Journal of Mathematical Modeling
Vol. 4, No. 2, 2016, pp. 171-185

JMM

Mathematical analysis and pricing of the European continuous installment call option

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Abstract. In this paper we consider the European continuous installment call option. Then its linear complementarity formulation is given. Writing the resulted problem in variational form, we prove the existence and uniqueness of its weak solution. Finally finite element method is applied to price the European continuous installment call option.

Keywords: installment option, Black-Scholes model, free boundary problem, variational inequality, finite element method.
AMS Subject Classification: 91Gxx, 91G20, 91G80, 35R35.

1 Introduction

One of the important instruments in the market of financial derivatives is option. In the recent years, the role and the complexity of financial contracts have grown tremendously, causing a dramatic change in the financial industry. Issuers, investors, and government regulators have increased their reliance on derivative instruments to augment the liquidity of markets, to reallocate financial risks among market participants, and to take advantage of differences in costs and returns between these markets. One of these instruments is installment derivatives which have two important features differentiating them from other types of derivatives: the premium is paid periodically at pre-specified dates, and the holder has the right

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Received: 10 January 2016 / Revised: 24 September 2016 / Accepted: 17 October 2016.

to stop making the payments, thereby terminating the contract. Installment options introduce flexibility in the liquidity management of portfolio strategies. Instead of paying a lump sum for a derivative instrument, the holder of the installment option will pay the installments as long as the need for being long in the option is present. After the payment of all of the installments, installment option will become a vanilla option. A number of other contracts can be considered as installment options such as: some life insurance contracts and capital investment projects [13], [7], installment warrant [3], [5], some contracts in pharmacy [14] and employee stock options [12].

Discrete installment options are investigated in [2]. Whereas in the case of continuous installment options a few work exist. Alobidi used the integral transform to price European continuous installment options [1]. Kimura investigated European continuous installment options using Laplace-Carson transform [11]. Closed form solutions for pricing European continuous installment options are determined by Kimura [11].

In this paper, we will consider the European continuous installment call option. To describe the share price, Black-Scholes model is applied. None of the mentioned papers have discussed existence and uniqueness of the solution of installment option pricing problem. To do this, we have introduced the complementarity problem formulation for instalment option and have presented, for the first time, the appropriate spaces for the weak solution of installment option. In continuation, the existence and uniqueness of the weak solution for the free boundary problem resulted from the modeling stage are proved. Using finite element method, we get the price of the mentioned option.

The rest of the paper is organized as follows: Section 2 presents the modeling of European continuous installment option under Black-Scholes model. In Section 3, the linear complementarity formulation of the European continuous installment call option under the Black-Scholes underlying asset model will be given. Section 4, will present the variational inequality formulation of call option and existence and uniqueness of the variational problem. In Section 5, the valuation of European installment call option will be discussed using finite element method. Numerical solution of the variational problems will be presented in Section 6. At last, Section 7 is devoted to the result analysis of the mentioned problem.

2 The model

Setting up a portfolio Π_t consisting of a European continuous installment option and Δ units of underlying asset, we get

$$\Pi_t = V(S_t, t; q) - \Delta S_t, \tag{1}$$

where $V(S_t, t; q)$ is the value of the European continuous installment option and q is the rate of installment that must be paid per unit time continuously and S_t is the value of underlying asset evolving according to the following stochastic differential equation, called Black-Scholes model,

$$dS_t = (r - \delta)S_t dt + \sigma S_t dW_t, \tag{2}$$

in which r is the interest rate, δ is the dividend yield, σ is a positive constant called volatility and W_t is a one dimensional Wienner process.

The dynamic of the portfolio Π_t is given by

$$d\Pi_t = dV(S_t, t; q) - \Delta dS_t - \Delta(S_t \delta dt). \tag{3}$$

Applying Ito's lemma to $V(S_t, t; q)$ yields

$$dV = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + (r - \delta)S_t \frac{\partial V}{\partial S_t} - q)dt + \sigma S_t \frac{\partial V}{\partial S_t} dW_t. \tag{4}$$

Substituting from (2) and (4) into (3), one can get

$$\begin{aligned} d\Pi_t = (&\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + (r-\delta)S_t \frac{\partial V}{\partial S_t} + (r - \delta)S_t(\frac{\partial V}{\partial S_t} - \Delta) \\ &- \delta \Delta S_t - q)dt + \sigma S_t(\frac{\partial V}{\partial S_t} - \Delta)dW_t. \end{aligned} \tag{5}$$

To avoid arbitrage opportunities, the portfolio must satisfy $d\Pi_t = r\Pi_t dt$. On the other hand the portfolio must be riskless $\frac{\partial V}{\partial S_t} = \Delta$. Substituting from these relations into (5), we obtain

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + (r - \delta)S_t \frac{\partial V}{\partial S_t} - rV = q. \tag{6}$$

The only difference between the above partial differential equation (PDE) and the one arising from the modeling of European vanilla option is the nonhomogeneous term q which is the rate of installment.

3 Linear complementarity formulation

Let $c(S_t, t; q)$ be the value of the European installment call option with the maturity T, the exercise price K and the payoff function $\max(S_T - K, 0)$. In this case an optimal stopping problem arises because of the opportunity to terminate the contract at any time $t \in [0, T]$. Hence, one should find such points (S_t, t) that optimally terminates the contract. The value of call option can be computed as the solution of the following optimal time stopping problem [11]

$$\begin{aligned} c(S_t, t; q) \; = \; & esssup_{\tau \in [t,T]} E[\chi_{\{\tau \geq T\}} e^{-r(T-t)} max(S_T - K, 0) \\ & - \frac{q}{r}(1 - e^{-r(\tau \wedge T - t)}) | \mathcal{F}_t], \end{aligned} \tag{7}$$

where $\tau \wedge T = \min(\tau, T)$ and $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}_t, \mathbf{P})$ is a filtered probability space and τ is a stopping time of its filtration. The time at which the above relation gets its supremum is called an optimal stopping time $\tau \in [0, T]$. The domain of definition is $\mathcal{D} = [0, T] \times [0, \infty)$. Let us denote the stopping region and the continuation region by $\mathcal{S}$ and $\mathcal{C}$, respectively. Then, the stopping region is

$$\mathcal{S} = \{(S_t, t) \in \mathcal{D} \mid c(S_t, t; q) = 0\}. \tag{8}$$

The optimal stopping time τ^* is characterized by

$$\tau^* = inf\{\tau \in [t, T] | (S_\tau, \tau) \in \mathcal{S}\}. \tag{9}$$

Since the continuation region is the complement of the stopping region in $\mathcal{D}$, it is given by

$$\mathcal{C} = \{(S_t, t) \in \mathcal{D} \mid c(S_t, t; q) > 0\}. \tag{10}$$

The boundary at which the regions $\mathcal{S}$ and $\mathcal{C}$ separated from each other is called stopping boundary

$$S_f(t) = \inf\{S_t \in [0, \infty) \mid c(S_t, t; q) > 0\}, \; t \in [0, T]. \tag{11}$$

The valuation of European call option can be done through the solution of the following inhomogeneous PDE [11]

$$\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 c}{\partial S^2} + (r - \delta) S \frac{\partial c}{\partial S} - rc = q, \tag{12}$$

subject to the terminal condition

$$c(S_T, T; q) = \max(S_T - K), \tag{13}$$

and along with boundary conditions

$$\lim_{S_t \to S_f(t)} c(S_t, t; q) = 0, \lim_{S_t \to S_f(t)} \frac{\partial c}{\partial S} = 0, \lim_{S_t \to \infty} \frac{\partial c}{\partial S} < \infty. \tag{14}$$

Now, we want to reformulate this problem as a linear complementarity problem. When the asset price falls within the stopping region, the European call option should be exercised optimally so that its value is given by $c(S_t, t; q) = 0$. Substituting this relation in the partial differential equation (PDE) (12), one can see that this equation is not satisfied so it is strictly less than q in stopping region. We then conclude that

$$\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 c}{\partial S^2} + (r - \delta) S \frac{\partial c}{\partial S} - rc \leq q, \; S_t \geq 0, \; t \in [0, T]. \tag{15}$$

On the other hand, the European call value is always positive when $S_t > S_f(t)$ and equal to zero when $S_t \leq S_f(t)$, that is,

$$c(S_t, t; q) \geq 0, \; S_t \geq 0, \; t \in [0, T]. \tag{16}$$

Since (S_t, t) is either in the continuation region or stopping region, equality holds in one of the above pair of inequalities. We then deduce that

$$(\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 c}{\partial S^2} + (r - \delta) S \frac{\partial c}{\partial S} - rc - q)c = 0, \; S_t \geq 0, \; t \in [0, T]. \tag{17}$$

Changing the variables by $\tau = T - t$ and $x = \ln S_t$ and setting $u(x, \tau) = c(S_t, t)$, the linear complementarity formulation [10] of the European call is given by

$$\begin{aligned} &\frac{\partial u}{\partial \tau} - \mathcal{A}u \geq l, \; x \in \mathbb{R}, \; \tau \in [0, T] \\ &u(x, \tau; q) \geq 0, \; x \in \mathbb{R}, \; \tau \in [0, T], \\ &(\frac{\partial u}{\partial \tau} - \mathcal{A}u - l)u(x, \tau; q) = 0, \; x \in \mathbb{R}, \; \tau \in [0, T], \end{aligned} \tag{18}$$

where

$$\begin{aligned} \mathcal{A}u &= \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial u}{\partial x} - ru, \\ l(x) &= -q, \; \mu = r - \delta - \frac{1}{2}\sigma^2, \end{aligned} \tag{19}$$

with the initial condition

$$u(x, 0; q) = g(x), \tag{20}$$

where

$$g(x) = \max(e^x - K, 0). \tag{21}$$

4 Existence and uniqueness

In this section, we derive the variational formulation to (18). We observe that the pay-off function (13) or initial condition (20) does not belong to $L^2(\mathbb{R})$. Moreover, since we switched to logarithmic price, this function has an exponential growth at infinity, therefore we cannot use standard Sobolev spaces as function spaces for this problem. We introduce weighted Sobolev spaces to account for the exponential growth of solutions at infinity. Before stating the variational inequality, we introduce some function spaces. Assume that $0 < \nu < \infty$, m is nonnegative integer and $1 \leq p \leq \infty$. We define $W^{m,p,\nu}(\mathbb{R})$ as the set of functions v in $L^p(\mathbb{R}, e^{-\nu|x|}dx)$ whose weak derivatives up to m exist and belong to $L^p(\mathbb{R}, e^{-\nu|x|}dx)$ [10]. By this definition we can write the following weighted space

$$H^m_\nu(\mathbb{R}) = W^{m,2,\nu}(\mathbb{R}). \tag{22}$$

For simplicity, we also set $L^2_\nu(\mathbb{R}) = W^{0,2,\nu}(\mathbb{R})$. The set of admissible solutions for the problem (18) is defined by

$$\mathcal{K}_\nu = \{v \in H^1_\nu(\mathbb{R}) : v \geq 0 \ a.e.\ x \in \mathbb{R}\}. \tag{23}$$

Clearly, this set is convex and closed and $0 \in \mathcal{K}_\nu$. Then variational inequality form of problem (18) is given by

$$\begin{aligned} &\text{Find } u \in L^2(J; H^1_\nu(\mathbb{R})) \cap H^1(J; L^2_\nu(\mathbb{R})) \text{ such that } u(t,.) \in \mathcal{K}_\nu \text{ and,}\\ &(\frac{\partial u}{\partial t}, v-u)_\nu + a_\nu(u, v-u) \geq (l, v-u)_\nu,\ \forall v \in \mathcal{K}_\nu\\ &u(x,0) = u_0(x), \end{aligned} \tag{24}$$

where $u_0(x) = g(x)$ and the bilinear form $a_\nu(.,.) : H^1_\nu(\mathbb{R}) \times H^1_\nu(\mathbb{R}) \to \mathbb{R}$ is given by

$$\begin{aligned} a_\nu(u,v) = &\frac{1}{2}\sigma^2 \int_{\mathbb{R}} \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} e^{-2\nu|x|}dx\\ &- \int_{\mathbb{R}} (\mu + \nu\sigma^2 \text{sign}(x))\frac{\partial u}{\partial x} v e^{-2\nu|x|}dx + r\int_{\mathbb{R}} uv e^{-2\nu|x|}dx, \end{aligned} \tag{25}$$

and $(.,.)_\nu$ is the inner product in $L^2_\nu(\mathbb{R})$ and defined by

$$(u,v)_\nu = \int_{\mathbb{R}} uv e^{-2\nu|x|}dx.$$

Theorem 1. *For $\nu > 0$, the bilinear form $a_\nu(.,.)$ is bounded and satisfies the Garding inequality,i.e.*

$$
\begin{aligned}
|a_\nu(u,v)| &\leq C_1\|u\|_{H^1_\nu(\mathbb{R})}\|v\|_{H^1_\nu(\mathbb{R})}, \forall u, v \in H^1_\nu(\mathbb{R}),\\
a_\nu(v,v) + C_2\|v\|^2_{L^2_\nu(\mathbb{R})} &\geq C_3\|v\|^2_{H^1_\nu(\mathbb{R})}, \forall v \in H^1_\nu(\mathbb{R}),
\end{aligned} \tag{26}
$$

where $C_i > 0$, $i = 1, 2, 3$ are constants.

Proof. First we prove the boundedness of the bilinear form $a_\nu(.,.)$. Let u, $v \in H^1_\nu(\mathbb{R})$. Using the Cauchy-Schawrz inequality, one can easily get

$$
\begin{aligned}
|a_\nu(u,v)| \leq & \frac{\sigma^2}{2}\|\frac{\partial u}{\partial x}\|_{L^2_\nu(\mathbb{R})}\|\frac{\partial v}{\partial x}\|_{L^2_\nu(\mathbb{R})} + C\|\frac{\partial u}{\partial x}\|_{L^2_\nu(\mathbb{R})}\|v\|_{L^2_\nu(\mathbb{R})}\\
&+ r\|u\|_{L^2_\nu(\mathbb{R})}\|v\|_{L^2_\nu(\mathbb{R})}\\
&\leq C_1\|u\|_{H^1_\nu(\mathbb{R})}\|v\|_{H^1_\nu(\mathbb{R})}.
\end{aligned} \tag{27}
$$

This proves the boundedness. To prove that the bilinear form $a_\nu(.,)$ satisfies Garding inequality, we consider $v \in H^1_\nu(\mathbb{R})$

$$
a_\nu(v,v) \geq \frac{\sigma^2}{2}\|\frac{\partial v}{\partial x}\|^2_{L^2_\nu(\mathbb{R})} - \beta\|\frac{\partial v}{\partial x}\|_{L^2_\nu(\mathbb{R})}\|v\|_{L^2_\nu(\mathbb{R})} + r\|v\|^2_{L^2_\nu(\mathbb{R})}. \tag{28}
$$

Using the inequality

$$
ab \leq \frac{1}{2\epsilon^2}a^2 + \frac{1}{2}\epsilon^2 b^2,\ \epsilon > 0, \tag{29}
$$

we can obtain that

$$
a_\nu(v,v) + C_2\|v\|^2_{L^2_\nu(\mathbb{R})} \geq C_3\|v\|^2_{H^1_\nu(\mathbb{R})}. \tag{30}
$$

□

At this moment we will show that the variational problem (24) admits a unique weak solution.

Theorem 2. *The variational problem* (24) *has a unique weak solution.*

Proof. Let $\mathcal{V} = H^1_\nu(\mathbb{R})$ and $\mathcal{H} = L^2_\nu(\mathbb{R})$. Clearly, we have $u_0 = g(x) \in \mathcal{H}$ and $l = -q \in L^2(J; \mathcal{V}^\star)$. Using [9, Theorem B.2.2], one can deduce that the variational inequality (24) admits a unique weak solution u for every $(u_0, l) \in \mathcal{H} \times L^2(J; \mathcal{V}^\star)$, where $\mathcal{V}^\star$ denotes the dual space of $\mathcal{V}$. □

5 Valuation

In order to compute the price of European continuous installment call option using finite element method, we must localize the problem (18) in a bounded domain then reformulate it as a variational inequality. Let R_1, $R_2 \in \mathbb{R}$ with $R_1 < R_2$ and set $\Omega = (R_1, R_2)$. Moreover assume that $\overline{u}(x,t) = u(x,\tau;q)$, $t = \tau$. Now, we consider the new linear complementarity problem

$$\begin{aligned}
&\frac{\partial \overline{u}}{\partial t} - \mathcal{A}\overline{u} \geq l,\ x \in \Omega,\ t \in [0,T] \\
&\overline{u}(x,t) \geq 0,\ x \in \Omega,\ t \in [0,T], \\
&(\frac{\partial \overline{u}}{\partial t} - \mathcal{A}\overline{u} - l)\overline{u}(x,t) = 0,\ x \in \Omega,\ t \in [0,T],
\end{aligned} \tag{31}$$

with initial condition (20) and boundary conditions

$$\overline{u}(x,t) = g(x),\ x \in \partial\Omega. \tag{32}$$

Let φ be a C^2 function with $\varphi = g$ in an open neibghorhood of $\partial\Omega$. Setting $u = \overline{u} - \varphi$ yields, for $x \in \Omega$ and $t \in [0,T]$, the following LCP

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \mathcal{A}u \geq f, \\
&u \geq -\varphi, \\
&(\frac{\partial u}{\partial t} - \mathcal{A}u - f)(u + \varphi) = 0,
\end{aligned} \tag{33}$$

with the initial and null Dirichlet boundary conditions

$$u(x,0) = u_0(x),\ x \in \Omega, \tag{34}$$

$$u(x,t) = 0,\ x \in \partial\Omega,\ t \in [0,T], \tag{35}$$

where $f = l - \mathcal{A}\varphi$ and $u_0(x) = g(x) - \varphi(x)$. To form the variational inequality problem for the above LCP, one needs to write the admissible set of solutions. Assume that this set is given by

$$\mathcal{K}_\varphi = \{v \in H_0^1(\Omega) : v \geq -\varphi\ a.e.\ x \in \Omega\}. \tag{36}$$

By multiplying the relation (33) by $\phi \in H_0^1(\Omega)$ and integrating over Ω, one can write the variational inequality problem for European installment call option as follows

$$\begin{aligned}
&\text{Find } u \in L^2(J; H_0^1(\Omega)) \cap H^1(J; L^2(\Omega)) \text{ such that } u(t,.) \in \mathcal{K}_\varphi \text{ and,} \\
&(\frac{\partial u}{\partial t}, v - u) + a(u, v - u) \geq (f, v - u),\ \forall v \in \mathcal{K}_\varphi, \\
&u(x,0) = u_0(x),
\end{aligned} \tag{37}$$

where the bilinear form $a(.,.): H^1(\Omega)\times H^1(\Omega)\to\mathbb{R}$ is given by

$$a(u,v)=\frac{1}{2}\sigma^2\int_\Omega \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}dx-\mu\int_\Omega \frac{\partial u}{\partial x}vdx+r\int_\Omega uvdx, \tag{38}$$

and $(.,.)$ is the inner product in $L^2(\Omega)$ and is given by $(u,v)=\int_\Omega uvdx$. Setting $\mathcal{V}=H_0^1(\Omega)$ and $\mathcal{H}=L^2(\Omega)$ and using a similar argument as in previous section, one can prove that the variational inequality problem (37) has a unique weak solution.

In the next step, we want to solve the mentioned variational problem (37) numerically using the finite element method. Let $N>0$ be integer and consider $h=\frac{R_2-R_1}{N+1}$ as the step length. We discretize the domain Ω by step length h into $N+1$ subintervals. Assuming that $\phi(x)=(x+1)\chi_{\{-1\le x\le 0\}}+(1-x)\chi_{\{0<x\le 1\}}$, one can define the basis functions as $\phi_{h,i}(x)=\phi(\frac{x-x_i}{h})$, $1\le i\le N$. Then the finite element space can be constructed as $V_h=Span(\phi_{h,1},\phi_{h,2},\cdots,\phi_{h,N})$. Therefore the approximate solution is defined by

$$u_h(x,t)=\sum_{i=1}^{N}u_i(t)\phi_{h,i}(x),\ t\in[0,T], \tag{39}$$

subsequently approximated set of admissible solutions is given by

$$\mathcal{K}_h=\{u_h\in V_h: u_i(t)\ge -\varphi(x_i),\ 1\le i\le N\}. \tag{40}$$

Let $u_h,v_h\in\mathcal{K}_h$ with $\mathbf{u}(t)=(u_1(t),u_2(t),\cdots,u_N(t))^T$ and $\mathbf{v}(t)=(v_1(t),v_2(t),\cdots,v_N(t))^T$ as their coefficient vectors. Substituting these functions in (37) yields

$$\begin{aligned}&\text{Find } u_h\in C^1(J;V_h) \text{ such that } u(t,.)\in\mathcal{K}_h,\\ &(\frac{\partial u_h}{\partial t},v_h-u_h)+a(u_h,v_h-u_h)\ge(f,v_h-u_h),\ \forall v_h\in\mathcal{K}_h,\\ &u_h(x,0)=u_0(x),\end{aligned} \tag{41}$$

where $J=[0,T]$. This, in turn, gives

$$(\mathbf{v}-\mathbf{u}(t))^T[\mathbf{M}\dot{\mathbf{u}}(t)+\mathbf{A}\mathbf{u}(t)+\mathbf{f}]\ge 0,\ \forall\mathbf{v}\ge-\varphi_h, \tag{42}$$

where $\varphi_h=(\varphi(x_1),\varphi(x_2),\cdots,\varphi(x_N))$, $\mathbf{M}=(\mathbf{M}_{ij})$ with $\mathbf{M}_{ij}=(\phi_{h,j},\phi_{h,i})$ is the mass matrix, $\mathbf{A}=(\mathbf{A}_{ij})$ with $\mathbf{A}_{ij}=a(\phi_{h,j},\phi_{h,i})$ is the stiffness matrix and $\mathbf{f}=(f_1,f_2,\cdots,f_M)^T$ with $f_i=(q,\phi_{h,i})+a(\varphi,\phi_{h,i})$ is the load vector. Note that $\mathbf{M}$ and $\mathbf{A}$ are tridiagonal matrices and are given by

$$\mathbf{M} = \frac{h}{6}\begin{pmatrix} 4 & 1 & & \\ 1 & 4 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 4 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} \beta & \gamma & & \\ \alpha & \beta & \ddots & \\ & \ddots & \ddots & \gamma \\ & & \alpha & \beta \end{pmatrix},$$

where

$$\alpha = a(\phi_{h,i}, \phi_{h,i+1}) = \frac{1}{2}\mu + \frac{1}{6}rh - \frac{1}{2h}\sigma^2,$$
$$\beta = a(\phi_{h,i}, \phi_{h,i+1}) = \frac{2}{3}rh + \frac{1}{h}\sigma^2,$$
$$\gamma = a(\phi_{h,i}, \phi_{h,i-1}) = -\frac{1}{2}\mu + \frac{1}{6}rh - \frac{1}{2h}\sigma^2.$$

Let $M > 0$ be an integer number and set $k = \frac{T}{M+1}$. Using k as time step to discretize the interval $[0, T]$ and applying θ-scheme to the relation (42), one can get

$$(\mathbf{v} - \mathbf{u}^{j+1})^T[(\mathbf{M} + k\theta\mathbf{A})\mathbf{u}^{j+1} - (\mathbf{M} - k(1-\theta)\mathbf{A})\mathbf{u}^j + k\mathbf{b}] \geq 0, \ \forall \mathbf{v} \geq -\varphi_h, \quad (43)$$

$$\mathbf{u}^0 = u_0, \ 0 \leq j \leq M, \quad (44)$$

where $\mathbf{u}^j = \mathbf{u}(t_j)$ and $u_0 = (u_0(x_1), u_0(x_2), \cdots, u_0(x_N))$. This is equivalent to [10]

$$(\mathbf{M} + k\theta\mathbf{A})\mathbf{u}^{j+1} - (\mathbf{M} - k(1-\theta)\mathbf{A})\mathbf{u}^j + k\mathbf{b} \geq 0, \quad (45)$$

$$\mathbf{u}^{j+1} \geq -\varphi_h, \quad (46)$$

$$[(\mathbf{M} + k\theta\mathbf{A})\mathbf{u}^{j+1} - (\mathbf{M} - k(1-\theta)\mathbf{A})\mathbf{u}^j + k\mathbf{b}](\mathbf{u}^{j+1} + \varphi_h) = 0, \quad (47)$$

Thus, for any j, we have an LCP whose general form is

$$\begin{cases} \mathbf{A}\mathbf{x} - \mathbf{b} \geq 0, \\ \mathbf{x} \geq \mathbf{c}, \\ (\mathbf{A}\mathbf{x} - \mathbf{b}, \mathbf{x} - \mathbf{c}) = 0, \end{cases} \quad (48)$$

where the $m \times m$ matrix $\mathbf{A}$ and the m-vectors $\mathbf{b}$ and $\mathbf{c}$ are constant, and $\mathbf{x}$ is the vector of unknowns. To compute the solution, projected successive over relaxation method (PSOR) will be applied. In the context of option pricing problems resulting to LCP, the most popular method is the projected SOR method [4, 15]. For other methods in this relation see [4, 15]. Description of this algorithm (Algorithm 1) is as follows

Algorithm 1: Projected SOR Method

1. Choose an initial guess $\mathbf{x}^0$.
2. Choose $\omega \in (0,1]$ and $\epsilon > 0$.
3. For $k = 0, 1, 2, \ldots,$ Do
4. For $i = 1, 2, \ldots, M$ Do
5. $\widetilde{\mathbf{x}}_i^{k+1} := \frac{1}{\mathbf{A}_{ii}}(\mathbf{b}_i - \sum_{j<i} \mathbf{A}_{ij}\mathbf{x}_j^{k+1} - \sum_{j\geq i} \mathbf{A}_{ij}\mathbf{x}_j^k))$
6. $\mathbf{x}_i^{k+1} := \max(\mathbf{c}_i, \mathbf{x}_i^k + \omega(\widetilde{\mathbf{x}}_i^{k+1} - \mathbf{x}_i^k))$
7. EndDo
8. If $\|\mathbf{x}^{k+1} - \mathbf{x}^k\|_2 < \epsilon$ stop else
9. EndDo

6 Numerical experiments

This section deals with the report of the numerical results related to pricing European continuous installment call option under Black-Scholes model. To implement the finite element method, one needs to determine the values of the parameters of the problem. Let these values be given as in the following table

Table 1: The values of the parameters

Parameter	K	T	σ	r	δ
Value	100	0.25,1	0.2,0.3	0.03	0.05

At this time we choose $\varphi(x) = u_0(x)$. By this choice the initial condition (34) becomes null. In the next step we will use the θ-scheme in the implicit case $\theta = 1$. Substituting this value of θ in (45) yields

$$Z\mathbf{u}^{j+1} - Y_j \geq 0, \tag{49}$$

$$\mathbf{u}^{j+1} \geq -u_0, \tag{50}$$

$$(Z\mathbf{u}^{j+1} - Y_j, \mathbf{u}^{j+1} + u_0) = 0, \tag{51}$$

where

$$Z = \mathbf{M} + k\mathbf{A}, \; Y_j = \mathbf{M}\mathbf{u}^j - k\mathbf{b} \geq 0. \tag{52}$$

Since $u_0 \in H^1(\Omega)$, $a(u, \phi_{h,i})$ is well-defined. Therefore, using integration by parts, for $\phi \in H_0^1(\Omega)$, one can get

$$a(u_0, \phi) = -\frac{1}{2}K\sigma^2\phi(\ln K) + \delta \int_{\ln K}^{R_2} e^x \phi dx + rK \int_{\ln K}^{R_2} \phi dx, \qquad (53)$$

which defines a functional in $H^{-1}(\Omega)$, dual space of $H_0^1(\Omega)$. The ends of the computational domain Ω are chosen as $R_1 = \frac{K}{2}$ and $R_2 = 2K$, where K is strike price. To solve a LCP using PSOR method, an initial guess is required. In this paper, we set the initial time step zero. Subsequently, in the j^{th} time step, the solution $\mathbf{u}^{j-1}$, obtained in the $(j-1)^{th}$ time step, is used as initial guess for $\mathbf{u}^j$. Since PSOR is an iterative scheme, a stopping criteria is required. When two successive iterations satisfy $\|\mathbf{x}^{k+1} - \mathbf{x}^k\| \leq \epsilon$, for $\epsilon = 10^{-9}$, the PSOR algorithm is terminated. Also relaxation parameter for PSOR algorithm is chosen as $\omega = 0.5$. Setting $N + 1 = 1000$ and $M + 1 = 200$, we discretize the interval $\Omega = (R_1, R_2)$ and the interval $[0, T]$ by step length $h = 0.5$ and time step $k = 0.05$, respectively. At this moment one can run PSOR method to get the fair price of the European continuous installment call option. Choosing some stock price S_t, some installment rate q and applying PSOR algorithm to solve the sequence of LCP in (49), we obtain the numerical result shown in Tables 2-5. For the prices in these tables, two values are chosen for both volatility $\sigma \in \{0.2, 0.3\}$ and maturity $T \in \{0.25, 1\}$.

Table 2: Installment call option prices for $\sigma = 0.2$ and $T = 0.25$

q	S_0	Price
1	95	0.5124
	105	5.0859
	115	11.6488
3	95	0.3119
	105	4.4296
	115	10.9597
6	95	0.5832
	105	3.8679
	115	9.3643

To investigate the effect of installment rate q, non-homogeneous term in PDE (12), we have chosen some value of this parameter. Comparing the prices with the same stock price S_0 and different installment rate q

Table 3: Installment call option prices $\sigma = 0.2$ and $T = 1$

q	S_0	Price
1	95	3.7069
	105	8.3989
	115	14.8534
3	95	2.2278
	105	6.6390
	115	12.9679
6	95	0.6760
	105	4.2751
	115	10.2529

Table 4: Installment call option prices for $\sigma = 0.3$ and $T = 0.25$

q	S_0	Price
1	95	2.3563
	105	7.0561
	115	13.5534
3	95	2.1369
	105	6.4593
	115	12.8482
6	95	2.1458
	105	5.7576
	115	11.2566

shows that the increase in installment rate causes the decrease in the value of the European continuous installment call option. As installment rate q tends to zero the installment option price approaches the price of its counterpart vanilla option. On the other hand, by Table 2, decreasing the rate of installment q increases the price of installment call option. This proves that the premium of the vanilla option is greater than the premium of the installment option.

Table 5: Installment call option prices $\sigma = 0.3$ and $T = 1$

q	S_0	Price
1	95	7.4035
	105	12.1847
	115	18.6498
3	95	5.8456
	105	10.3410
	115	16.7589
6	95	6.1711
	105	4.2751
	115	13.9618

7 Conclusion

We formulated the European Continuous installment call option in the complementarity form and proved existence and uniqueness of its weak solution using variational inequality form. In conclusion we believe that this approach gives a simple and straightforward framework to survey the problems arisen in option pricing in mathematical finance.

As future research, one can generalize the underlying asset model to other stochastic processes such as local volatility models, stochastic volatility models and jump-diffusion models. It is also possible to consider exotic options whose premiums are paid by a sequence of installment (discrete or continuous) but note that their boundary conditions may changed.

Acknowledgements

We are very grateful to the reviewers for their useful comments and suggestions that led to improvement of our manuscript.

References

[1] G. Alobaidi, R. Mallier and S. Deakin, *Laplace transforms and installment options*, Math. Models Methods Appl. Sci. **14** (2004) 1167–1189.

[2] G. Alobaidi and R. Mallier, *Installment options close to expiry*, J. Appl. Math. Stoch. Anal. Article ID 60824, **2006**, (2006) 1–9.

[3] H. Ben-Ameur, M. Breton and P. Francois, *Pricing ASX installment warrants under GARCH*, Working Paper G-2005-42, GERAD, 2005.

[4] R.W. Cottle, J.S. Pang and R.E. Stone, *The Linear Complementarity Problem*, Academic Press, London, 1992.

[5] M. Davis, W. Schachermayer and R. Tompkins, *Pricing, no-arbitrage bounds and robust hedging of installment options*, Quant. Finance **1** (2001) 597–610.

[6] M. Davis, W. Schachermayer and R. Tompkins, *Installment options and static hedging*, J. Risk Finance **3** (2002) 46–52.

[7] A.K. Dixit and R.S. Pindyck, *Investment Under Uncertainty*, Princeton University Press, New Jersey, 1994.

[8] L. Feng and X. Lin, *Pricing Bermudan options in Levy process models*, Working paper University of Illinois at Urbana-Champaign, 2009.

[9] N. Hilber, O. Reichmann, C. Schwab and C. Winter, *Computational Methods for Quantitative Finance*, Springer-Verlag, Berlin, 2013.

[10] P. Jaillet, D. Lamberton and B. Lapeyre, *Variational inequalities and the pricing of American options*, Acta Appl. Math. (1990) 263–289.

[11] T. Kimura, *Valuing continuous-installment options*, European J. Oper. Res. **201** (2010) 222–230.

[12] C.D. MacRae, *The Employee Stock Option: An Installment Option*, Available at SSRN: http://ssrn.com/abstract=1286928, 2008.

[13] S. Majd and R.S. Pindyck, *Time to build, option value, and investment decisions*, J. Financ. Econ. **18** (1978) 7–27.

[14] L. Thomassen and M. Van Wouwe, *The Influence of a Stochastic Interest Rate on the n- fold Compound option*, Part of the series Stat. Ind. Technol., pp. 343–353, Birkhauser Boston, Boston, 2004.

[15] S.J. Wright, *Primal-Dual Interior-Point Methods*, SIAM, Philadelphia, 1997.