Journal of Mathematical Modeling
Vol. 6, No. 2, 2016, pp. 211-232

JMM

Numerical method for a system of second order singularly perturbed turning point problems

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Abstract. In this paper, a parameter uniform numerical method based on Shishkin mesh is suggested to solve a system of second order singularly perturbed differential equations with a turning point exhibiting boundary layers. It is assumed that both equations have a turning point at the same point. An appropriate piecewise uniform mesh is considered and a classical finite difference scheme is applied on this mesh. An error estimate is derived by using supremum norm which is $O(N^{-1}(\ln N)^2)$. Numerical examples are given to validate theoretical results.

Keywords: singularly perturbed turning point problems, boundary value problems, finite difference scheme, Shishkin mesh and parameter uniform.
AMS Subject Classification: 65L10, 65L11, 65L12, 65L20.

1 Introduction

Singular Perturbation Problems (SPPs) (differential equations with small positive parameter ε multiplying the highest derivatives) arise in many branches of applied mathematics like fluid flow problems involving high Reynolds number, mathematical models of liquid crystal materials and chemical reactions, control theory, electrical networks etc. [4, 8]. The presence of small parameter in these problems prevents us from obtaining

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Received: 19 May 2016 / Revised: 8 July 2016 / Accepted: 9 November 2016.

parameter uniform numerical solutions. Therefore we seek a numerical method, which is uniformly convergent with respect to the parameter. We refer to [8, 10, 18, 24, 26] for detailed review of SPPs.

Most of the numerical methods are used to find the numerical solutions for singularly perturbed differential equations of second order. Only few authors have considered higher order and system of equations. Robust parameter uniform numerical methods for a system of singularly perturbed ordinary differential equations have been examined by a few authors [3, 7, 15–17, 20, 28]. In [7], a parameter uniform numerical method for a system of coupled singularly perturbed convection diffusion equations is presented. In [15] and [16, 17], a system of reaction-convection-diffusion type problem is discussed. In [25], the authors proposed a finite difference method for singularly perturbed linear reaction-diffusion system with discontinuous source terms and they assumed that the singular perturbation parameters are distinct. In [28], a numerical method for singularly perturbed weakly coupled system of two second order ordinary differential equations with discontinuous source term is considered. In [29], the authors proposed an asymptotic initial value method for a system of singularly perturbed second order ordinary differential equations. Approximation of derivative in a system of singularly perturbed convection-diffusion equations is discussed in [20].

Singularly Perturbed Turning Point Problems(SPTPPs) arise in various fields of applied mathematics like one dimensional version of stationary convection diffusion problems with a dominant convective term, speed field that changes its sign in the catch basin, geophysics and modeling thermal boundary layers in laminar flow [30].

In the past few years, few authors applied numerical methods for singularly perturbed second order ordinary differential equations with turning points. For example, Wasow [30], O. Malley [24], Watts [31] and Roos et.al [26] studied the qualitative aspects of turning point problems. Abrahammson [1] derived a priori estimates for the solution and its derivatives of SPPs with a turning point. Farrell [9] and Berger et al. [5] obtained a general sufficient condition for a uniformly convergent scheme for a second order turning point problem. Natesan and Ramanujam suggested a computational method and parameter uniform numerical method for second order SPTPP using classical and exponentially fitted difference schemes in [21,23]. Also they applied another technique known as initial value technique in [22] for the same problem. In [14], Kadalbajoo and Patidar applied a numerical method based on cubic spline with nonuniform grid for a second order SPTPP. In [6], the authors proposed a Richardson extrapolation technique for a second order SPTPP. For more detail one may refer [27] and the references therein.

Existence of solution for third order semi linear differential equation with turning point is proved in [13]. An asymptotic expansion of solution for the third order SPTPP was constructed by Jia-qi Mo et al. [19]. Parameter uniform numerical method for a third order SPTPPs is given in [11]. In [30, Page 5], Wasow have discussed a modified form of Orr-Summerfeld equation which is a fourth order singularly perturbed turning point problem.

System of singularly perturbed turning point problems arise in spherical shells and shallow cap dimpling [2]. In [12], the author suggested an asymptotic numerical method for solving a perturbed nonlinear system with turning points that consists of replacing the continuous problem with a sequence of constant coefficient problems on abutting intervals.

In this paper we extend the results of [23] to a weakly coupled system of SPTPP, having turning points on both equations. Our objective is to propose a parameter uniform approximation for the solution of weakly coupled system. In the proposed numerical method, the classical finite difference scheme on piecewise uniform mesh is used to obtain the desired results. The rest of the paper is organized as follows. In Section 2, the problem under study is stated. Method of steps,maximum principle and stability result are discussed in Section 3. Some analytical results are derived in Section 4. In Section 5, a mesh selection strategy is explained. Further, an upwind finite difference scheme is given and the discrete maximum principle is proved.The error analysis is carried out in Section 6. Numerical examples are given in Section 7 to validate our theoretical results.

2 Statement of the problem

Motivated by the works of F. A. Howes [12] and S. Natesan, J. Jayakumar, J. Vigo-Aguiar [23], the following system of second order singularly perturbed boundary value problem with a turning point at $x = 0$ is considered: Find $\bar{u} = (u_1, u_2)^T \in Y = C^0(\bar{\Omega}) \cap C^2(\Omega)$ such that

$$L_1\bar{u}(x) = \varepsilon u_1^{''} + a_1(x)u_1^{'} + b_{11}(x)u_1(x) + b_{12}(x)u_2(x) = f_1(x), \quad x \in \Omega, \tag{1a}$$
$$L_2\bar{u}(x) = \varepsilon u_2^{''} + a_2(x)u_2^{'} + b_{21}(x)u_1(x) + b_{22}(x)u_2(x) = f_2(x), \quad x \in \Omega, \tag{1b}$$
$$u_1(-1) = l_1, \quad u_2(-1) = l_2, \quad u_1(1) = l_3, \quad u_2(1) = l_4, \tag{1c}$$

$$\begin{cases} b_{12} \geq 0, \quad b_{21} \geq 0, \quad b_{11} + b_{12} \leq \beta_1 < 0, \quad b_{22} + b_{21} \leq \beta_2 < 0 \\ |a_k(x)| \leq \alpha_k > 0, \quad \text{for} \quad 0 < |x| \leq 1, \quad a_k(0) = 0, \quad a_k^{'}(0) < 0, \\ \quad \alpha_k + \beta_k < 0 \text{ and } |a_k^{'}(x)| \geq |a_k^{'}(0)|/2 \quad \forall x \in \bar{\Omega}, \text{ for } k = 1, 2, \end{cases} \tag{2}$$

where the functions $a_1(x)$, $a_2(x)$, $b_{11}(x)$, $b_{12}(x)$, $b_{21}(x)$, $b_{22}(x)$, $f_1(x)$ and $f_2(x)$ are sufficiently smooth on $\bar{\Omega}$ and $0 < \varepsilon \ll 1$. The above system can be written in the vector form as

$$\begin{aligned} \bar{L}\bar{u}(x) &= \begin{pmatrix} L_1\bar{u}(x) \\ L_2\bar{u}(x) \end{pmatrix} = \begin{pmatrix} \varepsilon\frac{d^2}{dx^2} & 0 \\ 0 & \varepsilon\frac{d^2}{dx^2} \end{pmatrix} \bar{u}(x) + \begin{pmatrix} a_1(x)\frac{d}{dx} & 0 \\ 0 & a_2(x)\frac{d}{dx} \end{pmatrix} \bar{u}(x) \\ &\quad + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \bar{u}(x) = \bar{f}(x), \quad x \in \Omega, \\ \bar{u}(-1) &= (l_1, l_2)^T, \quad \bar{u}(1) = (l_3, l_4)^T, \end{aligned}$$

where $\bar{f}(x) = (f_1(x), f_2(x))^T$.

Throughout the paper C and C_1 denote generic positive constants independent of the singular perturbation parameter ε and the discretization parameter N of the discrete problem. Let $y : D \to \mathbb{R}$. The appropriate norm for studying the convergence of numerical solution to the exact solution is the maximum norm $||w||_D = \sup_{x\in D} |w(x)|$. In case of vectors $\bar{w} = (w_1, w_2)^T$, we define $|\bar{w}(x)| = (|w_1(x)|, |w_2(x)|)^T$ and $||\bar{w}||_D = \max\{||w_1||_D, ||w_2||_D\}$.

3 Maximum principle and stability result

This section presents the maximum principle and stability result on the solution for the problem (1)-(2). Further, the derivative estimates are derived.

Theorem 1. ***(Maximum principle)*** *Let $\bar{w}(x) = (w_1(x), w_2(x))^T \in Y$ be any function satisfying $L_1\bar{w} \leq 0$, $L_2\bar{w} \leq 0$, $w_1(-1) \geq 0$, $w_2(-1) \geq 0$, $w_1(1) \geq 0$ and $w_2(1) \geq 0$. Then $\bar{w}(x) \geq \bar{0}$, $\forall x \in \bar{\Omega}$.*

Proof. Define $\bar{s}(x) = (s_1(x), s_2(x))^T$ where $s_1(x) = 2+x$ and $s_2(x) = 2+x$. Then $\bar{s}(x) > \bar{0}$, for all $x \in \bar{\Omega}$ and $\bar{L}\bar{s}(x) < \bar{0}, x \in \Omega$. Further we define

$$\mu = \max\left\{\max_{x\in\bar{\Omega}}\left(\frac{-w_1(x)}{s_1(x)}\right), \max_{x\in\bar{\Omega}}\left(\frac{-w_2(x)}{s_2(x)}\right)\right\}.$$

Assume that the theorem is not true. Then $\mu > 0$ and there exists a point $x_0 \in \Omega$, such that either $\left(\frac{-w_1(x_0)}{s_1(x_0)}\right) = \mu$ or $\left(\frac{-w_2(x_0)}{s_2(x_0)}\right) = \mu$ or both. Also $(\bar{w} + \mu\bar{s})(x) \geq \bar{0}, \quad \forall x \in \bar{\Omega}$.

Case1: Assume $\left(\dfrac{-w_1(x_0)}{s_1(x_0)}\right) = \mu$, that is $(w_1 + \mu s_1)(x_0) = 0$. Therefore $(w_1 + \mu s_1)$ attains its minimum at $x = x_0$. Then,

$$\begin{aligned} 0 &> L_1(\bar{w} + \mu\bar{s})(x_0) \\ &= \varepsilon(w_1 + \mu s_1)''(x_0) + a_1(x_0)(w_1 + \mu s_1)'(x_0) \\ &+ b_{11}(x_0)(w_1 + \mu s_1)(x_0) + b_{12}(x_0)(w_2 + \mu s_2)(x_0) \geq 0. \end{aligned}$$

which is a contradiction.

Case2: Assume $\left(\dfrac{-w_2(x_0)}{s_2(x_0)}\right) = \mu$, that is $(w_2 + \mu s_2)(x_0) = 0$. Therefore $(w_2 + \mu s_2)$ attains its minimum at $x = x_0$. Then,

$$\begin{aligned} 0 &> L_2(\bar{w} + \mu\bar{s})(x_0) \\ &= \varepsilon(w_2 + \mu s_2)''(x_0) + a_2(x_0)(w_2 + \mu s_2)'(x_0) \\ &+ b_{21}(x_0)(w_2 + \mu s_2)(x_0) + b_{22}(x_0)(w_1 + \mu s_1)(x_0) \geq 0. \end{aligned}$$

which is a contradiction. Hence $\bar{w}(x) \geq \bar{0}, \ \forall x \in \bar{\Omega}$. □

Lemma 1. ***(Stability Result)*** *If $u_1, u_2 \in Y$, then for $i = 1, 2$*

$$\begin{aligned} |u_i(x)| \leq C \max\{&\max\{|u_1(-1)|, |u_2(-1)|\}, \max\{|u_1(1)|, |u_2(1)|\}, \\ &||L_1\bar{u}||_{x\in\Omega}, ||L_2\bar{u}||_{x\in\Omega}\}, \quad \forall x \in \bar{\Omega}. \end{aligned}$$

Proof. Define $\bar{\Psi}^{\pm} = (\Psi_1^{\pm}, \Psi_2^{\pm})$

$$\Psi_1^{\pm}(x) = C(2 + x)C_1 \pm u_1(x) \text{ and } \Psi_2^{\pm}(x) = C(2 + x)C_1 \pm u_2(x),$$

$C_1 = \max\{\max\{|u_1(-1)|, |u_2(-1)|\}, \max\{|u_1(1)|, |u_2(1)|\}, ||L_1\bar{u}||_{x\in\Omega}, ||L_2\bar{u}$ $||_{x\in\Omega}\}$. Note that $\psi_1^{\pm}(-1) \geq 0$, $\psi_2^{\pm}(-1) \geq 0$, $\psi_1^{\pm}(1) \geq 0$, $\psi_2^{\pm}(1) \geq 0$ for a proper choice of $C > 0$. It is easy to see that, $L_1(\bar{\Psi}^{\pm}(x)) \leq 0$ and $L_2(\bar{\Psi}^{\pm}(x)) \leq 0, \ \forall \ x \in \Omega$. Then by the maximum principle we get the desired result. □

Note: Since the operators $L_j, \ j = 1, 2$ satisfy the above maximum principle, the solution $\bar{u}(x)$ of the BVP (1)-(2) is unique if it exists.

4 Analytical results

In this section, we present some analytical results for the solution $\bar{u}(x)$ and its derivatives. Herein after we shall denote the subdomains of $\bar{\Omega} = [-1, 1]$ as $\Omega_1 = [-1, -\delta]$, $\Omega_2 = [-\delta, \delta]$ and $\Omega_3 = [\delta, 1]$, $0 < \delta \leq 1/2$. The choice of $\delta = 1/2$ can be found in [5]. And $|a_k(x)| \geq \alpha > 0$ for $\delta < |x| \leq 1$.

The following lemma gives estimates for $\bar{u}(x)$ and its derivatives in the intervals Ω_1 and Ω_3 which exclude the turning point $x = 0$.

Lemma 2. *Let $\bar{u}(x) = (u_1, u_2)^T$ be the solution of (1)-(2). Then for j=1,2*

$$||u_j^{(k)}(x)|| \leq \begin{cases} C\varepsilon^{-(k)} \ \max\{||f_j||, ||\bar{u}||\}, \ \textit{for } k = 1, 2, \\ C\varepsilon^{-(k)} \ \max\{||f_j||, ||f_j'||, ||\bar{u}||\}, \ \textit{for } k = 3, \end{cases} \quad \forall x \in \Omega_1 \cup \Omega_3,$$

where C depends on $||a_1||, ||a_2||, ||b_{11}||, ||b_{12}||, ||b_{21}||, ||b_{22}||, ||a_1'||$ and $||a_2'||$.

Proof. Using the technique adopted in [10] the present lemma can be proved in the subdomain Ω_1. Note that

$$\left| \int_{-1}^{x} (f_1 - a_1 u_1' - b_{11} u_1 - b_{12} u_2)(t) dt \right| \leq ||f_1|| + C||\bar{u}|| \tag{3}$$

where C depends on $||a_1||, ||b_{11}||, ||b_{12}||, ||a_1'||$. By the Mean Value Theorem, there exists a point $z \in (-1, -1+\varepsilon)$ such that

$$\begin{aligned} u_1'(z) &= (u_1(-1+\varepsilon) - u_1(-1))/\varepsilon, \\ |\varepsilon u_1'(z)| &= |u_1(-1+\varepsilon) - u_1(-1)| \leq 2||u_1||. \end{aligned} \tag{4}$$

By integrating the differential equation (1a) we get

$$\begin{aligned} \int_{-1}^{z} u_1''(t) dt &= \varepsilon u_1'(x) - \varepsilon u_1'(-1) \\ &= \int_{-1}^{z} (f_1(t) - a_1(t) u_1'(t) + b_{11}(t) u_1(t) + b_{12}(t) u_2(t)) dt. \end{aligned} \tag{5}$$

Using (3) and (4) in (5), we get $|\varepsilon u_1'(-1)| \leq ||f_1|| + ||\bar{u}||$. Using equation (5) with $z = x$, we have

$$|\varepsilon u_1'(x)| \leq ||f_1|| + C||\bar{u}|| \ \ \forall \ \ x \in \Omega_1,$$

and hence

$$|u_1'(x)| \leq C\varepsilon^{-1} \ \max\{||f_1||, ||\bar{u}||\}.$$

Similarly

$$|u_2'(x)| \leq C\varepsilon^{-1} \ \max\{||f_2||, ||\bar{u}||\},$$

which are the desired results. Consider directly in component form of the first and second equation of the system as

$$\begin{aligned} \varepsilon u_1''(x) = f_1(x) - a_1(x) u_1'(x) - b_{11}(x) u_1(x) - b_{12}(x) u_2(x), \\ \varepsilon u_2''(x) = f_2(x) - a_2(x) u_2'(x) - b_{21}(x) u_1(x) - b_{22}(x) u_2(x), \end{aligned}$$

from which we obtain the required bounds on the second and third derivatives. In a similar way one can prove an analogous result in the subdomain Ω_3. □

Let us denote $\beta_i = b_{ii}(0)/a_i^{'}(0)$ for $i = 1, 2$. And also note that $\beta_1, \beta_2 < 0$ always. The following lemma gives estimates for $\bar{u}(x)$ and its derivatives in the interval Ω_2 which includes the turning point $x = 0$.

Lemma 3. *Let $\bar{u}(x) = (u_1, u_2)^T$ be the solution of (1)-(2). Then for $k = 1, 2, 3$,*

$$\|u_j^{(k)}(x)\| \leq C, \text{ for } j = 1, 2 \text{ and} \forall x \in \Omega_2$$

where C depends on $\|a_1\|$, $\|a_2\|$, $\|b_{11}\|$, $\|b_{12}\|$, $\|b_{21}\|$, $\|b_{22}\|$, $\|a_1'\|$, $\|a_2'\|$ and β_k.

Proof. We prove this lemma by adopting the technique as in Berger et. al. [5].
From the Mean Value Theorem and the assumptions in (2), we have

$$|a_k(x)| = |a_k(x) - a_k(0)| = |x||a_k^{'}(\zeta)| \geq |x||a_k^{'}(0)|/2 \geq \frac{|x|}{2\beta_k} \text{ for } k = 1, 2.$$

Then by the previous lemma the bound for $\bar{u}(x)$ and its derivatives at $x = \pm 1/2$ are found where C, depends on $||a_1||, ||a_2||, ||b_{11}||, ||b_{12}||, ||b_{21}||, ||b_{22}||, ||a_1'||, ||a_2'||$ and β_k. If equations (1a) and (1b) are differentiated k times, one finds that the differential equation satisfied by $\bar{z}(x) = (\bar{u})^{(k)}(x)$ is

$$\varepsilon z_1^{''}(x) + a_1(x) z_1^{'}(x) + [b_{11}(x) + k(a_1^{'}(x))] z_1(x) + b_{12}^{'}(x) z_2(x) = g_1(x) \quad (6)$$
$$\varepsilon z_2^{''}(x) + a_2(x) z_2^{'}(x) + [b_{22}(x) + k(a_2^{'}(x))] z_2(x) + b_{21}^{'}(x) z_1(x) = g_2(x) \quad (7)$$

where $\bar{g}$ depends on $\bar{u}, \cdots (\bar{u})^{(k-1)}$ and on the k^{th} order derivatives of $a_1, a_2, b_{11}, b_{12}, b_{21}, b_{22}$. Applying Lemma 1 with b_{ii} is replaced by $b_{ii} + k(a_i^{'})$ for $i = 1, 2$, b_{12} is replaced by $b_{12}^{'}$ and b_{21} is replaced by $b_{21}^{'}$ respectively, we obtain the required result by using an inductive argument. □

To obtain the sharper bounds of solution $\bar{u}(x)$ and its derivatives we decompose the solution $\bar{u}(x)$ into regular and singular components as, $\bar{u}(x) = \bar{v}(x) + \bar{w}(x)$, where $\bar{v}(x) = (v_1(x), v_2(x))^T$ and $\bar{w}(x) = (w_1(x), w_2(x))^T$. The regular component $\bar{v}(x)$ can be written in the form of $\bar{v} = \bar{v}_0 + \varepsilon \bar{v}_1 + \varepsilon^2 \bar{v}_2$, where $\bar{v}_0 = (v_{01}, v_{02})^T$, $\bar{v}_1 = (v_{11}, v_{12})^T$ and $\bar{v}_2 = (v_{21}, v_{22})^T$ which are defined respectively to be the solutions of the problems:

$$\begin{pmatrix} a_1\frac{d}{dx} & 0 \\ 0 & a_2\frac{d}{dx} \end{pmatrix}\bar{v}_0 + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\bar{v}_0 = \bar{f}, v_{01}(-1) = A_1,\ v_{02}(-1) = A_2$$

$$\begin{pmatrix} a_1\frac{d}{dx} & 0 \\ 0 & a_2\frac{d}{dx} \end{pmatrix}\bar{v}_1 + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\bar{v}_1 = \begin{pmatrix} \frac{d^2}{dx^2} & 0 \\ 0 & \frac{d^2}{dx^2} \end{pmatrix}\bar{v}_0, \bar{v}_1(-1) = \bar{0}$$

and $\bar{L}(\bar{v}_2) = \begin{pmatrix} \frac{d^2}{dx^2} & 0 \\ 0 & \frac{d^2}{dx^2} \end{pmatrix}\bar{v}_2, \bar{v}_2(-1) = 0,\ \bar{v}_2(1) = 0.$

Thus the regular component $\bar{v}(x)$ is the solution of

$$\bar{L}(\bar{v}) = \bar{f}, \tag{8}$$

$$\bar{v}(-1) = \bar{v}_0(-1) + \varepsilon\bar{v}_1(-1) + \varepsilon^2\bar{v}_2(-1),\ \bar{v}(1) = \bar{v}_0(1) + \varepsilon\bar{v}_1(1) + \varepsilon^2\bar{v}_2(1),$$

and the singular component $\bar{w}(x)$ is the solution of

$$\begin{aligned} &\bar{L}(\bar{w}) = \bar{0}, \\ &\bar{w}(-1) = \bar{u}(-1) - \bar{v}(-1), \quad \bar{w}(1) = \bar{u}(1) - \bar{v}(1). \end{aligned} \tag{9}$$

The following lemma provides the bound on the derivatives of the regular and singular components of the solution $\bar{u}(x)$.

Lemma 4. *The smooth component $\bar{v}$ and singular component $\bar{w}$ and their derivatives satisfy the bounds for $k = 0, 1, 2, 3,$ and $j = 1, 2$,*

$$||v_j^{(k)}(x)|| \le C(1 + \varepsilon^{2-k}), \quad \forall\ x \in \Omega_1 \cup \Omega_3,$$

and

$$|w_j^{(k)}(x)| \le \begin{cases} C\varepsilon^{-k}e^{-\alpha(1+x)/\varepsilon} & \forall\ x \in \Omega_1, \\ C\varepsilon^{-k}e^{-\alpha(1-x)/\varepsilon} & \forall\ x \in \Omega_3, \end{cases}$$

where $a_j(x) > 0$ for $x \in \Omega_1$ and $a_j(x) < 0$ for $x \in \Omega_3$.

Proof. Using appropriate barrier functions, applying Theorem 1 and adopting the method of proof used in [[10], p.44], the present lemma can be proved. □

Theorem 2. *The smooth component $\bar{v}$ and singular component $\bar{w}$ and their derivatives satisfy the bounds for $k = 0, 1, 2, 3,$ $j = 1, 2$*

$$\begin{aligned} ||v_j^{(k)}(x)|| &\le C(1 + \varepsilon^{2-k}), \quad \textit{and} \\ |w_j^{(k)}(x)| &\le C\varepsilon^{-k}(e^{-\alpha(1+x)/\varepsilon} + e^{-\alpha(1-x)/\varepsilon}), \quad \forall x \in \bar{\Omega} \end{aligned}$$

Proof. Lemma 3 guarantees that the solution of the SPTPP (1)-(2) and its derivatives are smooth in the domain Ω_2. Hence, the proof is an immediate consequence of the previous estimates on $\bar{v}^{(k)}(x)$ and $\bar{w}^{(k)}(x)$. □

5 Discrete problem

5.1 Mesh selection strategy

In this section, the system (1)-(2) is discretized using classical finite difference scheme on piecewise uniform meshes (Shishkin mesh). Consider the classical upwind scheme on a piecewise uniform mesh $\bar{\Omega}_\varepsilon^N$, $N \geq 4$ which is constructed by dividing the domain $\bar{\Omega}$ into three subintervals $\Omega_L = [-1, -1+\tau]$, $\Omega_C = [-1+\tau, 1-\tau]$ and $\Omega_R = [1-\tau, 1]$ such that $\bar{\Omega} = \Omega_L \cup \Omega_C \cup \Omega_R$. The transition parameter τ is chosen to be $\min\left\{\frac{1}{2}, \frac{2\varepsilon lnN}{\alpha}\right\}$.

5.2 Finite difference method for the problem (1a)-(1b)

The domain $\bar{\Omega}_\varepsilon^N$ is obtained by putting a uniform mesh with $N/4$ mesh elements in both Ω_L and Ω_R and a uniform mesh with $N/2$ elements in Ω_C. The resulting fitted finite difference scheme is to find $\bar{U}(x_i) = (U_1(x_i), U_2(x_i))^T$ for $i = 0, 1, 2, \cdots N$ such that for $x_i \in \bar{\Omega}_\varepsilon^N$,

$$\begin{aligned} L_1^N \bar{U}(x_i) &:= \varepsilon\delta^2 U_1(x_i) + a_1(x_i) D^* U_1(x_i) + b_{11}(x_i) U_1(x_i) \qquad (10)\\ &\quad + b_{12}(x_i) U_2(x_i) = f_1(x_i) \qquad i = 1(1)N-1 \\ L_2^N \bar{U}(x_i) &:= \varepsilon\delta^2 U_2(x_i) + a_2(x_i) D^* U_2(x_i) + b_{21}(x_i) U_1(x_i) \qquad (11)\\ &\quad + b_{22}(x_i) U_2(x_i) = f_2(x_i), \qquad i = 1(1)N-1 \\ U_1(x_0) &= u_1(-1), \quad U_1(x_N) = u_1(1), \\ U_2(x_0) &= u_2(-1), \quad U_2(x_N) = u_2(1), \end{aligned}$$

where

$$D^+ U_j(x_i) = \frac{U_j(x_{i+1}) - U_j(x_i)}{x_{i+1} - x_i}, D^- U_j(x_i) = \frac{U_j(x_i) - U_j(x_{i-1})}{x_i - x_{i-1}},$$

$$\delta^2 U_j(x_i) = \frac{D^+ U_j(x_i) - D^- U_j(x_i)}{(x_{i+1} - x_{i-1})/2}, D^* U_j(x_i) = \begin{cases} D^+ U_j(x_i) \text{ if } a_j(x_i) > 0, \\ D^- U_j(x_i) \text{ if } a_j(x_i) < 0. \end{cases}$$

6 Numerical solution estimates

Analogous to the results stated in Theorem 1 and Lemma 1 we prove the following results.

Theorem 3. *Let $\bar{\Psi}(x_i) = (\Psi_1(x_i), \Psi_2(x_i))^T$ be any mesh function satisfying $\bar{\Psi}(x_0) \geq \bar{0}, \bar{\Psi}(x_N) \geq \bar{0}$, $L_1^N(\bar{\Psi}(x_i)) \leq 0$, $\forall i = 1(1)N-1$ and $L_2^N(\bar{\Psi}(x_i)) \leq 0$ $\forall i = 1(1)N-1$. Then $\bar{\Psi}(x_i) \geq \bar{0}$, $\forall x_i \in \bar{\Omega}_\varepsilon^N$.*

Proof. Define $\bar{s}(x_i) = (s_1(x_i), s_2(x_i))^T$ as $s_1(x_i) = 2 + x_i$ and $s_2(x_i) = 2 + x_i$. Then, $\bar{s}(x_i) > \bar{0}$, for all $x_i \in \bar{\Omega}_\varepsilon^N$. Further we define

$$\xi = \max\left\{\max_{x_i\in\bar{\Omega}_\varepsilon^N}\left(\frac{-\Psi_1}{s_1}\right)(x_i), \max_{x_i\in\bar{\Omega}_\varepsilon^N}\left(\frac{-\Psi_2}{s_2}\right)(x_i)\right\}.$$

Assume that the theorem is not true. Then $\xi > 0$ and we have $(\bar{\psi}+\xi\bar{s})(x_i) \geq \bar{0}$ for $x_i \in \bar{\Omega}_\varepsilon^N$. For some $i = k$, we may have either $(\Psi_1 + \xi s_1)(x_k) = 0$ (or) $(\Psi_2 + \xi s_2)(x_k) = 0$ or both.
Case (i):$(\Psi_1 + \xi s_1)(x_k) = 0$. Then

$$\begin{aligned} 0 &\geq L_1^N(\bar{\Psi} + \xi\bar{s})(x_k) \\ &= \begin{cases} \varepsilon\delta^2(\Psi_1 + \xi s_1)(x_k) + a_1(x_k)D^+(\Psi_1 + \xi s_1)(x_k) \\ +b_{11}(x_k)(\Psi_1 + \xi s_1)(x_k) + b_{12}(x_k)(\Psi_2 + \xi s_2)(x_k) & \text{if } a_1(x_k) > 0 \\ \varepsilon\delta^2(\Psi_1 + \xi s_1)(x_k) + a_1(x_k)D^-(\Psi_1 + \xi s_1)(x_k) \\ +b_{11}(x_k)(\Psi_1 + \xi s_1)(x_k) + b_{12}(x_k)(\Psi_2 + \xi s_2)(x_k) & \text{if } a_1(x_k) < 0 \end{cases} \\ &> 0, \end{aligned}$$

which is a contradiction.
Case (ii): $(\Psi_2+\xi s_2)(x_k) = 0$. Similar to Case (i) it leads to a contradiction. Hence $\bar{\Psi}(x_i) \geq \bar{0} \quad \forall x_i \in \bar{\Omega}_\varepsilon^N$. □

Lemma 5. *Consider the scheme (10)-(11). If $\bar{z}(x_i) = (z_1(x_i), z_2(x_i))^T$ is any mesh function, then for all $x_i \in \bar{\Omega}_\varepsilon^N$,*

$$\begin{aligned} |z_j(x_i)| \leq C\max\{&\max\{|z_1(x_0)|, |z_1(x_N)|\}, \max\{|z_2(x_0)|, |z_2(x_N)|\}, \\ &\max_{1\leq i\leq N-1}|L_1^N\bar{z}(x_i)|, \max_{1\leq i\leq N-1}|L_2^N\bar{z}(x_i)|\}, \quad j = 1, 2. \end{aligned}$$

Proof. Let

$$\begin{aligned} C_1 = C\max\{&\max\{|z_1(x_0)|, |z_1(x_N)|\}, \max\{|z_2(x_0)|, |z_2(x_N)|\}, \\ &\left.\max_{1\leq i\leq N-1}|L_1^N\bar{z}(x_i)|, \max_{1\leq i\leq N-1}|L_2^N\bar{z}(x_i)|\right\}. \end{aligned}$$

Define the mesh functions $\bar{\psi}^{\pm}(x_i)$ as

$$\bar{\psi}^{\pm}(x_i) = \begin{pmatrix} C_1(2 + x_i) \\ C_1(2 + x_i) \end{pmatrix} \pm \bar{z}(x_i).$$

Then we have $\bar{\psi}_1^{\pm}(x_0) \geq \bar{0}$, $\bar{\psi}_2^{\pm}(x_N) \geq 0$ and $\bar{L}^N\bar{\psi}^{\pm}(x_i) \leq \bar{0}$. By Theorem 3 we get the required result. □

The discrete solution $\bar{U}(x_i)$ can be decomposed into the sum as $\bar{U}(x_i) = \bar{V}(x_i)+\bar{W}(x_i)$ where $\bar{V}(x_i)$ and $\bar{W}(x_i)$ are regular and singular components respectively defined as

$$\bar{L}^N\bar{V}(x_i) = \bar{f},\ i = 1,2,\dots,N-1,\ \bar{V}(-1) = \bar{v}(-1),\quad \bar{V}(1) = \bar{v}(1), \quad (12)$$
$$\bar{L}^N\bar{W}(x_i) = \bar{0},\ i = 1,2,\dots,N-1,\ \bar{W}(-1) = \bar{w}(-1),\ \bar{W}(1) = \bar{w}(1). \quad (13)$$

The error in the numerical solution can be written in the form

$$(\bar{U}-\bar{u})(x_i) = (\bar{V}-\bar{v})(x_i) + (\bar{W}-\bar{w})(x_i).$$

Lemma 6. *At each mesh point $x_i \in \bar{\Omega}^N_\varepsilon$, the error of the regular component satisfies the estimate*

$$|(\bar{V}-\bar{v})(x_i)| \le \begin{pmatrix} CN^{-1} \\ CN^{-1} \end{pmatrix},$$

Proof. This is obtained by using the following standard stability and consistency argument. We consider the local truncation error,

$$\bar{L}(\bar{V}-\bar{v}) = (\bar{L}-\bar{L}^N)\bar{v} = \begin{cases} \varepsilon(\frac{d^2}{dx^2}-\delta^2)v_1 + a_1(x_i)(\frac{d}{dx}-D^*)v_1, \\ \varepsilon(\frac{d^2}{dx^2}-\delta^2)v_2 + a_2(x_i)(\frac{d}{dx}-D^*)v_2. \end{cases} \quad (14)$$

Then by local truncation error estimates and Theorem 2, we obtain

$$|\bar{L}^N(\bar{V}-\bar{v})(x_i)|$$
$$\le \begin{pmatrix} \begin{cases} \frac{\varepsilon}{3}(x_{i+1}-x_{i-1})|v_1^{(3)}| + \frac{a_1(x_i)}{2}(x_{i+1}-x_i)|v_1^{(2)}| \text{ if } a_1(x_i) > 0 \\ \frac{\varepsilon}{3}(x_{i+1}-x_{i-1})|v_1^{(3)}| + \frac{a_1(x_i)}{2}(x_i-x_{i-1})|v_1^{(2)}| \text{ if } a_1(x_i) < 0 \end{cases} \\ \\ \begin{cases} \frac{\varepsilon}{3}(x_{i+1}-x_{i-1})|v_2^{(3)}| + \frac{a_2(x_i)}{2}(x_{i+1}-x_i)|v_2^{(2)}| \text{ if } a_2(x_i) > 0 \\ \frac{\varepsilon}{3}(x_{i+1}-x_{i-1})|v_2^{(3)}| + \frac{a_2(x_i)}{2}(x_i-x_{i-1})|v_2^{(2)}| \text{ if } a_2(x_i) < 0 \end{cases} \end{pmatrix}$$
$$\le \begin{pmatrix} CN^{-1} \\ CN^{-1}. \end{pmatrix}$$

Now applying Lemma 5 to the mesh functions $(\bar{V}-\bar{v})(x_i)$, we can easily obtain

$$|(\bar{V}-\bar{v})(x_i)| \le \begin{pmatrix} CN^{-1} \\ CN^{-1} \end{pmatrix}.$$

□

Lemma 7. *At each mesh point $x_i \in \bar{\Omega}_\varepsilon^N$ the error of the singular component satisfies the estimate*

$$|(\bar{W}-\bar{w})(x_i)| \leq \begin{pmatrix} CN^{-1}(\ln\ N)^2 \\ CN^{-1}(\ln\ N)^2 \end{pmatrix}.$$

Proof. We consider first the case $\tau = 1/2$ and so $\varepsilon^{-1} \leq ClnN$ and $h = N^{-1}$. By classical argument and using Theorem 2, we obtain

$$|\bar{L}^N(\bar{W}-\bar{w})(x_i)| \quad \leq \quad \begin{cases} C\varepsilon^{-2}N^{-1}(e^{-\alpha(1+x_i)/\varepsilon} + e^{-\alpha(1-x_i)/\varepsilon}), \\ C\varepsilon^{-2}N^{-1}(e^{-\alpha(1+x_i)/\varepsilon} + e^{-\alpha(1-x_i)/\varepsilon}), \end{cases} \tag{15}$$

and using $\varepsilon^{-1} \leq ClnN$ in the above inequality, we get

$$|\bar{L}^N(\bar{W}-\bar{w})(x_i)| \quad \leq \quad \left\{ \begin{pmatrix} CN^{-1}(lnN)^2 \\ CN^{-1}(lnN)^2 \end{pmatrix} \right. . \tag{16}$$

Applying Lemma 5 to the mesh function $(\bar{W}-\bar{w})(x_i)$, we have

$$|(\bar{W}-\bar{w})(x_i)| \quad \leq \quad \begin{pmatrix} CN^{-1}(lnN)^2 \\ CN^{-1}(lnN)^2 \end{pmatrix}. \tag{17}$$

We now consider the case $\tau = \dfrac{2\varepsilon}{\alpha} lnN$.

In this case the mesh is piecewise uniform with the mesh spacing $4\tau/N$ in the subintervals Ω_L, Ω_R and $2\tau/N$ in the subinterval Ω_C. We give separate proofs for coarse and fine mesh subintervals.

The subinterval Ω_C has no boundary layer, both W and w are small, and by the triangular inequality we have

$$|(\bar{W}-\bar{w})(x_i)| \leq |\bar{W}(x_i)| + |\bar{w}(x_i)|. \tag{18}$$

It suffices to bound $\bar{W}(x_i)$ and $\bar{w}(x_i)$ separately. Now we consider the subinterval $[-1+\tau, 0]$ for our discussion since one can obtain a similar proof for the subinterval $[0, 1-\tau]$.
Using Lemma 4 we have

$$|\bar{w}(x_i)| \leq \begin{cases} CN^{-1}, \\ CN^{-1}. \end{cases} \tag{19}$$

To obtain a similar bound for $\bar{W}(x_i)$, we introduce the mesh functions $\bar{Y} = (Y_1, Y_2)^T$, where $Y_1(x_i)$ is the solution of the problem (13), $a_1(x)$ is

Table 1: Values of D_1^N, p_1^N for the solution component U_1 for Example 1.

	Number of mesh points N				
ε	64	128	256	512	1024
2^{-1}	5.4178e-3	2.7828e-3	1.4121e-3	7.1157e-4	3.5721e-4
2^{-2}	1.7492e-2	8.9578e-3	4.5245e-3	2.2730e-3	1.1391e-3
2^{-3}	1.6283e-2	8.4688e-3	4.3592e-3	2.2171e-3	1.1188e-3
2^{-4}	1.7884e-2	1.1958e-2	7.7704e-3	4.8450e-3	2.4762e-3
2^{-5}	1.9992e-2	1.2900e-2	8.0884e-3	4.7619e-3	2.7232e-3
2^{-6}	2.1360e-2	1.3515e-2	8.4430e-3	4.9400e-3	2.8178e-3
2^{-7}	2.2234e-2	1.3972e-2	8.6691e-3	5.0464e-3	2.8731e-3
2^{-8}	2.2781e-2	1.4245e-2	8.8138e-3	5.1139e-3	2.9058e-3
2^{-9}	2.3062e-2	1.4393e-2	8.8993e-3	5.1573e-3	2.9272e-3
2^{-10}	2.3207e-2	1.4469e-2	8.9450e-3	5.1829e-3	2.9412e-3
2^{-11}	2.3279e-2	1.4506e-2	8.9680e-3	5.1965e-3	2.9496e-3
2^{-12}	2.3315e-2	1.4525e-2	8.9792e-3	5.2033e-3	2.9540e-3
2^{-13}	2.3333e-2	1.4534e-2	8.9848e-3	5.2066e-3	2.9562e-3
2^{-14}	2.3342e-2	1.4539e-2	8.9876e-3	5.2082e-3	2.9573e-3
2^{-15}	2.3346e-2	1.4541e-2	8.9890e-3	5.2090e-3	2.9578e-3
2^{-16}	2.3348e-2	1.4543e-2	8.9900e-3	5.2096e-3	2.9581e-3
2^{-17}	2.3350e-2	1.4543e-2	8.9902e-3	5.2097e-3	2.9582e-3
2^{-18}	2.3350e-2	1.4544e-2	8.9902e-3	5.2098e-3	2.9582e-3
2^{-19}	2.3351e-2	1.4544e-2	8.9904e-3	5.2097e-3	2.9596e-3
2^{-20}	2.3351e-2	1.4544e-2	8.9904e-3	5.2097e-3	2.9596e-3
D_1^N	2.3351e-2	1.4544e-2	8.9904e-3	5.2098e-3	2.9596e-3
p_1^N	6.8307e-1	6.9394e-1	7.8715e-1	8.1584e-1	-

replaced by α under boundary conditions that are the same as those used for W_1, and $Y_2(x_i) = Y_1(x_i)$. From Lemma 7.5 of [18],

$$|\bar{W}(x_i)| \leq |\bar{Y}(x_i)|, \quad 0 \leq i \leq N \tag{20}$$

and applying Lemma 7.3 of [18],

$$|\bar{Y}(x_i)| \leq \begin{pmatrix} CN^{-1} \\ CN^{-1} \end{pmatrix}. \tag{21}$$

Substituting (21) in the inequality (20) gives

$$|\bar{W}(x_i)| \leq \begin{pmatrix} CN^{-1} \\ CN^{-1} \end{pmatrix}. \tag{22}$$

Table 2: Values of D_2^N, p_2^N for the solution component U_2 for Example 1.

	Number of mesh points N				
ε	64	128	256	512	1024
2^{-1}	2.5727e-2	1.3337e-2	6.7963e-3	3.4316e-3	1.7243e-3
2^{-2}	8.0236e-2	4.1792e-2	2.1304e-2	1.0753e-2	5.4020e-3
2^{-3}	4.9948e-2	2.6596e-2	1.3924e-2	7.1492e-3	3.6251e-3
2^{-4}	3.6226e-2	2.6808e-2	1.8529e-2	1.1884e-2	6.1696e-3
2^{-5}	3.4341e-2	2.5371e-2	1.6817e-2	1.0183e-2	5.9781e-3
2^{-6}	3.3183e-2	2.4606e-2	1.6335e-2	9.8902e-3	5.8084e-3
2^{-7}	3.2654e-2	2.4253e-2	1.6108e-2	9.7520e-3	5.7281e-3
2^{-8}	3.2405e-2	2.4085e-2	1.6000e-2	9.6851e-3	5.6892e-3
2^{-9}	3.2284e-2	2.4004e-2	1.5947e-2	9.6523e-3	5.6701e-3
2^{-10}	3.2225e-2	2.3964e-2	1.5921e-2	9.6362e-3	5.6607e-3
2^{-11}	3.2195e-2	2.3934e-2	1.5901e-2	9.6242e-3	5.6537e-3
2^{-12}	3.2173e-2	2.3929e-2	1.5898e-2	9.6222e-3	5.6525e-3
2^{-13}	3.2169e-2	2.3926e-2	1.5896e-2	9.6212e-3	5.6519e-3
2^{-14}	3.2168e-2	2.3925e-2	1.5896e-2	9.6208e-3	5.6517e-3
2^{-15}	3.2167e-2	2.3924e-2	1.5895e-2	9.6205e-3	5.6515e-3
2^{-16}	3.2166e-2	2.3924e-2	1.5895e-2	9.6205e-3	5.6513e-3
2^{-17}	3.2166e-2	2.3924e-2	1.5895e-2	9.6203e-3	5.6515e-3
2^{-18}	3.2166e-2	2.3924e-2	1.5894e-2	9.6208e-3	5.6512e-3
2^{-19}	3.2166e-2	2.3924e-2	1.5895e-2	9.6203e-3	5.6574e-3
2^{-20}	3.2166e-2	2.3924e-2	1.5895e-2	9.6203e-3	5.6574e-3
D_2^N	8.0236e-2	4.1792e-2	2.1304e-2	1.1884e-2	6.1696e-3
p_2^N	9.4103e-1	9.7207e-1	8.4213e-1	9.4575e-1	-

Using inequalities (19) and (22) in (18), we get

$$|\bar{W}(x_i) - \bar{w}(x_i)| \leq \begin{cases} CN^{-1}, \\ CN^{-1}, \end{cases} \quad \forall\ N/4 \leq i \leq N/2. \tag{23}$$

Proceeding analogously on $[0, 1-\tau]$, we get

$$|\bar{W}(x_i) - \bar{w}(x_i)| \leq \begin{cases} CN^{-1}, \\ CN^{-1}, \end{cases} \quad \forall\ N/2 \leq i \leq 3N/4. \tag{24}$$

It remains to prove the results for $x_i \in \Omega_L$ and $x_i \in \Omega_R$. Let $x_i \in \Omega_L$. For $i = 0$ there is nothing to prove. For $x_i \in \Omega_L$ the proof follows the same lines as for the case $\tau = 1/2$ except that we use the discrete maximum principle on Ω_L and the already established bound $|\bar{W}(x_{N/4})| \leq \begin{cases} CN^{-1}, \\ CN^{-1}. \end{cases}$ In this case, we have

$$|\bar{L}^N(\bar{W} - \bar{w})(x_i)| \leq \begin{cases} C\tau\varepsilon^{-2}N^{-1}e^{-\alpha(1+x_i)/\varepsilon}, \\ C\tau\varepsilon^{-2}N^{-1}e^{-\alpha(1+x_i)/\varepsilon}, \end{cases} \quad \forall i, 0 \leq i \leq N/4. \quad (25)$$

We introduce the mesh functions

$$\Psi_1^{\pm}(x_i) = \frac{Ce^{2\gamma h/\varepsilon}}{\gamma(\alpha - \gamma)}\tau\varepsilon^{-1}N^{-1}Y_1(x_i) + C'N^{-1} \pm (\bar{W} - \bar{w})(x_i), \quad (26)$$

$$\Psi_2^{\pm}(x_i) = \frac{Ce^{2\gamma h/\varepsilon}}{\gamma(\alpha - \gamma)}\tau\varepsilon^{-1}N^{-1}Y_2(x_i) + C'N^{-1} \pm (\bar{W} - \bar{w})(x_i), \quad (27)$$

where γ is any constant satisfying $\alpha > \gamma > 0$ and $\forall\ i,\ 0 \leq i \leq N/4$, $Y_1(x_i) = \left(\dfrac{\lambda^{N/4-i} - 1}{\lambda^{N/4} - 1}\right)$, where $\lambda = 1 + \frac{\gamma h}{\varepsilon}$, $D^+Y_1(x_i) \leq -\gamma/\varepsilon e^{-\gamma(1+x_{i+1})/\varepsilon}$. Let $Y_2(x_i) = Y_1(x_i)$. It is easy to see that $\bar{\Psi}^{\pm}(x_0) > \bar{0}$, $\bar{\Psi}^{\pm}(x_{N/4}) \geq \bar{0}$ and $\bar{L}^N\bar{\Psi}^{\pm} \leq \bar{0}$ for $1 \leq i \leq N/4$. Then by the discrete maximum principle we conclude that $\Psi_i^{\pm} \geq 0,\ \forall x_i \in \Omega_L$. That is

$$|(\bar{W} - \bar{w})(x_i)| \leq \begin{pmatrix} CN^{-1}\ln\ N \\ CN^{-1}\ln\ N \end{pmatrix}, \quad 0 \leq i \leq N/4$$

Similarly the proof follows for $x_i \in \Omega_R$. Combining the estimates for the singular components in different regions, we obtain

$$|(\bar{W} - \bar{w})(x_i)| \leq \begin{pmatrix} CN^{-1}(\ln\ N)^2 \\ CN^{-1}(\ln\ N)^2 \end{pmatrix}, \quad 0 \leq i \leq N,$$

as required. □

Theorem 4. *Let $\bar{u}(x) = (u_1(x), u_2(x))^T$ for all $x \in \bar{\Omega}$ be the solution of (1)-(2)and let $\bar{U}(x_i) = (U_1(x_i), U_2(x_i))^T$ for all $x_i \in \bar{\Omega}_\varepsilon^N$ be the numerical solution of problem (10)-(11). Then we have*

$$\sup_{0<\varepsilon\leq 1} ||U_1 - u_1||_{\bar{\Omega}_\varepsilon^N} \leq CN^{-1}(lnN)^2 \ \textit{and} \ \sup_{0<\varepsilon\leq 1} ||U_2 - u_2||_{\bar{\Omega}_\varepsilon^N} \leq CN^{-1}(lnN)^2.$$

Proof. It follows immediately, if one applies Lemma 6 and Lemma 7 to $\bar{U} - \bar{u} = \bar{V} - \bar{v} + \bar{W} - \bar{w}$. □

Table 3: Values of D_1^N, p_1^N for the solution component U_1 for Example 2.

	Number of mesh points N				
ε	64	128	256	512	1024
2^{-1}	4.1021e-2	2.1350e-2	1.0918e-2	5.5237e-3	2.7788e-3
2^{-21}	3.1884e-2	1.5615e-2	7.8108e-3	3.9175e-3	1.9632e-3
2^{-3}	3.2748e-2	2.0570e-2	1.1591e-2	6.1597e-3	3.1837e-3
2^{-4}	2.5445e-2	2.2022e-2	1.6722e-2	1.1631e-2	6.1843e-3
2^{-5}	2.6795e-2	2.2758e-2	1.6300e-2	1.0853e-2	6.3287e-3
2^{-6}	2.7451e-2	2.3109e-2	1.6477e-2	1.0947e-2	6.3815e-3
2^{-7}	2.7773e-2	2.3282e-2	1.6564e-2	1.0994e-2	6.4076e-3
2^{-8}	2.7935e-2	2.3369e-2	1.6608e-2	1.1017e-2	6.4207e-3
2^{-9}	2.8020e-2	2.3413e-2	1.6630e-2	1.1029e-2	6.4272e-3
2^{-10}	2.8065e-2	2.3436e-2	1.6641e-2	1.1035e-2	6.4306e-3
2^{-11}	2.8090e-2	2.3448e-2	1.6647e-2	1.1038e-2	6.4323e-3
2^{-12}	2.8102e-2	2.3455e-2	1.6650e-2	1.1040e-2	6.4332e-3
2^{-13}	2.8108e-2	2.3459e-2	1.6652e-2	1.1041e-2	6.4337e-3
2^{-14}	2.8110e-2	2.3461e-2	1.6654e-2	1.1041e-2	6.4340e-3
2^{-15}	2.8111e-2	2.3462e-2	1.6654e-2	1.1042e-2	6.4341e-3
2^{-16}	2.8112e-2	2.3463e-2	1.6655e-2	1.1042e-2	6.4342e-3
2^{-17}	2.8112e-2	2.3463e-2	1.6655e-2	1.1042e-2	6.4341e-3
2^{-18}	2.8112e-2	2.3463e-2	1.6655e-2	1.1042e-2	6.4344e-3
2^{-19}	2.8112e-2	2.3463e-2	1.6655e-2	1.1042e-2	6.4346e-3
2^{-20}	2.8112e-2	2.3463e-2	1.6655e-2	1.1041e-2	6.4350e-3
D_1^N	4.1021e-2	2.3463e-2	1.6722e-2	1.1631e-2	6.4350e-3
p_1^N	8.0597e-1	4.8866e-1	5.2370e-1	8.5402e-1	-

7 Numerical results

In this section, two examples are given to illustrate the numerical method discussed in this paper. We use the double mesh principle to estimate the error and compute the rate of convergence in our computed solution. Let U^{2N} be the piecewise linear interpolants of the numerical solution U^N on the mesh Ω_{2N} , where $N, 2N$ are the number of mesh points.
Define the double mesh differences to be

$$D_{\varepsilon,j}^N = \left\{ \max_{x_i \in \bar{\Omega}_\varepsilon^N} |U_j^N(x_i) - U_j^{2N}(x_i)| \right\}, \quad j = 1,2 \text{ and } D_j^N = \max_\varepsilon D_{\varepsilon,j}^N,$$

where $U_j^N(x_i)$ and $U_j^{2N}(x_i)$ respectively, denote the numerical solution obtained using N and $2N$ mesh intervals. Further, we calculate the parameter robust order of convergence as

$$p_j = log_2\left(\frac{D_j^N}{D_j^{2N}}\right), \text{ for } j = 1, 2.$$

The following examples have a turning point at $x = 1/2$. The numerical results are presented for various values of the perturbation parameter $\varepsilon \in \{2^{-20}, 2^{-19}, \ldots, 2^{-1}\}$.

Table 4: Values of D_2^N, p_2^N for the solution component U_2 for Example 2.

	Number of mesh points N				
ε	64	128	256	512	1024
2^{-1}	1.3970e-1	7.0640e-2	3.5578e-2	1.7865e-2	8.9528e-3
2^{-2}	1.6114e-1	7.2911e-2	3.4704e-2	1.6933e-2	8.3636e-3
2^{-3}	4.7313e-2	2.7199e-2	1.4573e-2	7.9083e-3	3.8371e-3
2^{-4}	4.9447e-2	3.4826e-2	2.3091e-2	1.4504e-2	7.4398e-3
2^{-5}	4.3795e-2	3.2327e-2	1.9970e-2	1.2381e-2	7.1476e-3
2^{-6}	4.3061e-2	3.2216e-2	2.0015e-2	1.2455e-2	7.2046e-3
2^{-7}	4.2689e-2	3.2160e-2	2.0065e-2	1.2491e-2	7.2327e-3
2^{-8}	4.2486e-2	3.2127e-2	2.0089e-2	1.2508e-2	7.2462e-3
2^{-9}	4.2364e-2	3.2106e-2	2.0100e-2	1.2516e-2	7.2525e-3
2^{-10}	4.2286e-2	3.2090e-2	2.0103e-2	1.2520e-2	7.2554e-3
2^{-11}	4.2238e-2	3.2076e-2	2.0104e-2	1.2521e-2	7.2565e-3
2^{-12}	4.2212e-2	3.2064e-2	2.0103e-2	1.2521e-2	7.2569e-3
2^{-13}	4.2201e-2	3.2056e-2	2.0101e-2	1.2520e-2	7.2569e-3
2^{-14}	4.2197e-2	3.2051e-2	2.0098e-2	1.2519e-2	7.2567e-3
2^{-15}	4.2196e-2	3.2049e-2	2.0096e-2	1.2518e-2	7.2563e-3
2^{-16}	4.2196e-2	3.2049e-2	2.0095e-2	1.2517e-2	7.2558e-3
2^{-17}	4.2196e-2	3.2049e-2	2.0095e-2	1.2517e-2	7.2562e-3
2^{-18}	4.2196e-2	3.2049e-2	2.0095e-2	1.2515e-2	7.2572e-3
2^{-19}	4.2196e-2	3.2049e-2	2.0094e-2	1.2518e-2	7.2495e-3
2^{-20}	4.2196e-2	3.2049e-2	2.0096e-2	1.2519e-2	7.2559e-3
D_2^N	1.6114e-1	7.2911e-2	3.5578e-2	1.7865e-2	8.9528e-3
p_2^N	1.1441	1.0351	9.9389e-1	9.9669e-1	-

Example 1. Consider the following system of singularly perturbed turning point problem

$$\begin{aligned}
\varepsilon u_1''(x) - 2(2x-1)u_1'(x) - 9u_1'(x) + 2u_2(x) = 0, \quad x \in (0,1),\\
\varepsilon u_2''(x) - 4(2x-1)u_2'(x) - 6u_2'(x) + u_1(x) = 0, \quad x \in (0,1),\\
u_1(0) = 1, \quad u_2(0) = 1, \quad u_1(1) = 1, \quad u_2(1) = 1.
\end{aligned}$$

The computed maximum pointwise errors D_1^N, D_2^N and the computed order of convergence p_1^N, p_2^N for the Example 1 are presented in Tables 1 and 2. The graph of the numerical solution is given in Figure 7, which shows the layer occur at both the end points.The loglog plot of the maximum pointwise errors for U_1 and U_2 are given in Figure 2, which validate the theoretical error bound given in Theorem 6.5.

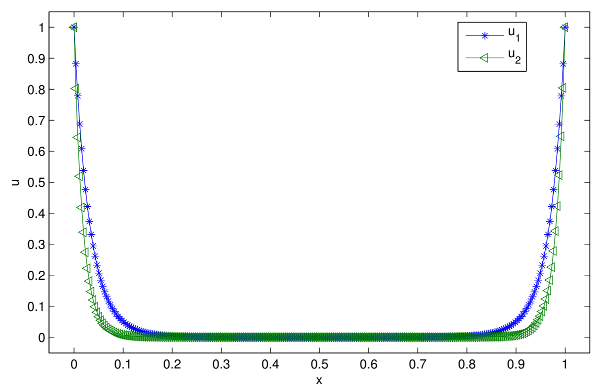

Figure 1: Solution graph of Example 1 for $\varepsilon = 2^{-4}$ and $N = 2^7$.

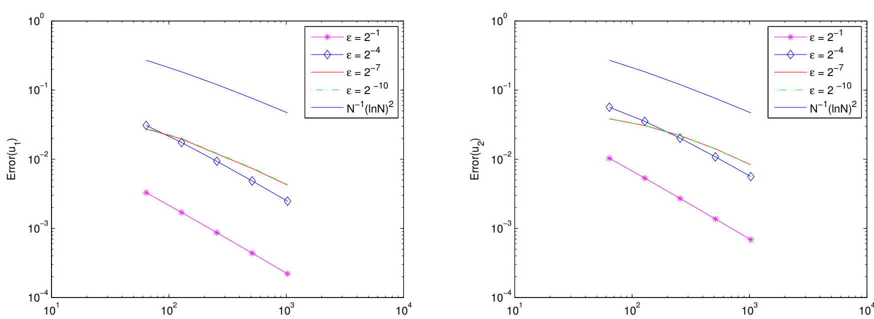

Figure 2: Maximum pointwise errors as a function of N and ε for the solution U_1 and U_2 for Example 1.

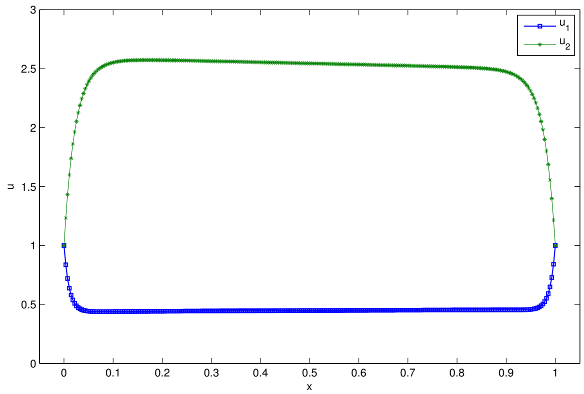

Figure 3: Solution graph of Example 2 for $\varepsilon = 2^{-4}$ and $N = 2^7$

Example 2. Consider the following system of singularly perturbed turning point problem

$$\begin{aligned} \varepsilon u_1''(x) - 7(2x-1)u_1'(x) - 10u_1'(x) + 2u_2(x) &= -e^{-x}, \quad x \in (0,1), \\ \varepsilon u_2''(x) - 3(2x-1)u_2'(x) - 7u_2'(x) + 3u_1(x) &= x+5, \quad x \in (0,1), \\ u_1(0) = 1, \quad u_2(0) = 2, \quad u_1(1) = 1, \quad u_2(1) &= 2. \end{aligned}$$

The computed maximum pointwise errors D_1^N, D_2^N and the computed order of convergence p_1^N, p_2^N for the Example 2 are presented in Tables 3 and 4. Figure 3 represents the numerical solution of Example 2. The loglog plot of the maximum pointwise errors for U_1 and U_2 are given in Figure 4.

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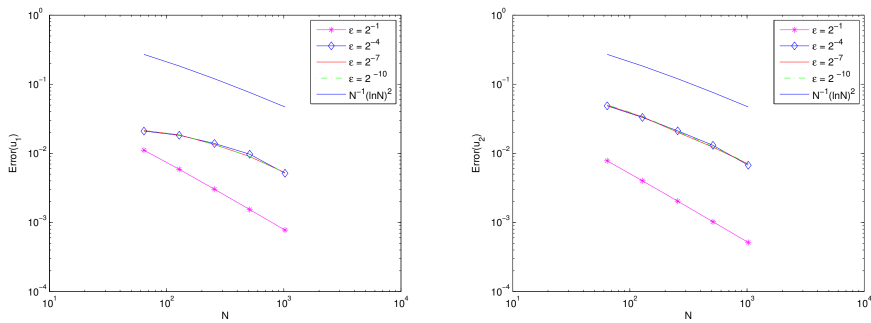

Figure 4: Maximum pointwise errors as a function of N and ε for the solution U_1 and U_2 for Example 2

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