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# Degenerate kernel approximation method for solving Hammerstein system of Fredholm integral equations of the second kind

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**Abstract.** Degenerate kernel approximation method is generalized to solve Hammerstein system of Fredholm integral equations of the second kind. This method approximates the system of integral equations by constructing degenerate kernel approximations and then the problem is reduced to the solution of a system of algebraic equations. Convergence analysis is investigated and on some test problems, the proposed method is examined.

*Keywords*: systems of nonlinear integral equations, degenerate kernel, Taylor-series expansion, nonlinear equations.
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## 1 Introduction

We consider the numerical solution of Hammerstein system of Fredholm integral equations (HSFIEs) of the second kind:

$$x_i(t) = f_i(t) + \lambda \sum_{j=1}^{m} \int_0^1 K_{ij}(t,s)\psi_{ij}(s, x_j(s))ds, \quad i = 1, \ldots, m, \qquad (1)$$

where $0 \le s, t \le 1$, $f_i$, $K_{ij}$ and $\psi_{ij}$, $i, j = 1, \ldots, m$, are given and $x_j(t)$, $j = 1, \ldots, m$ are the solution to be determined. For more details about

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such problems and solution approach, we refer to [2, 6, 7] and references therein. Solution of Eq. (1) is easily reduced to the solution of a system of $m^2 n$ algebraic equations, if the kernels are in the form

$$K_{ij}(t,s) = \sum_{\kappa=1}^{n} a_{ij}^{\kappa}(t) b_{ij}^{\kappa}(s), \qquad i,j = 1,\ldots,m,$$

called separable or degenerate kernels [7]. The idea behind the degenerate kernel approximation method is to replace the given kernels by degenerate kernels. This method was introduced and analyzed by Kanekoand and Xu in [3] for a single Hammerstein integral equation. The goal of this paper is to generalize this method and its convergence results [3] to HSFIEs (1). We define the Hilbert space $H$ and its inner product as follows

$$H = \underbrace{L^2[0,1] \times L^2[0,1] \times \ldots \times L^2[0,1]}_{m \text{ times}},$$

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^1 \mathbf{f}(t)^{\mathcal{H}} \mathbf{g}(t) dt, \qquad \mathbf{f}, \mathbf{g} \in H,$$

where $L^2[0,1]$ is the vector space of all functions $u : [0,1] \to \mathbb{C}$ satisfying

$$\int_a^b |u(t)|^2 dt < \infty,$$

and $\mathbf{f}(t)^{\mathcal{H}}$ denotes the conjugate transpose of $\mathbf{f}(t)$. We denote by $\|.\|$ the associated norm of the inner product $\langle ., . \rangle$. Moreover, we define the operator $T : H \to H$ by

$$(T\boldsymbol{x})_i(t) = f_i(t) + \lambda \sum_{j=1}^{m} \int_0^1 K_{ij}(t,s) \psi_{ij}(s, x_j(s)) ds, \qquad i = 1,\ldots,m,$$

where $\boldsymbol{x} = [x_1 \ x_2 \ldots x_m]^T$, so that a solution of (1) is a fixed point of the operator $T$. Therefore to establish existence and uniqueness theorem for Eq.(1), it is enough to examine a number of conditions under which we can guarantee the existence and uniqueness of the fixed point for the operator $T$. This leads to the following theorem.

**Theorem 1.** *Let the individual kernels $K_{ij}(t,s)$, $i,j = 1,\ldots,m$ are continuous for all $t,s \in [0,1]$, $\psi_{ij}(t,s)$, $i,j = 1,\ldots,m$ are continuous for all $t \in [0,1]$ and all $s$ and*

$$\int_0^1 |\psi_{ij}(t, u(t))|^2 dt \le A_{ij}^2 \|u\|^2.$$

*Moreover, suppose that $\psi_{ij}(t,s)$, $i,j=1,\ldots,m$ satisfy the Lipschitz condition*

$$|\psi_{ij}(t,s_1)-\psi_{ij}(t,s_2)|\le B_{ij}|s_1-s_2|, \qquad i,j=1,\ldots,m, \tag{2}$$

*where $B_{ij}$, $i,j=1,\ldots,m$ are independent of $t$, and let there exist constants $C_{ij}$, $i,j=1,\ldots,m$ which*

$$|K_{ij}(t,s)|\le C_{ij}, \qquad i,j=1,\ldots,m. \tag{3}$$

*Then (1) has a unique solution in $H$ provided that*

$$|\lambda|<\frac{1}{\sigma}, \qquad \sigma=\sqrt{\sum_{i=1}^{m}B^{(i)}C^{(i)}},$$

*where $B^{(i)}=\sum\limits_{j=1}^{m}B_{ij}^2$ and $C^{(i)}=\sum\limits_{j=1}^{m}C_{ij}^2$.*

The proof of this theorem is similar to that of its discretized form given in Theorem 2.

The remainder of the paper is organized as follows: In Section 2, the degenerate kernel approximation method is described to the HSFIEs of the second kind. The convergence analysis of the proposed method is studied in Section 3. Section 4 deals with constructing degenerate kernel approximations. In this section, the Taylor series expansion is used to construct degenerate kernel approximations and its convergence analysis is discussed. The results of numerical experiments on some examples are given in Section 5. Finally in Section 6, a brief conclusion is drawn.

## 2 Description of the method

We assume that we can approximate the kernels in Eq. (1) as follows

$$K_{ij}(t,s)\approx K_{ij}^n(t,s)=\sum_{\kappa=1}^{n}b^{\kappa}(s)a_{ij}^{\kappa}(t), \quad i,j=1,\ldots,m. \tag{4}$$

Replacing this approximations in (1), we get the following system

$$x_i^n(t)=f_i(t)+\lambda\sum_{j=1}^{m}\sum_{\kappa=1}^{n}a_{ij}^{\kappa}(t)\int_0^1 b^{\kappa}(s)\psi_{ij}(s,x_j^n(s))ds, \quad i=1,\ldots,n. \tag{5}$$

Let

$$\beta_{ij}^{\kappa}=\int_0^1 b^{\kappa}(s)\psi_{ij}(s,x_j^n(s))ds, \quad i,j=1,\ldots,m, \quad \kappa=1,\ldots,n, \tag{6}$$

be unknown constants depending on unknown vector function

$$x^n(s) = (x_j^n(s))_{j=1,\ldots,m}.$$

Then Eq. (5) can be rewritten as follows

$$x_i^n(t) = f_i(t) + \lambda \sum_{j=1}^{m} \sum_{\kappa=1}^{n} \beta_{ij}^{\kappa} a_{ij}^{\kappa}(t), \quad i = 1, \ldots, m. \tag{7}$$

Substituting (7) into (6) gives

$$\beta_{ij}^{\kappa} = \int_0^1 b^{\kappa}(s)\psi_{ij}(s, f_j(s) + \lambda \sum_{r=1}^{m} \sum_{p=1}^{n} \beta_{jr}^{p} a_{jr}^{p}(s))ds, \tag{8}$$
$$i, j = 1, \ldots, m, \quad \kappa = 1, \ldots n,$$

which is a nonlinear system of $m^2 n$ algebraic equations that can be solved by well-known methods such as Newton and so on.

Let

$$\boldsymbol{\beta} = [\beta_1 \ \beta_2 \ldots \beta_n]^T, \quad \boldsymbol{F}(\boldsymbol{\beta}) = [F_1(\boldsymbol{\beta}) \ F_2(\boldsymbol{\beta}) \ldots F_n(\boldsymbol{\beta})]^T, \tag{9}$$

where for $\kappa = 1, \ldots n$,

$$\beta_\kappa = [\beta_{11}^{\kappa} \ \beta_{12}^{\kappa} \ldots \beta_{1m}^{\kappa} \ \beta_{21}^{\kappa} \ldots \beta_{2m}^{\kappa} \ldots \beta_{m1}^{\kappa} \ldots \beta_{mm}^{\kappa}], \tag{10}$$
$$F_\kappa(\boldsymbol{\beta}) = [F_{11}^{\kappa} \ F_{12}^{\kappa} \ldots F_{1m}^{\kappa} \ F_{21}^{\kappa} \ldots F_{2m}^{\kappa} \ldots F_{m1}^{\kappa} \ldots F_{mm}^{\kappa}], \tag{11}$$

and

$$F_{ij}^{\kappa} = \int_0^1 b^{\kappa}(s)\psi_{ij}(s, f_j(s) + \lambda \sum_{r=1}^{m} \sum_{p=1}^{n} \beta_{jr}^{p} a_{jr}^{p}(s))ds. \tag{12}$$

As a result the system (8) can be written in general form

$$\boldsymbol{\beta} = \boldsymbol{F}(\boldsymbol{\beta}). \tag{13}$$

Therefore, solving system (8) is equivalent to finding the fixed point of $\boldsymbol{F}$ which is a nonlinear operator from $\mathbb{R}^{m^2 n}$ into $\mathbb{R}^{m^2 n}$. Substituting the solution of this system in (7), the approximate solution of the system of integral equations (1) is obtained. It should be mentioned that often the definite integrals in (8) must be computed numerically. To approximate these integrals, we use some convergent quadrature method

$$\int_0^1 u(t)dt \approx \sum_{l=0}^{N} \omega_l u(t_l), \qquad t_l \in [0,1], \quad l = 0, 1, \ldots, N,$$

where $\omega_l > 0$, $l = 0, 1, \ldots, N$. As a result, Eq. (8) is replaced by

$$\beta_{ij}^{\kappa} = \sum_{l=0}^{N} \omega_l b^{\kappa}(t_l)\psi_{ij}(t_l, f_j(t_l) + \lambda \sum_{r=1}^{m}\sum_{p=1}^{n} \beta_{jr}^{p} a_{jr}^{p}(t_l)),$$
$$i, j = 1, \ldots, m, \quad \kappa = 1, \ldots n.$$

The above discussion leads to a discrete degenerate kernel approximation method.

# 3 Convergence analysis

In this section, we study the convergence of the degenerate kernels approximation method. The following theorem shows that the approximate Eqs. (5) has a unique solution.

**Theorem 2.** *Suppose that the hypotheses of Theorem 1 are fulfilled and let the individual approximate kernels $K_{ij}^n(t,s)$, $i, j = 1, \ldots m$ are continuous for all $t, s \in [0,1]$ and*

$$\left(\int_0^1 \int_0^1 |K_{ij}^n(t,s) - K_{ij}(t,s)|^2 dtds\right)^{\frac{1}{2}} \to 0 \text{ as } n \to \infty, \ i, j = 1, \ldots m. \tag{14}$$

*Then there exists $M \in \mathbb{N}$ such that for each $n \geq M$, Eqs. (5) has a unique solution $\boldsymbol{x}^n \in H$.*

*Proof.* Using (14) and (3), it is easy to show that there exists $M \in \mathbb{N}$ such that for $n \geq M$,

$$\left(\int_0^1 \int_0^1 |K_{ij}^n(t,s)|^2 dtds\right)^{\frac{1}{2}} \leq C_{ij}, \qquad i, j = 1, \ldots m. \tag{15}$$

Now, for $n \geq M$, it is sufficient to show the following operator, which is an approximation of the operator $T$, that a unique fixed point:

$$(\widetilde{T}\boldsymbol{x})_i(t) = f_i(t) + \lambda \sum_{j=1}^{m}\sum_{\kappa=1}^{n} \int_0^1 K_{ij}^n(t,s)\psi_{ij}(s, x_j(s))ds, \qquad i = 1, \ldots, m.$$

Let $\boldsymbol{x}, \boldsymbol{y} \in H$, then

$$\begin{aligned}
\|\widetilde{T}\boldsymbol{x} - \widetilde{T}\boldsymbol{y}\|^2 &= |\lambda|^2 \int_0^1 \sum_{i=1}^m \Big| \sum_{j=1}^m \int_0^1 K_{ij}^n(t,s)(\psi_{ij}(s, x_j(s)) \\
&\qquad\qquad - \psi_{ij}(s, y_j(s)))ds\Big|^2 dt \\
&\le |\lambda|^2 \int_0^1 \sum_{i=1}^m \Big( \int_0^1 \sum_{j=1}^m \Big|K_{ij}^n(t,s)\Big|\Big|\psi_{ij}(s, x_j(s)) \\
&\qquad\qquad - \psi_{ij}(s, y_j(s))\Big| ds\Big)^2 dt \\
&\le |\lambda|^2 \int_0^1 \sum_{i=1}^m \Big( \int_0^1 \sum_{j=1}^m |K_{ij}^n(t,s)|^2 ds\Big) \\
&\qquad \times \Big( \int_0^1 \sum_{j=1}^m |\psi_{ij}(s, x_j(s)) - \psi_{ij}(s, y_j(s))|^2 ds\Big) dt \\
&= |\lambda|^2 \sum_{i=1}^m \Big( \int_0^1 \sum_{j=1}^m |\psi_{ij}(s, x_j(s)) - \psi_{ij}(s, y_j(s))|^2 ds\Big) \\
&\qquad\qquad \times \sum_{j=1}^m \int_0^1 \int_0^1 |K_{ij}^n(t,s)|^2 ds dt \\
&\le |\lambda|^2 \sum_{i=1}^m \Big( \sum_{j=1}^m B_{ij}^2 \int_0^1 |x_j(s) - y_j(s)|^2 ds\Big) \sum_{j=1}^m C_{ij}^2,
\end{aligned}$$

where the second inequality have been obtained from the Cauchy-Schwarz inequality for the vector functions $[|K_{ij}(t,s)|]_{1\le j\le m}$ and $[|\psi_{ij}(t,s)|]_{1\le j\le m}$, $i = 1, \dots, m$ in $H$ and the last inequality is the result of Lipschitz condition (2). On the other hand,

$$\int_0^1 |x_j(s) - y_j(s)|^2 ds \le \int_0^1 \sum_{j=1}^m |x_j(s) - y_j(s)|^2 ds = \|\boldsymbol{x} - \boldsymbol{y}\|^2, \quad j = 1, \dots, m.$$

Therefore

$$\|\widetilde{T}\boldsymbol{x} - \widetilde{T}\boldsymbol{y}\|^2 \le |\lambda|^2 \|\boldsymbol{x} - \boldsymbol{y}\|^2 \sum_{i=1}^m \Big( \sum_{j=1}^m B_{ij}^2 \sum_{j=1}^m D_{ij}^2 \Big) = |\lambda|^2 \sigma^2 \|\boldsymbol{x} - \boldsymbol{y}\|^2.$$

Thus by taking $|\lambda| < \frac{1}{\sigma}$, the operator $\widetilde{T}$ is contraction. □

Uniqueness of the solution of the system of algebraic equations (8) or (13) is the subject of the following theorem.

**Theorem 3.** *Assume that*

$$|\lambda| < \frac{1}{M},$$

*where*

$$M = \Big(\int_0^1 \sum_{\kappa=1}^{n} |b^{\kappa}(s)|^2 ds\Big)^{\frac{1}{2}} \Big(\sum_{i=1}^{m} B^{(i)} \sum_{l=1}^{n}\sum_{r=1}^{m} \int_0^1 |a_{ir}^l(s)|^2 ds\Big)^{\frac{1}{2}}.$$

*Then the system of nonlinear algebraic equations* (13) *has a unique solution.*

*Proof.* It is sufficient to prove that $F$ is a contraction operator. We use the Euclidean norm $\|.\|_2$ on $\mathbb{R}^{m^2 n}$. For $\boldsymbol{\beta} = [\beta_1\ \beta_2 \dots \beta_n]^T$ and $\boldsymbol{\gamma} = [\gamma_1\ \gamma_2 \dots \gamma_n]^T$, where $\beta_\kappa, \gamma_\kappa$, $1 \le \kappa \le n$ are in the form (10), we have

$$\|\boldsymbol{F}(\boldsymbol{\beta}) - \boldsymbol{F}(\boldsymbol{\gamma})\|_2^2 = \sum_{\kappa=1}^{n} \|F_\kappa(\boldsymbol{\beta}) - F_\kappa(\boldsymbol{\gamma})\|_2^2, \tag{16}$$

and by Equations (9)-(12)

$$\begin{aligned}
\|F_\kappa(\boldsymbol{\beta}) - F_\kappa(\boldsymbol{\gamma})\|_2^2 &= \sum_{i=1}^{m}\sum_{j=1}^{m}\Big|\int_0^1 b^\kappa(s)\Big(\psi_{ij}\big(f_j(s) + \lambda \sum_{r=1}^{m}\sum_{l=1}^{n} \beta_{ir}^l a_{ir}^l(s)\big) \\
&\qquad - \psi_{ij}\big(f_j(s) + \lambda \sum_{r=1}^{m}\sum_{l=1}^{n} \gamma_{ir}^l a_{ir}^l(s)\big)\Big) ds\Big|^2 \\
&\le |\lambda|^2 \sum_{i=1}^{m}\sum_{j=1}^{m} B_{ij}^2 \\
&\qquad \times \Big(\int_0^1 \big|b^\kappa(s)\big|\Big|\sum_{r=1}^{m}\sum_{l=1}^{n}(\beta_{ir}^l - \gamma_{ir}^l) a_{ir}^l(s)\Big| ds\Big)^2 \\
&\le |\lambda|^2 \int_0^1 |b^\kappa(s)|^2 ds \sum_{i=1}^{m}\sum_{j=1}^{m} B_{ij}^2 \\
&\qquad \times \int_0^1 \Big(\sum_{r=1}^{m}\sum_{l=1}^{n} |\beta_{ir}^l - \gamma_{ir}^l||a_{ir}^l(s)|\Big)^2 ds \\
&\le |\lambda|^2 \int_0^1 |b^\kappa(s)|^2 ds \sum_{i=1}^{m}\Big(\sum_{j=1}^{m} B_{ij}^2\Big)\Big(\sum_{r=1}^{m}\sum_{l=1}^{n} |\beta_{ir}^l - \gamma_{ir}^l|^2\Big) \\
&\qquad \times \Big(\sum_{r=1}^{m}\sum_{l=1}^{n}\int_0^1 |a_{ir}^l(s)|^2 ds\Big).
\end{aligned}$$

The first inequality have been obtained from the assumption (2). The second and third inequalities are the results of the Schwarz inequality for $\|.\|_{L^2[0,1]}$ and $\|.\|_2$ respectively. Moreover,

$$\sum_{r=1}^{m}\sum_{l=1}^{n} |\beta_{ir}^l - \gamma_{ir}^l|^2 \le \sum_{i=1}^{m}\sum_{r=1}^{m}\sum_{l=1}^{n} |\beta_{ir}^l - \gamma_{ir}^l|^2 = \|\boldsymbol{\beta} - \boldsymbol{\gamma}\|_2^2.$$

Therefore, for $\kappa = 1, \dots, n$,

$$\begin{aligned} \|F_\kappa(\boldsymbol{\beta}) - F_\kappa(\boldsymbol{\gamma})\|_2^2 \le |\lambda|^2 \int_0^1 |b^\kappa(s)|^2 ds \sum_{i=1}^m \left( \sum_{j=1}^m B_{ij}^2 \right) \\ \times \left( \sum_{r=1}^m \sum_{l=1}^n \int_0^1 \left| a_{ir}^l(s) \right|^2 ds \right) \|\boldsymbol{\beta} - \boldsymbol{\gamma}\|_2^2 . \end{aligned} \tag{17}$$

Substituting (17) into (16) gives

$$\|\boldsymbol{F}(\boldsymbol{\beta}) - \boldsymbol{F}(\boldsymbol{\gamma})\|_2^2 \le |\lambda|^2 M^2 \|\boldsymbol{\beta} - \boldsymbol{\gamma}\|_2^2.$$

□

The following theorem shows that the sequence, produced by the degenerate kernel method, converges to the exact solution of Eq. (1).

**Theorem 4.** *Let $\boldsymbol{x}$ and $\boldsymbol{x}^n$ be the solutions of Equations (1) and (5), respectively. Then*

$$\|\boldsymbol{x} - \boldsymbol{x}^n\| \le \frac{|\lambda| \boldsymbol{A} \|\boldsymbol{x}\|}{1 - |\lambda| \mu \|\boldsymbol{x}\|} \boldsymbol{K}, \tag{18}$$

$$\|\boldsymbol{x} - \boldsymbol{x}^n\| \le \frac{|\lambda| \boldsymbol{A} \|\boldsymbol{x}^{\boldsymbol{n}}\|}{1 - |\lambda| \mu \|\boldsymbol{x}^n\|} \boldsymbol{K}, \tag{19}$$

*where*

$$\boldsymbol{K} = \Big( \int_0^1 \int_0^1 \sum_{i=1}^m \sum_{j=1}^m |K_{ij}(t,s) - K_{ij}^n(t,s)|^2 dt ds \Big)^{\frac{1}{2}},$$

*and*

$$\boldsymbol{A} = \sum_{i=1}^m \sum_{j=1}^m A_{ij}^2, \qquad \mu = \sum_{i=1}^m \Big( \sum_{j=1}^m C_{ij}^2 \sum_{j=1}^m B_{ij}^2 \Big)^{\frac{1}{2}}.$$

*As a result, under condition (14), $\boldsymbol{x}^n \to \boldsymbol{x}$ as $n \to \infty$.*

*Proof.* We have

$$\|\boldsymbol{x} - \boldsymbol{x}^n\|^2 = \int_0^1 \sum_{i=1}^m |x_i(t) - x_i^n(t)|^2 dt = \sum_{i=1}^m \|x_i(t) - x_i^n(t)\|_{L^2}^2. \tag{20}$$

In addition

$$
\begin{aligned}
&\|\boldsymbol{x}_i(t)-\boldsymbol{x}_i^n(t)\|_{L^2}^2\\
&\quad\le |\lambda|^2\Big(\Big\|\int_0^1\sum_{j=1}^m \big(K_{ij}(t,s)-K_{ij}^n(t,s)\big)\psi_{ij}(s,x_j(s))ds\Big\|_{L2}\\
&\qquad+\Big\|\int_0^1\sum_{j=1}^m K_{ij}^n(t,s)\big(\psi_{ij}(s,x_j(s))-\psi_{ij}(s,x_j^n(s))\big)\Big\|_{L^2}\Big)^2\\
&\quad= |\lambda|^2\Big(\Big(\int_0^1\Big|\int_0^1\sum_{j=1}^m \big(K_{ij}(t,s)-K_{ij}^n(t,s)\big)\psi_{ij}(s,x_j(s))ds\Big|^2dt\Big)^{\frac12}\\
&\qquad+\Big(\int_0^1\Big|\int_0^1\sum_{j=1}^m K_{ij}^n(t,s)\big(\psi_{ij}(s,x_j(s))-\psi_{ij}(s,x_j^n(s))\big)ds\Big|^2dt\Big)^{\frac12}\Big)^2\\
&\quad\le |\lambda|^2\Big(\Big(\int_0^1\Big(\int_0^1\sum_{j=1}^m\big|K_{ij}(t,s)-K_{ij}^n(t,s)\big|^2ds\Big)\\
&\qquad\times\Big(\int_0^1\sum_{j=1}^m\big|\psi_{ij}(s,x_j(s))\big|^2ds\Big)dt\Big)^{\frac12}\\
&\qquad+\Big(\int_0^1\Big(\int_0^1\sum_{j=1}^m\big|K_{ij}(t,s)\big|^2ds\Big)\Big(\int_0^1\sum_{j=1}^m\big|\psi_{ij}(s,x_j(s))-\psi_{ij}(s,x_j^n(s))\big|^2ds\Big)dt\Big)^{\frac12}\Big)^2\\
&\quad\le |\lambda|^2\Big(\Big(\Big(\sum_{j=1}^m A_{ij}^2\int_0^1|x_j(s)|^2ds\Big)\times\Big(\int_0^1\int_0^1\big|K_{ij}(t,s)-K_{ij}^n(t,s)\big|^2dsdt\Big)\Big)^{\frac12}\\
&\qquad+\Big(\Big(\sum_{j=1}^m C_{ij}^2\Big)\Big(\sum_{j=1}^m B_{ij}^2\int_0^1|x_j(s)-x_j^n(s)|^2ds\Big)\Big)^{\frac12}\Big)^2.
\end{aligned}
$$

By substituting the above result in (20), we have

$$
\begin{aligned}
\|\boldsymbol{x}-\boldsymbol{x}^n\|^2 \le |\lambda|^2\Bigg(\sum_{i=1}^m\Big(&\Big(\Big(\sum_{j=1}^m A_{ij}^2\int_0^1|x_j(s)|^2ds\Big)\\
&\times\Big(\int_0^1\int_0^1\big|K_{ij}(t,s)-K_{ij}^n(t,s)\big|^2dsdt\Big)\Big)^{\frac12}\\
&+\Big(\Big(\sum_{j=1}^m C_{ij}^2\Big)\Big(\sum_{j=1}^m B_{ij}^2\int_0^1|x_j(s)-x_j^n(s)|^2ds\Big)\Big)^{\frac12}\Big)^2\Bigg).
\end{aligned}
$$

Now by using the Cauchy-Schwarz inequality for $\|.\|_2$ and

$$
\int_0^1|x_j(s)|^2ds\le\|\boldsymbol{x}\|^2,\qquad \int_0^1|x_j(s)-x_j^n(s)|^2ds\le\|\boldsymbol{x}-\boldsymbol{x}^n\|^2,
$$

we have

$$\|\boldsymbol{x}-\boldsymbol{x}^n\|^2 \le |\lambda|^2\|\boldsymbol{x}\|^2\Bigg(\Big(\sum_{i=1}^{m}\sum_{j=1}^{m}A_{ij}^2\Big)^{\frac{1}{2}} \times\Big(\int_0^1\int_0^1\sum_{i=1}^{m}\sum_{j=1}^{m}|K_{ij}(t,s)-K_{ij}^n(t,s)|^2dtds\Big)^{\frac{1}{2}} +\|\boldsymbol{x}-\boldsymbol{x}^n\|\sum_{i=1}^{m}\Big(\sum_{j=1}^{m}C_{ij}^2\sum_{j=1}^{m}B_{ij}^2\Big)^{\frac{1}{2}}\Bigg)^2,$$

which proves (18). Eq. (19) can be proved similarly. □

# 4 Construction of degenerate kernel

There are several methods for constructing degenerate kernel approximations. Three of the most important methods are Taylor series expansion, interpolation and orthonormal expansions [1, 4, 5]. Here the Taylor series expansion is used.

## 4.1 Taylor series expansion

We assume that the individual kernels $K_{ij}$, $i,j = 1,2,\ldots,m$ and their derivatives of any order with respect to variable $s$ exist in a neighborhood of a point $c \in [0,1]$. Then by using $n$th truncation of Taylor-series expansion of the individual kernels with respect to $s$ at the point $(t,c)$, we have

$$K_{ij}(t,s) \simeq K_{ij}^n(t,s) = \sum_{\kappa=1}^{n-1}\frac{(s-c)^{\kappa-1}}{(\kappa-1)!}\frac{\partial^{\kappa-1}K_{ij}}{\partial s^{\kappa-1}}(t,c), \qquad i,j=1,\ldots m.$$

By (4) $K_{ij}^n$, $i,j = 1,\ldots m$, are degenerate kernels with

$$b^\kappa(s) = \tfrac{(s-c)^{\kappa-1}}{(\kappa-1)!},\ a_{ij}^\kappa(t) = \tfrac{\partial^{\kappa-1}K_{ij}}{\partial s^{\kappa-1}}(t,c), \quad i,j=1,2,\ldots,m,\ \kappa=1,2,\ldots,n. \tag{21}$$

For the convergence analysis, it is sufficient to show that the condition (14) satisfies. By using Taylor's theorem, we have

$$K_{ij}(t,s) = K_{ij}^n(t,s) + R_{ij}^n(t,s), \qquad i,j=1,2,\ldots,m,$$

where

$$R_{ij}^n(t,s) = \frac{(s-c)^n}{n!}\frac{\partial^n K_{ij}}{\partial s^n}(t,\eta_{ij}^n), \qquad i,j=1,2,\ldots,m,$$

for some $\eta_{ij}^n$ between $s$ and $c$. Moreover, for each $s \in (c-\delta_{ij}, c+\delta_{ij})$,

$$R_{ij}^n(t,s) \to 0 \qquad \text{as} \qquad n \to \infty, \qquad i,j = 1,\dots,m,$$

where $\delta_{ij}$ is convergence radius of Taylor-series of $K_{ij}$ for variable $s$ about $s = c$. In this case

$$K_{ij}^n(t,s) \to K_{ij}(t,s) \qquad \text{as} \qquad n \to \infty, \qquad i,j = 1,\dots,m,$$

which results (14). Moreover, we have the following Proposition from [4].

**Proposition 1.** *Let* $\max\limits_{x\in[0,1]} |\frac{\partial^n K_{ij}}{\partial y^n}|_{(x,\eta_{ij}^n)} |= \mathcal{M}$, *then*

$$\min_{c\in[0,1]} \max_{(x,y)\in\Omega} |R_{ij}^n(x,y)| = \frac{\mathcal{M}}{2^n n!},$$

*where* $\Omega = [0,1] \times [0,1]$.

## 5 Numerical experiments

In this section, some examples are given to examine the proposed method numerically. We introduce the notations $e_i$ and $E_i$, $i = 1,2$ as follows

$$e_i = \max_{0\le j\le N} |x_i^n(t_j) - x_i(t_j)| \qquad t_j = j\Delta t, \quad j = 0,\dots N$$

$$E_i(t) = |x_i^n(t) - x_i(t)|, \qquad t \in [0,1],$$

where $\Delta t = 0.001$ and $x_i^n$, $x_i$ are the $i$th element of approximate and exact solutions of (1), respectively. All computations are performed in MATLAB with double precision. In addition, to approximate the involved definite integrals we have used 8 points Gauss-Legendre quadrature. Also, the resulting nonlinear algebraic system of equations obtained by the proposed method have been solved using the *solve* function in MATLAB.

**Example 1.** We consider the following HSFIEs

$$\begin{cases} x_1(t) = f_1(t) - \displaystyle\int_0^1 e^{ts} x_1^3(s)ds - \int_0^1 \cos(ts) x_2^2(s)ds, \\ x_2(t) = f_2(t) - \displaystyle\int_0^1 e^{t^2 s} x_1^2(s)ds - \int_0^1 e^{ts} x_2^3(s)ds, \end{cases}$$

where $f_1(t) = \frac{e^{t+3}-1}{t+3} + \frac{\sin t}{t} + e^t$ and $f_2(t) = \frac{e^{t^2+2}-1}{t^2+2} + \frac{e^t-1}{t} + 1$ with exact solutions $(x_1(t), x_2(t)) = (e^t, 1)$.

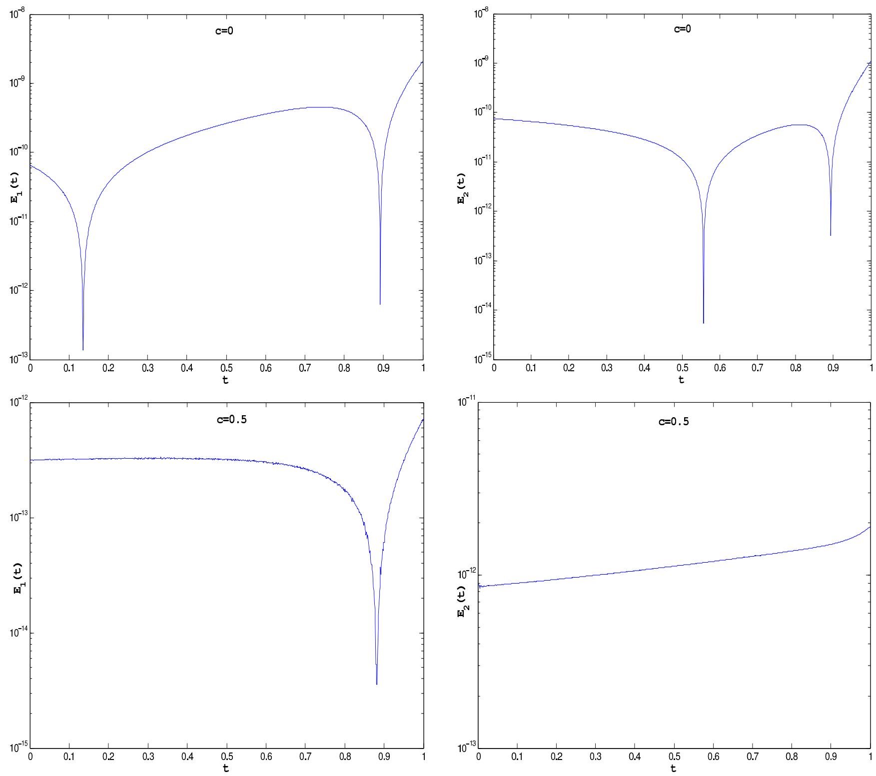


Figure 1: The error functions for Example 1 with $n = 12$ and various $c$.

**Example 2.** We consider the following HSFIEs

$$\begin{cases} x_1(t) = f_1(t) - \displaystyle\int_0^1 \cos(ts)x_1^2(s)ds - \int_0^1 \cos(ts)x_2^2(s)ds, \\ x_2(t) = f_2(t) - \displaystyle\int_0^1 e^{ts}e^{x_1(s)}ds - \int_0^1 e^{ts}e^{x_2(s)}ds, \end{cases}$$

where $f_1(t) = \frac{t+2(t^2\sin t - 2\sin t + 2t\cos t)}{t^3}$ and $f_2(t) = 2\frac{e^{t+1}-1}{t+1} + t$ with exact solutions $(x_1(t), x_2(t)) = (t, t)$.

**Example 3.** We consider the following HSFIEs

$$\begin{cases} x_1(t) = f_1(t) - \displaystyle\int_0^1 e^{t-s}x_1^2(s)ds - \int_0^1 e^{(t+2)s}e^{-x_2(s)}ds, \\ x_2(t) = f_2(t) - \displaystyle\int_0^1 e^{ts}x_1^2(s)ds - \int_0^1 e^{t+s}e^{-x_2(s)}ds, \end{cases}$$

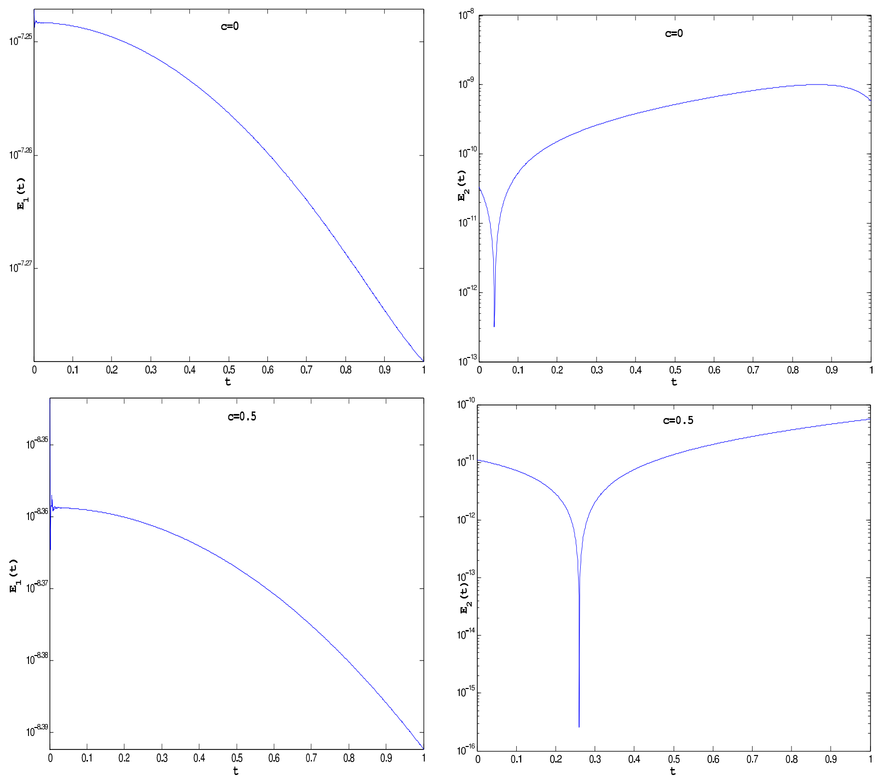


Figure 2: The error functions for Example 2 with $n = 12$ and various $c$.

where $f_1(t) = \frac{e^{t+1}-1}{t+1} + e^{t+1}$ and $f_2(t) = \frac{e^{t+2}-1}{t+2} + e^t + t$ with exact solutions $(x_1(t), x_2(t)) = (e^t, t)$.

Numerical results for Examples 1, 2 and 3 are given in Tables 1, 2 and 3 respectively. In these tables, the notations $c$ and $n$ are the same as the one introduced in Section 4. As we can see the errors $e_1$ and $e_2$ for $c = 0.5$ are more satisfactory than that of $c = 0$. This manner confirms Proposition 1. In addition, the error functions $E_i$, $i = 1, 2$, for $n = 12$ or $n = 15$ and various $c$ , for Examples 1, 2 and 3 are plotted in Figures 1, 2 and 3 respectively.

## 6 Conclusion

Degenerate kernels approximation method was proposed for solving Hammerstein type system of Fredholm integral equations of the second kind. Taylor series expansion was used to construct degenerate kernels. The

Table 1: Numerical results for Example 1.

| | $c=0$ | | $c=0.5$ | |
|---|---|---|---|---|
| $n$ | $e_1$ | $e_2$ | $e_1$ | $e_2$ |
| 3 | $2.45E-01$ | $2.07E-01$ | $2.59E-02$ | $1.55E-02$ |
| 6 | $1.61E-03$ | $1.17E-03$ | $2.49E-05$ | $1.97E-05$ |
| 9 | $2.94E-06$ | $1.79E-06$ | $4.85E-09$ | $2.21E-9$ |
| 12 | $2.07E-09$ | $1.10E-09$ | $7.25E-13$ | $1.91E-12$ |

Table 2: Numerical results for Example 2.

| | $c=0$ | | $c=0.5$ | |
|---|---|---|---|---|
| $n$ | $e_1$ | $e_2$ | $e_1$ | $e_2$ |
| 3 | $2.52E-02$ | $1.56E-01$ | $1.79E-03$ | $9.59E-03$ |
| 6 | $3.17E-04$ | $9.11E-04$ | $2.56E-06$ | $1.53E-05$ |
| 9 | $1.41E-08$ | $1.24E-06$ | $4.51E-09$ | $1.25E-09$ |
| 12 | $5.66E-08$ | $1.00E-09$ | $5.60E-09$ | $5.60E-11$ |

Table 3: Numerical results for Example 3.

| | $c=0$ | | $c=0.5$ | |
|---|---|---|---|---|
| $n$ | $e_1$ | $e_2$ | $e_1$ | $e_2$ |
| 3 | $3.17E-01$ | $4.54E-01$ | $1.05E-02$ | $5.53E-02$ |
| 6 | $3.36E-02$ | $6.96E-02$ | $2.50E-03$ | $4.47E-03$ |
| 9 | $1.40E-03$ | $2.00E-03$ | $4.64E-06$ | $6.17E-06$ |
| 12 | $2.39E-05$ | $2.64E-05$ | $3.56E-08$ | $3.61E-08$ |
| 15 | $2.02E-07$ | $1.84E-07$ | $1.17E-11$ | $1.43E-11$ |

method reduces the solution of a system of integral equations to the solution of a system of algebraic equations. Under certain conditions, it was shown that the approximate solution obtained by the proposed method converges to the exact solution. Numerical results verified the efficiency of the proposed method.

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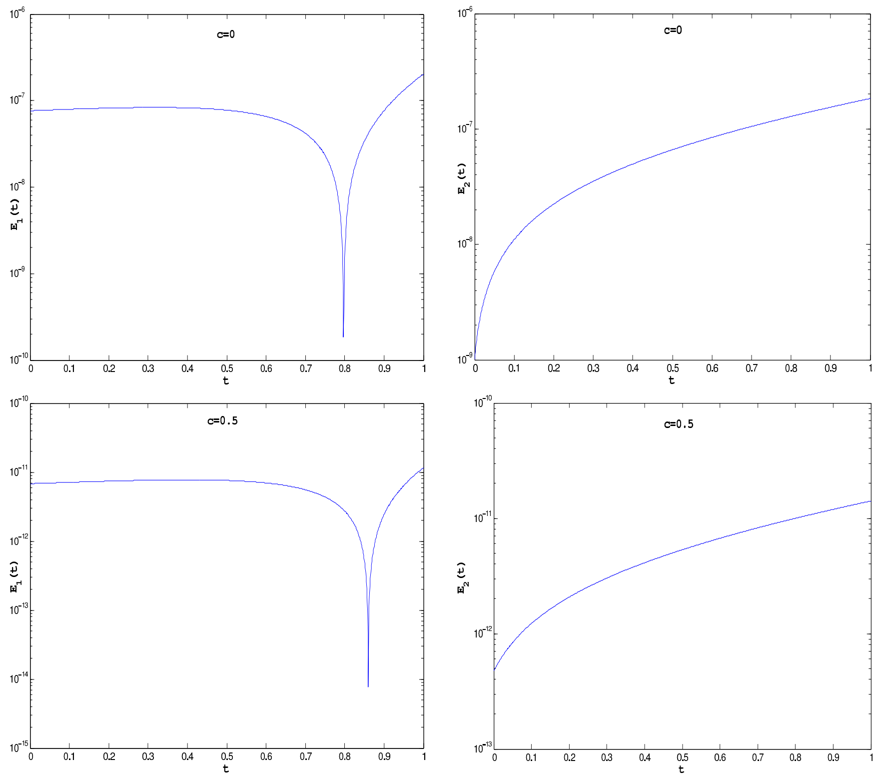


Figure 3: The error functions for Example 3 with $n = 15$ and various $c$.

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