padmanabha Reddy, C. Sateesha and S.H. Manjula, *Application of Haar wavelet collocation method to solve the fifth order ordinary differential equations*, Int. J. Math. Archi. **7** (2016) 53–62.

[20] K.D. Ruch and J.V. Patrik, *Wavelet theory:an elementary approach with applications*, John wiely and sons, New Jersey, USA, 2007.

[21] Y.I. Shen, *Hybrid damping through intelligent constrained layer layer treatments*, J. Vib. Acou. **116** (1994) 341-349.