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A numerical method for solving nonlinear partial differential equations based on Sinc-Galerkin method

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Abstract. In this paper, we consider two dimensional nonlinear elliptic equations of the form $-\mathrm{div}(a(u, \nabla u)) = f$. Then, in order to solve these equations on rectangular domains, we propose a numerical method based on Sinc-Galerkin method. Finally, the presented method is tested on some examples. Numerical results show the accuracy and reliability of the proposed method.

Keywords: Sinc-Galerkin method, elliptic partial differential equations, nonlinear problems, numerical solutions.
AMS Subject Classification: 35J60.

1 Introduction

Sinc methods have been increasingly used for finding a numerical solution of ordinary and partial differential equations. Using these methods in computational mathematics are similar to polynomials, splines, and Fourier polynomials. Each existing method excels in some particular class of problems. For example, polynomials excel in the approximation of analytic func-

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tions without singularities, splines are particularly good for approximating measured, or noisy data, while Fourier polynomials excel in the approximation of functions that are both smooth and periodic over the whole real line. Sinc methods excel for problems with singularities, for boundary-layer problems and for problems over infinite or semi-infinite ranges. They are typified by exponentially decaying errors and in special cases by optimal convergence rate [6, 15]. In addition, the run time of computer programs based on Sinc methods are usually considerably shorter than the corresponding ones based on classical methods of approximation [6, 15]. The Sinc-Galerkin method utilizes a modified Galerkin scheme to discretize ordinary and partial differential equations. The basis elements used in this approach are Sinc functions composed with a suitable conformal map [15]. The books [6] and [15] provide overviews of existing methods based on Sinc functions for solving ordinary differential equations (ODEs), partial differential equations (PDEs), and integral equations [9]. These methods have also been employed for some inverse problems [7, 10, 14].

The main aim of this paper is to use Sinc-Galerkin method to find an approximate solution of the following nonlinear elliptic problem

$$\begin{aligned} \aleph u(x,y) = -\operatorname{div}\left(a\left(u,\nabla u\right)\right) &= f(x,y), \qquad (x,y)\in\Omega, \\ u(x,y) &= 0, \qquad\qquad (x,y)\in\partial\Omega, \end{aligned} \tag{1}$$

where $a(u,\nabla u) = \overrightarrow{\nabla} u + \overrightarrow{F}(u)$ and $\Omega := \{(x,y)\,|\,0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1\}$. The method consists of reducing the problem (1) to a system of nonlinear equations by approximating $u = u(x,y)$ based on Sinc functions with unknown coefficients. In this approach, no mesh generation is required. Moreover, due to the well-posedness of the problem (1) and rapid convergence rate of Sinc-Galerkin method, the method does not suffer from the usual instability issues that typically occur in different methods (see [6,15]). In other words, we use a stabilized, mesh-free method to solve these kinds of nonlinear elliptic problems. As a special case, in finite element method, if we need a solution in $H^2(\Omega)\cap C^1(\Omega)$, we have to choose the standard full quintic finite element approximation for each triangle. Consequently, we obtain 21 unknown coefficients for the polynomials of degree 5 for each triangle. Although this method is convergent, we have to solve a large linear system of equations. But the proposed method without mesh generation provides an approximate solution which is analytic.

The paper is organized as follows. Section 2 is devoted to some existence and uniqueness theorems related to nonlinear elliptic problems. In Section 3, the basic formulation of the Sinc function required for our subsequent development is provided and Sinc-Galerkin method is used to approximate the solution of nonlinear elliptic problems. In Section 4, parameter selections are given to guarantee the exponential convergence. Finally, in Section 5, some numerical examples are given to show the efficiency and accuracy of the proposed numerical scheme.

2 Existence and uniqueness theorems

In this section, some existence and uniqueness theorems are given related to nonlinear elliptic partial differential equations. To do so, we consider the nonlinear elliptic problem (1) in a general form

$$\begin{aligned} -\operatorname{div}(a(\mathbf{x},u,\nabla u)) &= f(\mathbf{x}), \qquad \mathbf{x} \text{ in } \Omega, \\ u(\mathbf{x}) &= 0, \qquad \mathbf{x} \text{ on } \partial\Omega, \end{aligned} \tag{2}$$

where Ω is a bounded open set in $\mathbb{R}^N$, $N \geq 2$, $f \in L^1(\Omega)$ and $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Caratheodory function such that

$$a(\mathbf{x},s,\xi) \geqslant \alpha|\xi|^p, \qquad \alpha > 0, \tag{3}$$

$$|a(\mathbf{x},s,\xi)| \leqslant \left[|\xi|^{p-1} + |s|^{p-1} + a_0(\mathbf{x})\right], \quad a_0(\mathbf{x}) \in L^{p'}(\Omega), \tag{4}$$

$$(a(\mathbf{x},s,\xi) - a(\mathbf{x},s,\eta), \xi - \eta) > 0, \qquad \xi \neq \eta, \tag{5}$$

a.e. $\mathbf{x} \in \Omega$, $\forall\, s \in \mathbb{R}$, $\forall\, \xi, \eta \in \mathbb{R}^N$. The existence and uniqueness of solution to nonlinear elliptic problem (2) can be analyzed in different methods such as weak solution methods, duality methods, entropy solution and renormalized solution [1,4]. In this paper, we only consider entropy solution and renormalized solution.

In order to define the entropy solution, first the truncature operator $T_k(s)$ for a given constant $k > 0$ is introduced [4]

$$T_k(s) = \begin{cases} s, & \text{if } \ |s| \leqslant k, \\ k \ \operatorname{sign}(s), & \text{if } \ |s| > k. \end{cases}$$

A measurable function $u : \Omega \to \mathbb{R}$ satisfying the condition $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k > 0$ is an entropy solution for (2) if we have

$$\int_\Omega a(\mathbf{x}, \nabla u) \nabla T_k(u - \varphi) d\mathbf{x} \leqslant \int_\Omega T_k(u - \varphi) f d\mathbf{x}.$$

Theorem 1. *[2,4] Let $f \in L^1(\Omega)$ and assume that the conditions (3)-(5) are satisfied. Then there exists a unique entropy solution to problem (2).*

An equivalent notion of solution for (2) is so-called the renormalized solution. A function u is a renormalized solution if it satisfies the following conditions

(1) u is a measurable function, almost everywhere finite in Ω,

(2) $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$, for every k and $p > 1$,

(3)
$$\frac{1}{n}\lim_{n\to\infty}\int_{\{n\leqslant|u|\leqslant 2n\}} a(\mathbf{x},\nabla u).\nabla u d\mathbf{x} = 0,$$

(4) $\displaystyle\int_{\Omega} h(u)a(\mathbf{x},\nabla u)\nabla u d\mathbf{x} + \int_{\Omega} h'(u)a(\mathbf{x},\nabla u)\nabla u\ v d\mathbf{x} = \int_{\Omega} f\, h(u) v d\mathbf{x},$

for every $h \in W^{1,\infty}(\mathbb{R})$ with compact support in $\mathbb{R}$ and $v \in W_0^{1,p}(\Omega) \cap L^{\infty}(\Omega)$.

Theorem 2. *[3, 4] Let $f \in L^1(\Omega)$ and assume that the conditions (3)-(5) are satisfied. Then there exists a unique renormalized solution to problem (2).*

3 Mathematical formulations

3.1 Sinc function

The Sinc function is defined on the whole real line $-\infty < x < \infty$ by

$$Sinc(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

For h_x and h_y, the translated Sinc functions with evenly spaced nodes for space variables are given as

$$\begin{aligned} S(k,h_x)(x) &= Sinc(\frac{x-kh_x}{h_x}), \qquad k = 0,\pm1,\pm2,\ldots, \\ S(k,h_y)(y) &= Sinc(\frac{y-kh_y}{h_y}), \qquad k = 0,\pm1,\pm2,\ldots. \end{aligned}$$

In order to construct approximations on the interval $(0,1)$, which are used in this paper, we should apply the conformal map $\varphi(z) = \ln\left(\frac{z}{1-z}\right)$. In other words, the compositions $S_k(x) = S(k,h_x)o\varphi(x)$ and $S_k(y) = S(k,h_y)o\varphi(y)$ define the basis elements on the interval $(0,1)$, (see [6, 12, 13, 15]). Consequently, using tensor product, two dimensional Sinc basis functions are considered as follows

$$S_{k,\ell}(x,y) \equiv \{S(k,h_x)o\varphi(x)\}\{S(\ell,h_y)o\varphi(x)\} = S_k(x)S_\ell(y).$$

In what follows, we apply these functions as a basis function in Sinc-Galerkin method.

3.2 The Sinc-Galerkin method

Let

$$u_{m_x m_y}(x,y) = \sum_{\ell=-M_y}^{N_y} \sum_{k=-M_x}^{N_x} u(x_k,y_\ell) S_{k\ell}(x,y), \tag{6}$$

be an approximate solution of (1) in which $m_x = M_x + N_x + 1$, $m_y = M_y + N_y + 1$, $x_k = \varphi^{-1}(kh_x)$ and $y_\ell = \varphi^{-1}(\ell h_y)$. The unknown coefficients $u_{k\ell} = u(x_k, y_\ell)$, $k = -M_x, \ldots, N_x$, $\ell = -M_y, \ldots, N_y$ in (6) are determined by orthogonalizing the residual with respect to the functions $S_{k\ell}$, [15, 16]. This yields the discrete system

$$\left\langle \aleph u_{m_x m_y}, S_{k\ell} \right\rangle = \langle f, S_{k\ell} \rangle, \quad k = -M_x, \ldots N_x, \quad \ell = -M_y, \ldots N_y,$$

where the weighted inner product is defined by

$$\langle f, g \rangle = \int_0^1 \int_0^1 f(x,y) g(x,y) v(x) w(y) dx dy,$$

in which $v(x)w(y)$ is a product weight function. Consequently, we will have

$$\begin{aligned}
I &= \left\langle -\mathrm{div}\left(a\left(x,y,u_{m_x m_y}, \nabla u_{m_x m_y}\right)\right), S_{k\ell}\right\rangle \\
&= \int_0^1 \int_0^1 -\mathrm{div}\left(a\left(x,y,u_{m_x m_y}, \nabla u_{m_x m_y}\right)\right) S_{k\ell}(x,y) v(x) w(y) dx dy \\
&= \int_0^1 \int_0^1 \left(-\frac{\partial^2 u_{m_x m_y}(x,y)}{\partial x^2} - \frac{\partial^2 u_{m_x m_y}(x,y)}{\partial y^2} - \frac{\partial F_1\left(u_{m_x m_y}(x,y)\right)}{\partial x} \right. \\
&\quad \left. - \frac{\partial F_2\left(u_{m_x m_y}(x,y)\right)}{\partial y} \right) S_{k\ell}(x,y) v(x) w(y) dx dy,
\end{aligned}$$

where $\overrightarrow{F(u)} = (F_1(u), F_2(u))^T$. Using integrating by parts, all derivatives transfer from $u_{m_x m_y}$ to $S_{k\ell}$. Then by choosing $v(x) = \frac{1}{\varphi'(x)}$ and $w(y) =$

$\frac{1}{\varphi'(y)}$, we conclude that

$$\begin{aligned} I = & \int_0^1 \int_0^1 -u_{m_x m_y}(x,y) \frac{\partial^2}{\partial x^2} \left(S_k(x) v(x) \right) S_\ell(y) w(y) dx dy \\ & + \int_0^1 \int_0^1 -u_{m_x m_y}(x,y) \frac{\partial^2}{\partial y^2} \left(S_\ell(y) w(y) \right) S_k(x) v(x) dx dy \\ & + \int_0^1 \int_0^1 F_1(u_{m_x m_y}) \frac{\partial}{\partial x} \left(S_k(x) v(x) \right) S_\ell(y) w(y) dx dy \\ & + \int_0^1 \int_0^1 F_2(u_{m_x m_y}) \frac{\partial}{\partial y} \left(S_\ell(y) w(y) \right) S_k(x) v(x) dx dy. \end{aligned}$$

In the Sinc-Galerkin method, in order to approximate integrals [6], the following Sinc quadrature rules are used

$$\int_0^1 u(x) dx \simeq h_x \sum_{p=-M_x}^{N_x} \frac{u(x_p)}{\varphi'(x_p)},$$

and

$$\int_0^1 \int_0^1 u(x,y) dx dy \simeq h_x h_y \sum_{q=-M_y}^{N_y} \sum_{p=-M_x}^{N_x} \frac{u(x_p, y_q)}{\varphi'(x_p)\varphi'(y_q)}.$$

This yields

$$\begin{aligned} I \simeq h_x h_y \sum_{q=-M_y}^{N_y} \sum_{p=-M_x}^{N_x} \Bigg(& u_{m_x m_y}(x_p, y_q) \frac{-1}{\varphi'(x_p)} \frac{\partial^2}{\partial x^2} \left(S_k(x) v(x) \right) \big|_{x=x_p} \\ & \times \frac{1}{\varphi'(y_q)} S_\ell(y_q) w(y_q) \\ & + u_{m_x m_y}(x_p, y_q) \frac{-1}{\varphi'(y_q)} \frac{\partial^2}{\partial x^2} \left(S_\ell(y) w(y) \right) \big|_{y=y_q} \frac{1}{\varphi'(x_p)} S_k(x_p) v(x_p) \\ & + F_1 \left(u_{m_x m_y}(x_p, y_q) \right) \frac{-1}{\varphi'(x_p)} \frac{\partial}{\partial x} \left(S_k(x) v(x) \right) \big|_{x=x_p} \frac{1}{\varphi'(y_q)} S_\ell(y_q) w(y_q) \\ & + F_2 \left(u_{m_x m_y}(x_p, y_q) \right) \frac{-1}{\varphi'(y_q)} \frac{\partial}{\partial x} \left(S_\ell(y) w(y) \right) \big|_{y=y_q} \frac{1}{\varphi'(x_p)} S_k(x_p) v(x_p) \Bigg), \end{aligned}$$

and

$$\begin{aligned}\langle f, S_{k\ell}\rangle &= \int_0^1 \int_0^1 f(x,y) S_k(x) S_\ell(y) v(x) w(y) dx dy \\ &\simeq h_x h_y \sum_{q=-M_y}^{N_y} \sum_{p=-M_x}^{N_x} \frac{\delta_{kp}^{(0)} \delta_{\ell q}^{(0)} f(x_p, y_q) v(x_p) w(y_q)}{\varphi'(x_p)\varphi'(y_q)} \\ &= \frac{f(x_k, y_\ell) v(x_k) w(y_\ell)}{\varphi'(x_k)\varphi'(y_\ell)},\end{aligned}$$

where $\delta_{kp}^{(0)} = S_k(x_p)$ and $\delta_{\ell q}^{(0)} = S_\ell(y_q)$. Now, we have a nonlinear system of $m_x \times m_y$ equations of the $m_x \times m_y$ unknown coefficients $u_{k\ell}$. These coefficients are obtained by using Newton's method or many other different methods such as, conjugate gradient method, genetic algorithms, Steffensen's methods and so on (see [5,11]).

In particular, if we consider a linear partial differential equation of the form

$$\begin{aligned}&\mathbf{L}u(x,y) = f(x,y), \quad (x,y) \in \Omega = (0,1) \times (0,1), \\ &u(x,y) = 0, \qquad\qquad (x,y) \in \partial\Omega,\end{aligned}$$

where

$$\begin{aligned}\mathbf{L}u(x,y) &= a_1(x) b_1(y) \frac{\partial^2 u(x,y)}{\partial x^2} + a_2(x) b_2(y) \frac{\partial^2 u(x,y)}{\partial x \partial y} \\ &\quad + a_3(x) b_3(y) \frac{\partial^2 u(x,y)}{\partial y^2} + a_4(x) b_4(y) \frac{\partial u(x,y)}{\partial x} \\ &\quad + a_5(x) b_5(y) \frac{\partial u(x,y)}{\partial y} + a_6(x) b_6(y) u(x,y),\end{aligned}$$

we will get the following system of linear algebraic equations

$$A_1 U B_1^T + A_2 U B_2^T + \ldots + A_6 U B_6^T = G, \tag{7}$$

such that

$$\begin{aligned}U &= [U_{p,q}]_{m_x \times m_y}, \qquad A_n = \left[(A_n)_{k,p}\right]_{m_x \times m_x}, \\ B_n &= \left[(B_n)_{\ell,q}\right]_{m_y \times m_y}, \qquad G = [G_{k,\ell}]_{m_x \times m_y},\end{aligned}$$

and for $p, k = -M_x, \ldots, N_x$ and $q, \ell = -M_y, \ldots, N_y$,

$$U_{p,q} = u_{m_x m_y}(x_p, y_q),$$

Table 1: Values of s_n and t_n.

n	1	2	3	4	5	6
s_n	2	1	0	1	0	0
t_n	0	1	2	0	1	1

$$\begin{aligned}(A_n)_{k,p} &= \frac{(-1)^{s_n}}{\varphi'(x_p)}\frac{\partial^{s_n}}{\partial x^{s_n}}\left(a_n(x)S_k(x)v(x)\right)\big|_{x=x_p}\,,\\ (B_n)_{\ell,q} &= \frac{(-1)^{t_n}}{\varphi'(y_q)}\frac{\partial^{t_n}}{\partial y^{t_n}}\left(b_n(y)S_\ell(y)w(y)\right)\big|_{y=y_q}\,,\\ G_{k,\ell} &= \frac{f(x_k,y_\ell)v(x_k)w(y_\ell)}{\varphi'(x_k)\varphi'(y_\ell)},\end{aligned}$$

in which s_n and t_n are set to be 0, 1 and 2 according to Table 1. To solve (7), we refer to [16] which has proposed a computational method based on Kronecker product and vec-function.

4 Parameter selections for the Sinc-Galerkin method

The step sizes h_x, and h_y, and summation limits M_x, N_x, M_y, and N_y, are selected so that guarantee the exponential convergence [8, 15, 16]. Due to this, if the exact solution satisfies the condition

$$|u(x,y)| \leqslant Cx^{\alpha_s+\frac{1}{2}}(1-x)^{\beta_s+\frac{1}{2}}y^{\zeta_s+\frac{1}{2}}(1-y)^{\eta_s+\frac{1}{2}},$$

we should make the following selections

$$h_x = h_y = \sqrt{\frac{\pi d}{\alpha_s M_x}},$$

and

$$N_x = \left[\left|\alpha_s\frac{M_x}{\beta_s}+1\right|\right], \quad M_y = \left[\left|\alpha_s\frac{M_x}{\zeta_s}+1\right|\right], \quad N_y = \left[\left|\alpha_s\frac{M_x}{\eta_s}+1\right|\right].$$

Here $[|\,.\,|]$ denotes the greatest integer operation and C is a positive constant. Hence, we will have

$$\left\|u-u_{m_x m_y}\right\|_\infty \leqslant C_s M_x^2 \exp\left(-\sqrt{\pi d\alpha_s M_x}\right),$$

where C_s is a constant depending on u, p, and d.

5 Numerical results

In this section, two nonlinear problems are tested by using the Sinc-Galerkin method. All the experiments are performed in MATLAB by a system with this specification: Intel(R) Core(TM) i5-2430M CPU @ 2.40GHz.

Example 1. Consider the nonlinear elliptic problem of the form

$$\begin{aligned} -u_{xx} - u_{yy} + u^2 &= f(x,y), \quad (x,y) \in \Omega = (0,1) \times (0,1), \\ u(x,y) &= 0, \quad (x,y) \in \partial\Omega, \end{aligned} \tag{8}$$

where

$$\begin{aligned} f(x,y) = & -2x^2(1-x)^2(1-6y+6y^2) - 2y^2(1-y)^2(1-6x+6x^2) \\ & + x^4y^4(1-x)^4(1-y)^4. \end{aligned}$$

The exact solution is $u(x,y) = x^2y^2(1-x)^2(1-y)^2$. Using Sinc-Galerkin method, the problem (8) is converted into a system of nonlinear equations. To find the solution of these equations, Newton's method is used. In Table 2, the absolute errors between the exact and approximate solutions are given for

$$h_x = h_y = \frac{\pi}{\sqrt{3M_x}},$$

and different values of $M = M_x = N_x = M_y = N_y$ $(M = 2, 3, 6, 8, 16)$. In addition, run time (CPU time) and the maximum of condition numbers of Jacobian matrices appeared in Newton's iterations are provided in Table 2. Also, Figures 1 and 2 show the exact and approximate solutions of the problem (8) for $M = 16$ and Figure 3 demonstrates the plot of these solutions for $y = 0.5$ and $M = 2, 3, 6, 8, 16$. As we observe in Figure 3, the solution of nonlinear system of equations, which is obtained from Newton's method, does not influence the convergence rate of Sinc-Galerkin method, since the Jacobian matrices appeared in Newton's iterations are well-conditioned.

Note that, for $M = 16$ we have a nonlinear system of $33 \times 33 = 1089$ equations of the $33 \times 33 = 1089$ unknown coefficients. Run time and the condition number show the efficiency and accuracy of the method.

Example 2. For second example we consider the following problem

$$\begin{aligned} -u_{xx} - u_{yy} + \sin(u) = f(x,y), & \quad (x,y) \in \Omega = (0,1) \times (0,1), \\ u(x,y) = 0, & \quad (x,y) \in \partial\Omega, \end{aligned} \tag{9}$$

where

$$f(x,y) = -\frac{x\ln(x)}{y} - \frac{y\ln(y)}{x} + \sin(xy\ln(x)\ln(y)).$$

Table 2: Numerical results of Sinc-Galerkin method for Example 1.

$M_x = N_x$	$M_y = N_y$	h_x	$\|ES(h_s)(x,y)\|$	Condition number	Run Time (Sec)
2	2	1.2825	$5.0852e-05$	13.0	0.33
3	3	1.0472	$1.4804e-05$	24.9	0.64
6	6	0.7405	$2.2436e-06$	98.2	2.99
8	8	0.6413	$6.8924e-07$	191.1	5.06
16	16	0.4534	$1.3993e-08$	1203.9	23.09

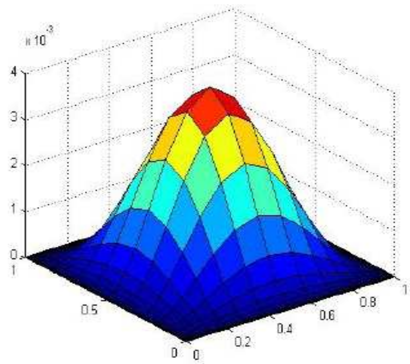

Figure 1: The exact solution $u(x,y) = x^2y^2(1-x)^2(1-y)^2$.

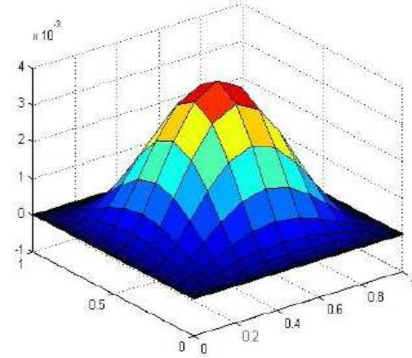

Figure 2: The approximate solution $u_{m_x m_y}(x,y)$ with $M = 16$.

The exact solution is $u(x,y) = xy\ln(x)\ln(y)$. The numerical results are reported in Table 3 for

$$h_x = h_y = \frac{\pi}{\sqrt{M_x}},$$

and different values of $M = M_x = N_x = M_y = N_y$ $(M = 2,3,6,8,16))$, that are the absolute error of exact and approximate solution, condition number of Jacobian matrix and run time (CPU time). Also, Figures 4 and 5 show the exact and approximate solutions of the problem (9) for $M = 16$ and Figure 6 demonstrates the plot of these solutions for $y = 0.5$ and $M = 2,3,6,8,16$. Similar to Example 1, the solution of nonlinear system of equations, which is obtained from Newton's method, does not influence the convergence rate of Sinc-Galerkin method.

6 Conclusion

In this paper, using Sinc-Galerkin method an approximate solution is derived for a two dimensional nonlinear elliptic problem on a rectangular

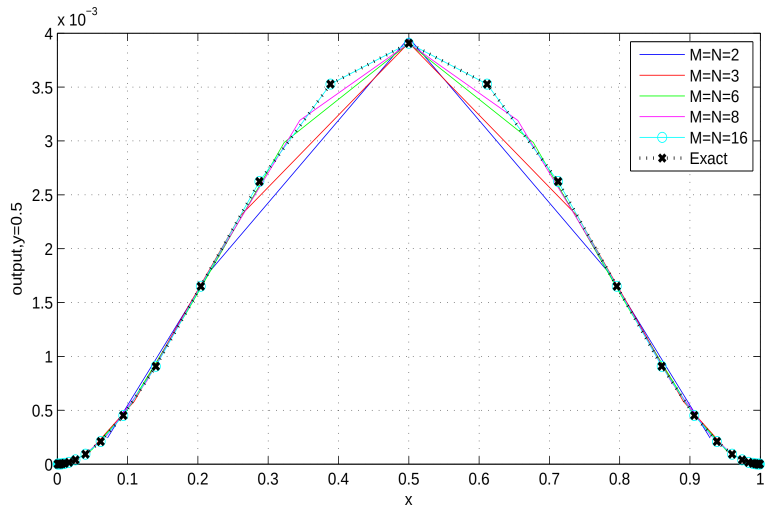

Figure 3: The approximate solution $u_{m_x m_y}(x, 0.5)$ and exact solution $u(x, 0.5)$ for $M = 2, 3, 6, 8, 16$.

Table 3: Numerical results of Sinc-Galerkin method for Example 2.

$M_x = N_x$	$M_y = N_y$	h_x	$\|ES(h_s)(x, y)\|$	Condition number	Run Time (Sec)
2	2	2.2214	$3.9000e-03$	11.7	0.28
3	3	1.8138	$2.3000e-03$	26.6	0.41
6	6	1.2825	$4.6597e-04$	158.7	2.16
8	8	1.1107	$1.8299e-04$	387.8	3.72
16	16	0.7854	$8.5640e-06$	5137	21.83

domain. The features of Sinc-Galerkin method, i.e., being mesh free and stability, provide us a suitable solutions with short run time on the rectangular domain. Numerical examples revealed the efficiency and accuracy of the proposed method. The method can also be extended to solve some inverse problems, since Sinc methods can be employed as forward solvers in the solution of inverse problems.

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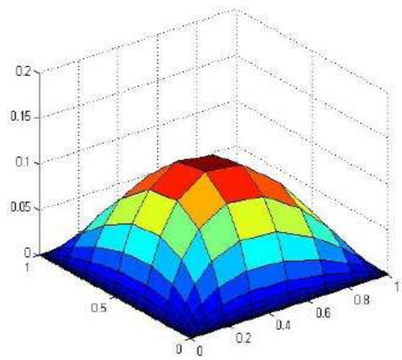

Figure 4: The exact solution $u(x,y) = xy\ln(x)\ln(y)$.

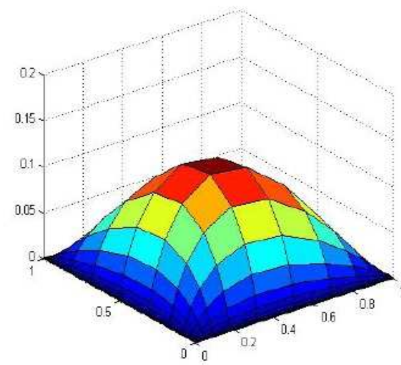

Figure 5: The approximate solution $u_{m_x m_y}(x,y)$ with $M = 16$.

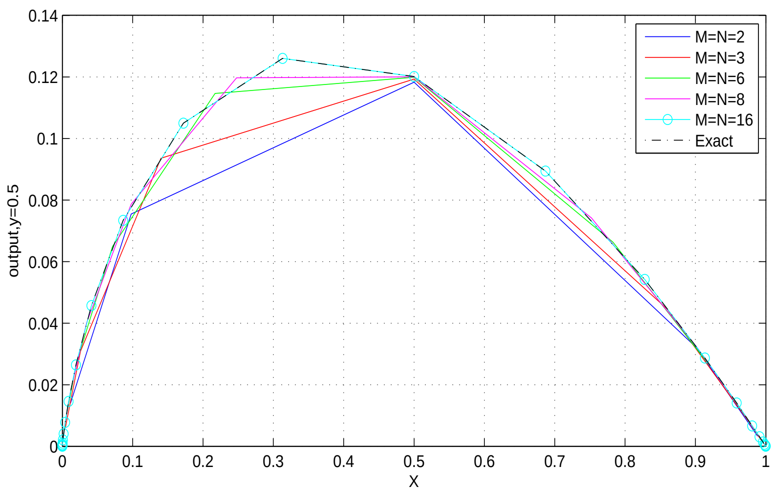

Figure 6: The approximate solution $u_{m_x m_y}(x, 0.5)$ and exact solution $u(x, 0.5)$ for $M = 2, 3, 6, 8, 16$.

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