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Approximation of stochastic advection diffusion equations with finite difference scheme

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Abstract. In this paper, a high-order and conditionally stable stochastic difference scheme is proposed for the numerical solution of Itô stochastic advection diffusion equation with one dimensional white noise process. We applied a finite difference approximation of fourth-order for discretizing space spatial derivative of this equation. The main properties of deterministic difference schemes, i.e. consistency, stability and convergence, are developed for the stochastic case. It is shown through analysis that the proposed scheme has these properties. Numerical results are given to demonstrate the computational efficiency of the stochastic scheme.

Keywords: stochastic partial differential equations, consistency, stability, convergence.
AMS Subject Classification: 34A12, 34D20.

1 Introduction

Physical phenomena of interest in science and technology are very often theoretically simulated by means of models which correspond to deterministic partial differential equations (PDEs). PDEs are widely used as models to describe complex physical phenomena in various fields of science, for example, chemical physics, fluid mechanics, solid-state physics, plasma physics,

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plasma wave, biology and economics [11, 16]. As in the case of deterministic PDEs, only a few, very simple PDE can be solved analytically, as a consequence, there is a need for the numerical schemes for approximating their solution. Moreover, most frequently some of the parameters and initial data are not known with complete certainty due to lack of information, uncertainty in the measurements or incomplete knowledge of the mechanism themselves and therefore, the behaviour of the system might be far away from the ideal deterministic representation. To compensate this lack of information and make description of the system more realistic, one introduces random inputs which may be random variables or stochastic processes. This leads to stochastic partial differential equations (SPDEs). Stochastic partial differential equations have many applications in chemistry, physics, engineering, mathematical biology and finance. Analytical solution can be obtained for very few SPDEs, see for example [6,7,12]. One hope is that using numerical methods to generate solutions to such equations will lead to better understanding of the equations. For numerical simulation of solution of SPDEs, some authors have used the finite element approximation [1,17] and others have used finite difference scheme [4,5,18]. Roth used an explicit finite difference method to approximate the solution of some stochastic hyperbolic equations [13]. Soheili et al. presented two methods for solving linear parabolic SPDEs based on the Saul'yev method and a high order finite difference scheme [14]. In [3], Soheili and Bishehniasar considered the approximation of stochastic advection diffusion equation using compact finite difference technique, and investigated their numerical results. Kamrani and Hosseini reported explicit and implicit finite difference method for general SPDE [8]. Some authors used spectral method for spatial variable discertization and solved the resulting system of stochastic ordinary differential equation (SODEs) via the Crank-Nicolson scheme or stochastic Runge-Kutta method [2,9]. In this paper, we extend one kind of the finite difference methods to stochastic case in order to approximate the solution of stochastic advection diffusion equation. This paper is organized as follows. An explicit finite difference scheme to approximate stochastic advection diffusion equations is introduced in Section 2. In addition, consistency, stability and convergence, important properties of a deterministic difference scheme, are developed for the stochastic scheme. In Section 3, consistency, stability and convergence of the proposed stochastic difference scheme is established. Finally, numerical results are given in Section 4.

2 Finite Difference Approximation for Advection Diffusion Equations

Consider the following stochastic advection diffusion equation

$$\begin{aligned} u_t(x,t) + \nu u_x(x,t) &= \gamma u_{xx}(x,t) + \sigma u(x,t)\ \dot{W}(t),\ x \in [0,X],\ t \in [0,T], \\ u(x,0) &= u_0(x), \quad u(0,t) = u(X,t) = 0, \end{aligned} \tag{1}$$

where ν, γ and σ are random variables such that $\mathbb{E}(\nu^2) < \infty$, $\mathbb{E}(\gamma^2) < \infty$, $\mathbb{E}(\sigma^2) < \infty$ and $W(t)$ is a one–dimensional standard Wiener process such that the white noise $\dot{W}(t)$ is a Gaussian distribution with zero mean [10]. Numerically, finite difference methods have vast applications in approximating the solution of SPDEs. These schemes discretize continuous space and time into an evenly distributed grid system, and the values of the state variables are evaluated at each node of the grid. Considering a uniform space grid Δx and time grid Δt in the time–space lattice, we can estimate the solution of the equation at the points of this lattice. The value of the approximate solution at the point $(k\Delta x, n\Delta t)$ will be denoted by u_k^n where n and k are integers. In the explicit method, the time and space derivatives in the SPDE are approximated by finite difference replacements in the following form [15]:

$$\begin{aligned} u_x(k\Delta x, n\Delta t) &\approx \frac{u_{k+1}^n - u_k^n}{\Delta x}, \qquad u_t(k\Delta x, n\Delta t) \approx \frac{u_k^{n+1} - u_k^n}{\Delta t}, \\ u_{xx}(x,t) &\approx \frac{1}{\Delta x^2}\Big(-\frac{1}{12}u(x-2\Delta x,t) + \frac{4}{3}u(x-\Delta x,t) - \frac{5}{2}u(x,t) \\ &\quad + \frac{4}{3}u(x+\Delta x,t) - \frac{1}{12}u(x+2\Delta x,t)\Big). \end{aligned} \tag{2}$$

In fact, (2) is a fourth order approximation of u_{xx}, with the truncation error being $O(\Delta x^4)$. Therefore, the scheme approximates the stochastic advection diffusion equation is given by

$$\begin{aligned} u_k^{n+1} &= \left(1 + \nu\lambda - \frac{5}{2}\gamma\rho\right) u_k^n + \left(\frac{4}{3}\gamma\rho - \nu\lambda\right) u_{k+1}^n \\ &+ \gamma\rho\left(-\frac{1}{12}u_{k-2}^n + \frac{4}{3}u_{k-1}^n - \frac{1}{12}u_{k+2}^n\right) + \sigma u_k^n \Delta W_n, \end{aligned} \tag{3}$$

where $\lambda = \frac{\Delta t}{\Delta x}$, $\rho = \frac{\Delta t}{\Delta x^2}$, $\Delta W_n = W((n+1)\Delta t) - W(n\Delta t)$ and ΔW_n is a Gaussian distribution with mean 0 and variance Δt, i.e., $\Delta W_n \sim N(0, \Delta t)$.

Remark 1. *For the proposed scheme, we assume that the random variables ν, γ and σ are independent of the Wiener process and the states u_k^n.*

Consistency, stability and convergence are important properties of interest in deterministic theory for the stochastic case and we aim to appropriate these concepts to the stochastic case. To get a higher degree of generality in the following definitions, it is useful to introduce the following notations. Consider an SPDE in the following form $Lv = G$, where L denotes the differential operator and $G \in L^2(\mathbb{R})$ is an inhomogeneity. Let u_k^n be a solution that is approximated by a stochastic finite difference scheme denoted by L_k^n, and applying the stochastic scheme to the SPDE, we have $L_k^n u_k^n = G_k^n$, where G_k^n is the approximation of the inhomogeneity. For consistency, stability and convergence, we will need a norm. Hence for a sequence $u = \{\dots, u_{-1}, u_0, u_1, \dots\}$, the sup–norm is defined as $\|u\|_\infty = \sqrt{\sup_k |u_k|^2}$. Based on [13], we propose the following definitions of stochastic difference scheme.

Definition 1. *A stochastic difference scheme $L_k^n u_k^n = G_k^n$ is pointwise consistent with the SPDE $Lv = G$ at point (x, t), if for any continuously differentiable function $\Phi = \Phi(x, t)$, in mean square*

$$\mathbb{E}\| \left(L\Phi - G\right)|_k^n - [L_k^n \Phi(k\Delta x, n\Delta t) - G_k^n] \|^2 \to 0,$$

as $\Delta x \to 0$, $\Delta t \to 0$, and $(k\Delta x, (n+1)\Delta t) \to (x, t)$.

Definition 2. *A stochastic difference scheme is said to be stable with respect to a norm in mean square if there exist some positive constants $\overline{\Delta x_0}$ and $\overline{\Delta t_0}$ and non–negative constants K and β such that*

$$\mathbb{E}\|u^{n+1}\|^2 \le K e^{\beta t} \mathbb{E}\|u^0\|^2,$$

for all $0 \le t = (n+1)\Delta t$, $0 \le \Delta x \le \overline{\Delta x_0}$, and $0 \le \Delta t \le \overline{\Delta t_0}$, where

$$u^{n+1} = \left(\dots, u_{k-2}^{n+1}, u_{k-1}^{n+1}, u_k^{n+1}, u_{k+1}^{n+1}, u_{k+2}^{n+1}, \dots\right)^T.$$

Definition 3. *A stochastic difference scheme $L_k^n u_k^n = G_k^n$ approximating the SPDE $Lv = G$ is convergent in mean square at time t, if as $(n+1)\Delta t$ converges to t, $\mathbb{E}\|u^{n+1} - v^{n+1}\|^2 \to 0$, for $(n+1)\Delta t = t$, $\Delta x \to 0$ and $\Delta t \to 0$.*

3 Convergence analysis of the stochastic scheme

Theorem 1. *The stochastic difference scheme (3) is consistent in mean square in the sense of Definition 1.*

Proof. Let $\Phi(x,t)$ be a smooth function, then we have:

$$\begin{aligned} L(\Phi)|_k^n &= \Phi(k\Delta x,(n+1)\Delta t) - \Phi(k\Delta x, n\Delta t) \\ &\quad + \nu \int_{n\Delta t}^{(n+1)\Delta t} \Phi_x(k\Delta x, s)\ ds \\ &\quad - \gamma \int_{n\Delta t}^{(n+1)\Delta t} \Phi_{xx}(k\Delta x, s)\,\mathrm{d}s - \sigma \int_{n\Delta t}^{(n+1)\Delta t} \Phi(k\Delta x, s)\ dW(s), \end{aligned}$$

and

$$\begin{aligned} L_k^n\Phi \;=\; & \Phi(k\Delta x,(n+1)\Delta t) - \Phi(k\Delta x, n\Delta t) \\ & +\nu\Delta t\left(\frac{\Phi((k+1)\Delta x, n\Delta t) - \Phi(k\Delta x, n\Delta t)}{\Delta x}\right) \\ & -\gamma\frac{\Delta t}{\Delta x^2}\left(-\frac{1}{12}\Phi((k-2)\Delta x, n\Delta t) + \frac{4}{3}\Phi((k-1)\Delta x, n\Delta t)\right. \\ & -\frac{5}{2}\Phi(k\Delta x, n\Delta t) + \frac{4}{3}\Phi((k+1)\Delta x, n\Delta t) \\ & \left. -\ \frac{1}{12}\Phi((k+2)\Delta x, n\Delta t)\right) \\ & -\sigma\Phi(k\Delta x, n\Delta t)(W((n+1)\Delta t) - W(n\Delta t)). \end{aligned}$$

Therefore, if we use the square property of Itô integral, then we obtain:

$$\begin{aligned} & \mathbb{E}|L(\Phi)|_k^n - L_k^n\Phi|^2 \\ & = \mathbb{E}\left|\nu\int_{n\Delta t}^{(n+1)\Delta t}\left(\Phi_x(k\Delta x, s) - \frac{\Phi((k+1)\Delta x, n\Delta t) - \Phi(k\Delta x, n\Delta t)}{\Delta x}\right)ds\right. \\ & \quad - \gamma\int_{n\Delta t}^{(n+1)\Delta t}\left(\Phi_{xx}(k\Delta x, s)\right. \\ & \quad - \frac{1}{\Delta x^2}\left[-\frac{1}{12}\Phi((k-2)\Delta x, n\Delta t) + \frac{4}{3}\Phi((k-1)\Delta x, n\Delta t)\right. \\ & \quad \left.\left. -\frac{5}{2}\Phi(k\Delta x, n\Delta t) + \frac{4}{3}\Phi((k+1)\Delta x, n\Delta t) - \frac{1}{12}\Phi((k+2)\Delta x, n\Delta t)\right]\right)ds \\ & \quad \left. - \sigma\int_{n\Delta t}^{(n+1)\Delta t}(\Phi(k\Delta x, s) - \Phi(k\Delta x, n\Delta t))\ dW(s)\right|^2 \\ & \le 4\mathbb{E}(\nu^2)\mathbb{E}\left|\int_{n\Delta t}^{(n+1)\Delta t}\left(\Phi_x(k\Delta x, s)\right.\right. \end{aligned}$$

$$
\begin{aligned}
&\left.- \frac{\Phi((k+1)\Delta x, n\Delta t) - \Phi(k\Delta x, n\Delta t)}{\Delta x}\right) ds\Bigg|^2 \\
&+ 4\mathbb{E}(\gamma^2)\mathbb{E}\Bigg|\int_{n\Delta t}^{(n+1)\Delta t}\Bigg(\Phi_{xx}(k\Delta x, s) \\
&- \frac{1}{\Delta x^2}\Bigg[-\frac{1}{12}\Phi((k-2)\Delta x, n\Delta t) + \frac{4}{3}\Phi((k-1)\Delta x, n\Delta t) \\
&- \frac{5}{2}\Phi(k\Delta x, n\Delta t) + \frac{4}{3}\Phi((k+1)\Delta x, n\Delta t) \\
&- \frac{1}{12}\Phi((k+2)\Delta x, n\Delta t)\Bigg]\Bigg)\ ds\Bigg|^2 \\
&+ 4\mathbb{E}(\sigma^2)\int_{n\Delta t}^{(n+1)\Delta t} \mathbb{E}|\Phi(k\Delta x, s) - \Phi(k\Delta x, n\Delta t)|^2\ ds.
\end{aligned}
$$

Since $\Phi(x,t)$ is a deterministic function, hence $\mathbb{E}|L(\Phi)|_k^n - L_k^n\Phi|^2 \to 0$, as $n,\, k \to \infty$. □

Theorem 2. *The stochastic difference scheme (3) with $t = (n+1)\Delta t$ and $\frac{3}{4}\nu\lambda \leq \gamma\rho \leq \frac{2}{5}(1+\nu\lambda)$, (note that $\nu\lambda \leq \frac{8}{7}$), is stable with respect to $\|\cdot\|_\infty = \sqrt{\sup_k |\cdot|^2}$.*

Proof. Applying $\mathbb{E}|\cdot|^2$ in (3) and using the independence of the Wiener process increments, we get

$$
\begin{aligned}
\mathbb{E}|u_k^{n+1}|^2 \quad = \quad & \mathbb{E}\Bigg|\left(1 + \nu\lambda - \frac{5}{2}\gamma\rho\right)u_k^n + \left(\frac{4}{3}\gamma\rho - \nu\lambda\right)u_{k+1}^n \\
& + \gamma\rho\left(-\frac{1}{12}u_{k-2}^n + \frac{4}{3}u_{k-1}^n - \frac{1}{12}u_{k+2}^n\right)\Bigg|^2 + \mathbb{E}(\sigma^2)\Delta t\mathbb{E}|u_k^n|^2.
\end{aligned}
$$

By using $\frac{3}{4}\nu\lambda \leq \gamma\rho \leq \frac{2}{5}(1+\nu\lambda)$, we arrive at

$$
\begin{aligned}
&\mathbb{E}|u_k^{n+1}|^2 \leq \mathbb{E}\left(\left(1 + \nu\lambda - \frac{5}{2}\gamma\rho\right)^2\right)\mathbb{E}|u_k^n|^2 + \mathbb{E}\left(\left(\frac{4}{3}\gamma\rho - \nu\lambda\right)^2\right)\mathbb{E}|u_{k+1}^n|^2 \\
&+ \mathbb{E}\left((\gamma\rho)^2\right)\mathbb{E}\left|-\frac{1}{12}u_{k-2}^n + \frac{4}{3}u_{k-1}^n - \frac{1}{12}u_{k+2}^n\right|^2 \\
&+ 2\mathbb{E}\left(\left(1 + \nu\lambda - \frac{5}{2}\gamma\rho\right)\left(\frac{4}{3}\gamma\rho - \nu\lambda\right)\right)\mathbb{E}|u_k^n u_{k+1}^n|
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\mathbb{E}\left(\left(1+\nu\lambda-\frac{5}{2}\gamma\rho\right)\gamma\rho\right)\mathbb{E}\left|u_k^n\left(-\frac{1}{12}u_{k-2}^n+\frac{4}{3}u_{k-1}^n-\frac{1}{12}u_{k+2}^n\right)\right| \\
&+ 2\mathbb{E}\left(\left(\frac{4}{3}\gamma\rho-\nu\lambda\right)\gamma\rho\right)\mathbb{E}\left|u_{k+1}^n\left(-\frac{1}{12}u_{k-2}^n+\frac{4}{3}u_{k-1}^n-\frac{1}{12}u_{k+2}^n\right)\right| \\
&+ \mathbb{E}(\sigma^2)\Delta t\,\mathbb{E}|u_k^n|^2.
\end{aligned}
$$

Also, we can use the following inequalities

$$
\begin{aligned}
\mathbb{E}|X+Y+Z| &\le \mathbb{E}|X|+\mathbb{E}|Y|+\mathbb{E}|Z|, \\
\mathbb{E}|X+Y+Z|^2 &\le 4(\mathbb{E}|X|^2+\mathbb{E}|Y|^2+\mathbb{E}|Z|^2), \qquad (4)
\end{aligned}
$$

so we conclude that

$$
\begin{aligned}
\mathbb{E}|u_k^{n+1}|^2 &\le \mathbb{E}\left(\left(1+\nu\lambda-\frac{5}{2}\gamma\rho\right)^2\right)\mathbb{E}|u_k^n|^2+\mathbb{E}\left(\left(\frac{4}{3}\gamma\rho-\nu\lambda\right)^2\right)\mathbb{E}|u_{k+1}^n|^2 \\
&+ \frac{\mathbb{E}\left((\gamma\rho)^2\right)}{144}\left(4\mathbb{E}|u_{k-2}^n|^2+1024\mathbb{E}|u_{k-1}^n|^2+4\mathbb{E}|u_{k+2}^n|^2\right) \\
&+ 2\mathbb{E}\left(\left(1+\nu\lambda-\frac{5}{2}\gamma\rho\right)\left(\frac{4}{3}\gamma\rho-\nu\lambda\right)\right)\mathbb{E}|u_k^n u_{k+1}^n| \\
&+ \frac{1}{6}\mathbb{E}\left(\left(1+\nu\lambda-\frac{5}{2}\gamma\rho\right)\gamma\rho\right)\left(\mathbb{E}|u_k^n u_{k-2}^n|+16\mathbb{E}|u_k^n u_{k-1}^n|+\mathbb{E}|u_k^n u_{k+2}^n|\right) \\
&+ \frac{1}{6}\mathbb{E}\left(\left(\frac{4}{3}\gamma\rho-\nu\lambda\right)\gamma\rho\right)\left(\mathbb{E}|u_{k+1}^n u_{k-2}^n|+16\mathbb{E}|u_{k+1}^n u_{k-1}^n|+\mathbb{E}|u_{k+1}^n u_{k+2}^n|\right) \\
&+ \mathbb{E}(\sigma^2)\Delta t\,\mathbb{E}|u_k^n|^2 \\
&\le \Bigg\{\mathbb{E}\left(\left(1+\nu\lambda-\frac{5}{2}\gamma\rho\right)^2\right)+\mathbb{E}\left(\left(\frac{4}{3}\gamma\rho-\nu\lambda\right)^2\right)+\frac{1032}{144}\mathbb{E}\left((\gamma\rho)^2\right) \\
&+ 2\mathbb{E}\left(\left(1+\nu\lambda-\frac{5}{2}\gamma\rho\right)\left(\frac{4}{3}\gamma\rho-\nu\lambda\right)\right)+3\mathbb{E}\left(\left(1+\nu\lambda-\frac{5}{2}\gamma\rho\right)\gamma\rho\right) \\
&+ 3\mathbb{E}\left(\left(\frac{4}{3}\gamma\rho-\nu\lambda\right)\gamma\rho\right)+\mathbb{E}(\sigma^2)\Delta t\Bigg\}\sup_k \mathbb{E}|u_k^n|^2 \\
&= \left\{1+\frac{2}{3}\mathbb{E}(\gamma\rho)+\frac{181}{36}\mathbb{E}\left((\gamma\rho)^2\right)+\mathbb{E}(\sigma^2)\Delta t\right\}\sup_k \mathbb{E}|u_k^n|^2,
\end{aligned}
$$

where the first inequality follows from (4). It is sufficient to select δ such that $\frac{2}{3}\mathbb{E}(\gamma\rho)+\frac{181}{36}\mathbb{E}\left((\gamma\rho)^2\right)+\mathbb{E}(\sigma^2)\Delta t \le \delta^2\Delta t$ holds, for all k. Therefore

$$
\sup_k \mathbb{E}|u_k^{n+1}|^2 \le (1+\delta^2\Delta t)\sup_k \mathbb{E}|u_k^n|^2 \le \cdots \le (1+\delta^2\Delta t)^{n+1}\sup_k \mathbb{E}|u_k^0|^2,
$$

and by substituting Δt with $\frac{t}{n+1}$,

$$
\mathbb{E}\|u^{n+1}\|_\infty^2 \le \left(1+\frac{\delta^2 t}{n+1}\right)^{n+1}\mathbb{E}\|u^0\|_\infty^2 \le e^{\delta^2 t}\mathbb{E}\|u^0\|_\infty^2. \qquad (5)
$$

So, the stochastic difference scheme (3) is stable for $\frac{3}{4}\nu\lambda \leq \gamma\rho \leq \frac{2}{5}(1+\nu\lambda)$, according to Definition (2), with $K = 1$ and $\beta = \delta^2$. □

Theorem 3. *The stochastic difference scheme (3) for*

$$\frac{3}{4}\nu\lambda \leq \gamma\rho \leq \frac{2}{5}(1+\nu\lambda),$$

is convergent with respect to $\|\cdot\|_\infty$–norm.

Proof. The stochastic finite difference scheme is given by

$$\begin{aligned} u_k^{n+1} &= u_k^n - \nu\Delta t \frac{u_{k+1}^n - u_k^n}{\Delta x} \\ &+ \gamma\frac{\Delta t}{\Delta x^2}\left(-\frac{1}{12}u_{k-2}^n + \frac{4}{3}u_{k-1}^n - \frac{5}{2}u_k^n + \frac{4}{3}u_{k+1}^n - \frac{1}{12}u_{k+2}^n\right) \\ &+ \sigma u_k^n(W((n+1)\Delta t) - W(n\Delta t)). \end{aligned}$$

The solution v_k^{n+1} can be represented by the Taylor expansion $v_{xx}(x,s)$ with respect to the space variable as

$$\begin{aligned} v_k^{n+1} &= v_k^n - \nu\int_{n\Delta t}^{(n+1)\Delta t} v_x(x,s)|_{x=x_k}\, ds + \gamma\int_{n\Delta t}^{(n+1)\Delta t} v_{xx}(x_k,s)\, ds \\ &\quad + \sigma\int_{n\Delta t}^{(n+1)\Delta t} v(x,s)|_{x=x_k}\, dW(s) \\ &= v_k^n - \nu\int_{n\Delta t}^{(n+1)\Delta t}\left(\frac{v_{k+1}^n - v_k^n}{\Delta x} - \frac{\Delta x}{2}v_{xx}((k+\alpha)\Delta x, s)\right) ds \\ &\quad + \gamma\int_{n\Delta t}^{(n+1)\Delta t}\left(\frac{1}{\Delta x^2}\left(-\frac{1}{12}v_{k-2}^n + \frac{4}{3}v_{k-1}^n - \frac{5}{2}v_k^n + \frac{4}{3}v_{k+1}^n - \frac{1}{12}v_{k+2}^n\right)\right. \\ &\quad + \frac{\Delta x^4}{135}\left(v^{(6)}((k+\theta)\Delta x, s) + v^{(6)}((k+\mu)\Delta x, s)\right) \\ &\quad \left. - \frac{\Delta x^4}{540}\left(v^{(6)}((k+\theta')\Delta x, s) + v^{(6)}((k+\mu')\Delta x, s)\right)\right) ds \\ &\quad + \sigma\int_{n\Delta t}^{(n+1)\Delta t} v(x,s)|_{x=x_k}\, dW(s), \end{aligned}$$

where α, θ, μ, θ', $\mu' \in (0,1)$. Let $z_k^n = v_k^n - u_k^n$, then we have

$$\begin{aligned} z_k^{n+1} &= \left(1 + \nu\lambda - \frac{5}{2}\gamma\rho\right) z_k^n + \left(\frac{4}{3}\gamma\rho - \nu\lambda\right) z_{k+1}^n \\ &+ \gamma\rho\left(-\frac{1}{12}z_{k-2}^n + \frac{4}{3}z_{k-1}^n - \frac{1}{12}z_{k+2}^n\right) \\ &+ \nu\frac{\Delta x}{2}\int_{n\Delta t}^{(n+1)\Delta t} v_{xx}((k+\alpha)\Delta x, s)\, ds \end{aligned}$$

$$+\gamma\int_{n\Delta t}^{(n+1)\Delta t}\left(\frac{\Delta x^4}{135}\left(v^{(6)}((k+\theta)\Delta x,s)+v^{(6)}((k+\mu)\Delta x,s)\right)\right.$$
$$\left.-\frac{\Delta x^4}{540}\left(v^{(6)}((k+\theta')\Delta x,s)+v^{(6)}((k+\mu')\Delta x,s)\right)\right)\ ds$$
$$+\sigma\int_{n\Delta t}^{(n+1)\Delta t}\left(v(x,s)|_{x=x_k}-u_k^n\right)\ dW(s). \tag{6}$$

Applying $\mathbb{E}|\cdot|^2$ to (6), we arrive at the following inequality

$$\mathbb{E}|z_k^{n+1}|^2 \leq 4\mathbb{E}\left|\left(1+\nu\lambda-\frac{5}{2}\gamma\rho\right)z_k^n+\left(\frac{4}{3}\gamma\rho-\nu\lambda\right)z_{k+1}^n\right.$$
$$\left.+\gamma\rho\left(-\frac{1}{12}z_{k-2}^n+\frac{4}{3}z_{k-1}^n-\frac{1}{12}z_{k+2}^n\right)\right|^2$$
$$+4\mathbb{E}\left|\nu\frac{\Delta x}{2}\int_{n\Delta t}^{(n+1)\Delta t}v_{xx}((k+\alpha)\Delta x,s)\ ds\right.$$
$$+\gamma\int_{n\Delta t}^{(n+1)\Delta t}\left(\frac{\Delta x^4}{135}\left(v^{(6)}((k+\theta)\Delta x,s)+v^{(6)}((k+\mu)\Delta x,s)\right)\right.$$
$$\left.\left.-\frac{\Delta x^4}{540}\left(v^{(6)}((k+\theta')\Delta x,s)+v^{(6)}((k+\mu')\Delta x,s)\right)\right)\ ds\right|^2$$
$$+2\mathbb{E}\left|\sigma\int_{n\Delta t}^{(n+1)\Delta t}\left(v(x,s)|_{x=x_k}-u_k^n\right)\ dW(s)\right|^2$$
$$\leq 4\mathbb{E}\left|\left(1+\nu\lambda-\frac{5}{2}\gamma\rho\right)z_k^n+\left(\frac{4}{3}\gamma\rho-\nu\lambda\right)z_{k+1}^n\right.$$
$$\left.+\gamma\rho\left(-\frac{1}{12}z_{k-2}^n+\frac{4}{3}z_{k-1}^n-\frac{1}{12}z_{k+2}^n\right)\right|^2$$
$$+4\mathbb{E}\left|\nu\frac{\Delta x}{2}\int_{n\Delta t}^{(n+1)\Delta t}v_{xx}((k+\alpha)\Delta x,s)\ ds\right.$$
$$+\gamma\int_{n\Delta t}^{(n+1)\Delta t}\left(\frac{\Delta x^4}{135}\left(v^{(6)}((k+\theta)\Delta x,s)+v^{(6)}((k+\mu)\Delta x,s)\right)\right.$$
$$\left.\left.-\frac{\Delta x^4}{540}\left(v^{(6)}((k+\theta')\Delta x,s)+v^{(6)}((k+\mu')\Delta x,s)\right)\right)\ ds\right|^2$$
$$+2\mathbb{E}(\sigma^2)\int_{n\Delta t}^{(n+1)\Delta t}\mathbb{E}|v(x,s)|_{x=x_k}-v_k^n+v_k^n-u_k^n|^2\ ds.$$

Assuming $\frac{3}{4}\nu\lambda \leq \gamma\rho \leq \frac{2}{5}(1+\nu\lambda)$, introducing the notation $\psi_{1k}=v_{xx}((k+\alpha)\Delta x,s)<\infty$, $\psi_{2k}=v^{(6)}((k+\theta)\Delta x,s)<\infty$, $\psi_{3k}=v^{(6)}((k+\mu)\Delta x,s)<\infty$,

$\psi_{4k} = v^{(6)}((k+\theta')\Delta x, s) < \infty$, $\psi_{5k} = v^{(6)}((k+\mu')\Delta x, s) < \infty$ and also

$$\int_{n\Delta t}^{(n+1)\Delta t} \mathbb{E}|v(x,s)|_{x=x_k} - v_k^n|^2 \; ds = \mathbb{E}\int_{n\Delta t}^{(n+1)\Delta t} |v(x,s)|_{x=x_k} - v_k^n|^2 \; ds$$

$$\leq \sup_{s\in[n\Delta t,(n+1)\Delta t]} |v(x,s)|_{x=x_k} - v(k\Delta x, n\Delta t)|^2 \Delta t \leq \psi_1 \Delta t,$$

we conclude that

$$\begin{aligned}
\mathbb{E}|z_k^{n+1}|^2 &\leq 4\left(1+\frac{2}{3}\mathbb{E}(\gamma\rho)+\frac{181}{36}\mathbb{E}\left((\gamma\rho)^2\right)\right)\sup_k \mathbb{E}|z_k^n|^2 \\
&+ 4\sup_k \mathbb{E}\left|\int_{n\Delta t}^{(n+1)\Delta t}\left(\nu\frac{\Delta x}{2}\psi_{1k}+\gamma\frac{\Delta x^4}{135}(\psi_{2k}+\psi_{3k})\right.\right. \\
&\left.\left.-\gamma\frac{\Delta x^4}{540}(\psi_{4k}+\psi_{5k})\right) ds\right|^2 \\
&+ 4\mathbb{E}(\sigma^2)\sup_k \int_{n\Delta t}^{(n+1)\Delta t} \mathbb{E}|v(x,s)|_{x=x_k} - v_k^n|^2 \; ds \\
&+ 4\mathbb{E}(\sigma^2)\sup_k \int_{n\Delta t}^{(n+1)\Delta t} \mathbb{E}|v_k^n - u_k^n|^2 \; ds \\
&\leq 4\left(1+\frac{2}{3}\mathbb{E}(\gamma\rho)+\frac{181}{36}\mathbb{E}\left((\gamma\rho)^2\right)+\mathbb{E}(\sigma^2)\Delta t\right)\sup_k \mathbb{E}|z_k^n|^2 \\
&+ 4\sup_k \mathbb{E}|\Psi_1\Delta t|^2 + \psi_2\Delta t \\
&\leq 4\left(1+\frac{2}{3}\mathbb{E}(\gamma\rho)+\frac{181}{36}\mathbb{E}\left((\gamma\rho)^2\right)+\mathbb{E}(\sigma^2)\Delta t\right)\sup_k \mathbb{E}|z_k^n|^2 + \Psi\Delta t.
\end{aligned}$$

By selecting δ such that

$$\frac{2}{3}\mathbb{E}(\gamma\rho)+\frac{181}{36}\mathbb{E}\left((\gamma\rho)^2\right)+\mathbb{E}(\sigma^2)\Delta t \leq \delta^2\Delta t,$$

yields

$$\mathbb{E}|z_k^{n+1}|^2 \leq 4(1+\delta^2\Delta t)\sup_k \mathbb{E}|z_k^n|^2 + \Psi\Delta t,$$

and

$$\sup_k \mathbb{E}|z_k^{n+1}|^2 \leq 4(1+\delta^2\Delta t)\sup_k \mathbb{E}|z_k^n|^2 + \Psi\Delta t.$$

It follows that

$$\begin{aligned}\mathbb{E}\|z^{n+1}\|_\infty^2 &\le 4(1+\delta^2\Delta t)\mathbb{E}\|z^n\|_\infty^2 + \Psi\Delta t\\ &\le \left(1+\delta^2\frac{t}{n+1}\right)^{n+1}\sum_{j=1}^{n}(4\Psi\Delta t)^j + \Psi\Delta t\\ &\le e^{\delta^2 t}\sum_{j=1}^{n}(4\Psi\Delta t)^j + \Psi\Delta t.\end{aligned}$$

When time step, i.e. Δt, tends to zero, we obtain

$$\begin{aligned}\mathbb{E}\|z^{n+1}\|_\infty^2 &\le (n-1)e^{\delta^2 t}(4\Psi\Delta t)^2 + 4e^{\delta^2 t}\Psi\Delta t + \Psi\Delta t\\ &\le te^{\delta^2 t}(4\Psi)^2\Delta t + 4e^{\delta^2 t}\Psi\Delta t + \Psi\Delta t\\ &= (te^{\delta^2 t}(4\Psi)^2 + 4e^{\delta^2 t}\Psi + \Psi)\Delta t,\end{aligned}$$

and consequently $\mathbb{E}\|z^{n+1}\|_\infty^2 \to 0$. □

4 Numerical results

In this section, we present the numerical results of the stochastic difference scheme (3) on two test problems. Also, the convergence and stability of the stochastic difference scheme (3) is numerically investigated. For computational purpose, it is useful to consider the discrete Brownian motion, where $W(t)$ is specified at discrete t values.

Example 1. Consider

$$u_t(x,t) = 0.001u_{xx}(x,t) - u(x,t)\dot{W}(t), \qquad x\in[0,1], \qquad t\in[0,1],$$

with $u(x,0) = x^2(1-x)^2$ as the initial condition and the boundary conditions $u(0,t) = u(1,t) = 0$. The discrete form of the stochastic difference scheme is

$$\begin{aligned}u_k^{n+1} = u_k^n + \frac{\rho}{1000}(-\frac{1}{12}u_{k-2}^n + \frac{4}{3}u_{k-1}^n - \frac{5}{2}u_k^n + \frac{4}{3}u_{k+1}^n - \frac{1}{12}u_{k+2}^n)\\ -u_k^n(W((n+1)\Delta t) - W(n\Delta t)),\end{aligned}$$

where $\Delta t = \frac{1}{N}$ and $\Delta x = \frac{1}{M}$, for some positive integer N and M. The above form is conditionally stable with $\beta = \delta^2$ and $K = 1$ and for $\gamma\rho \le \frac{2}{5}$, $\gamma \ge 0$. Therefore, if $M = 150$, then for the stability (or convergence) condition, we must have $\Delta t \le \frac{1}{56}$ or $N \ge 56$. We have shown this in Figure 1.

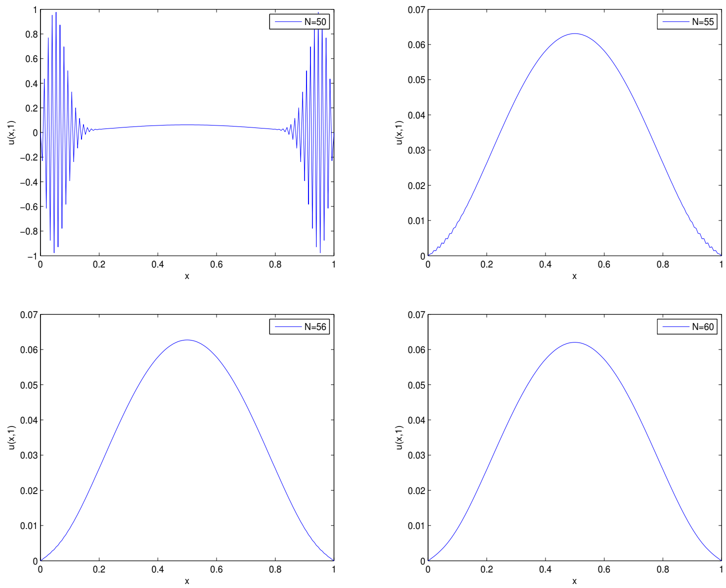

Figure 1: Representation of conditional convergence, $u(x,1)$ for different values of N.

Table 1: δ^2 for stability.

N	5×10^2	10^3	2×10^3	4×10^3	8×10^3	16×10^3
δ^2	21.0906	18.5453	17.2727	16.6363	16.3182	16.1591

In the proof of Theorem 3, we assumed that $\frac{2}{3}\mathbb{E}(\gamma\rho)+\frac{181}{36}\mathbb{E}\left((\gamma\rho)^2\right)+\mathbb{E}(\sigma^2)\Delta t\leq\delta^2\Delta t$, and for different values of N, we obtained the least value of δ^2 in Table 1. Figure 2 shows that the approximation of the stochastic advection diffusion equation using the stochastic difference scheme on a 150 by 1000 grid during the time interval $[0,1]$. On the other hand, in (5) we had

$$\mathbb{E}\|u^{n+1}\|_\infty^2\leq e^{\delta^2 t}\mathbb{E}\|u^0\|_\infty^2\Rightarrow y=ln\left(\frac{\mathbb{E}\|u^{n+1}\|_\infty^2}{\mathbb{E}\|u^0\|_\infty^2}\right)\leq\delta^2 t,\quad (n+1)\Delta t=t. \tag{7}$$

According to (7) and Figure 3 (or Figure 1) and Table 1, the stability condition is satisfied for $N\geq 56$. In Figure 4, we investigate the convergence of the solutions. We do not have the exact solution for this example, and

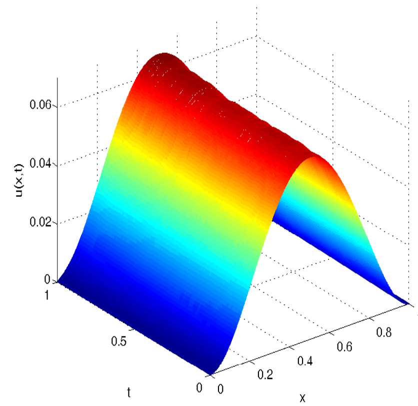

Figure 2: Mean solutions of stochastic advection diffusion equation using stochastic difference scheme.

so the numerical approximation at $t = 1$, for $N = 120$, is chosen as a basic fixed solution (Figure 4, left). The right hand side of Figure 4, gives the log–scale of the difference between the numerical approximations with $N = 60$ and $N = 500$ having the basic fixed solution at the mesh points.

Example 2. Consider the equation

$$u_t(x,t) = 0.001u_{xx}(x,t) + u_x(x,t) - 2u(x,t)\dot{W}(t), \quad x \in [0,1], \quad t \in [0,1],$$

subject to the following initial condition

$$u(x,0) = x^2(1-x)^2, \qquad x \in [0,1],$$

with the boundary conditions

$$u(0,t) = u(1,t) = 0, \qquad t \in [0,1].$$

Therefore, the stochastic difference scheme can be written as:

$$\begin{aligned} u_k^{n+1} = (1-\lambda)u_k^n + \lambda u_{k+1}^n + \gamma\rho(-\frac{1}{12}u_{k-2}^n + \frac{4}{3}u_{k-1}^n - \frac{5}{2}u_k^n + \frac{4}{3}u_{k+1}^n - \frac{1}{12}u_{k+2}^n) \\ -2u_k^n(W((n+1)\Delta t) - W(n\Delta t)). \end{aligned} \tag{8}$$

Let M and N be the total number of grid points for the space and time discretizations, respectively. The above form is conditionally stable for

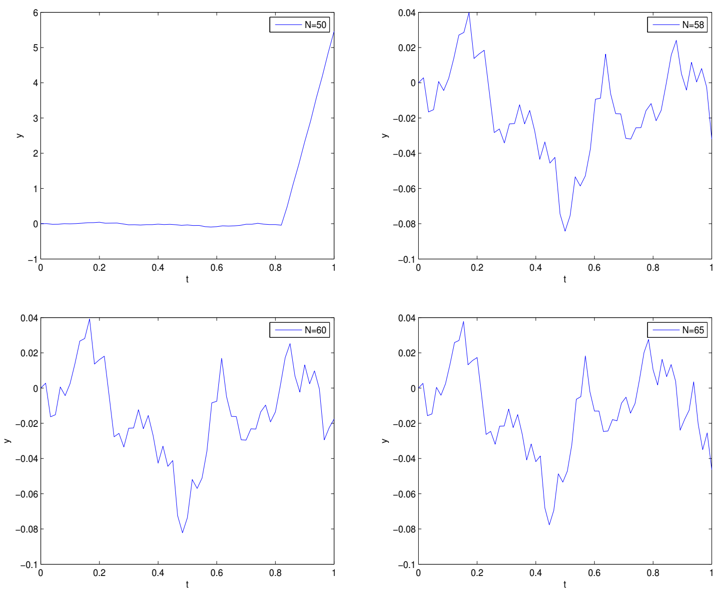

Figure 3: Figures of y in (7) against t.

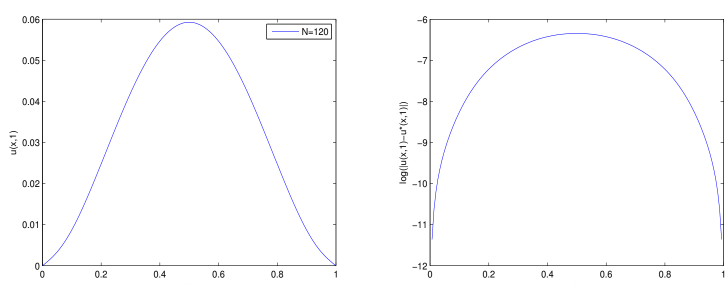

Figure 4: Log difference numerical approximation (right figure) for $N = 60$ and $N = 500$ with $N = 120$ (left figure) and common value $M = 150$.

$\gamma\rho \leq \frac{2}{5}(1-\lambda), \gamma \geq 0$, and $0 < \lambda \leq 1$. Therefore, if $M = 200$, then for the stability (or convergence) condition, we must have $N \geq 300$. The convergence of the scheme at the end of time interval $t = 1$, for the fixed space grid points $M = 200$ and various time grid points $N = 300, 350, 400, 450, 500$ is considered. Note that the scheme is unstable for $N = 250$.

It is clear from Figures 5–7 that the numerical solution obtained for the stochastic advection diffusion equation for the different time steps is convergent at time $t = 1$. In the proof of Theorem 3, we assumed that

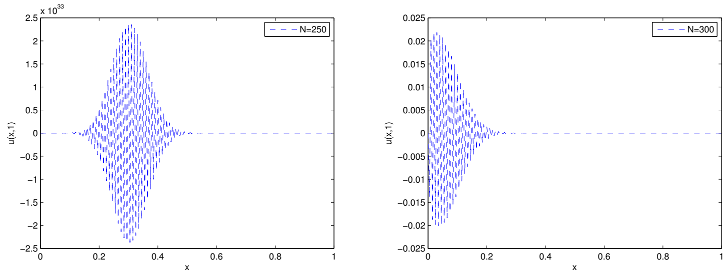

Figure 5: Representation of conditional convergence, $u(x,1)$ for different values of N.

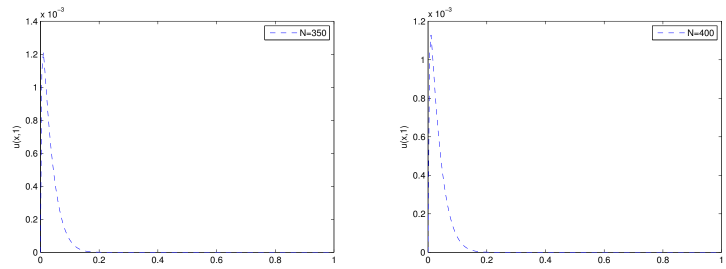

Figure 6: Representation of conditional convergence, $u(x,1)$ for different values of N.

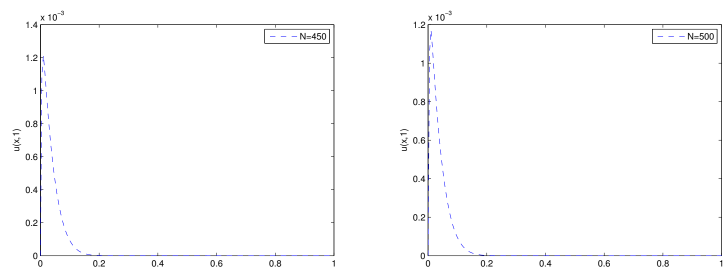

Figure 7: Representation of conditional convergence, $u(x,1)$ for different values of N.

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$\frac{2}{3}\mathbb{E}(\gamma\rho) + \frac{181}{36}\mathbb{E}\left((\gamma\rho)^2\right) + \mathbb{E}(\sigma^2)\Delta t \leq \delta^2\Delta t$, and for different values of N, we obtained the least value of δ^2 in Table 2. In order to qualify the numerical results of the considered stochastic advection diffusion equation, we plot, in Figure 8, the stochastic solution using stochastic scheme (8) on a 200 by 1000 grid during the time interval $[0, 1]$.

Table 2: δ^2 for stability.

N	5×10^2	10^3	2×10^3	4×10^3	8×10^3	16×10^3
δ^2	46.7556	38.7111	34.6889	32.6778	31.6277	31.1694

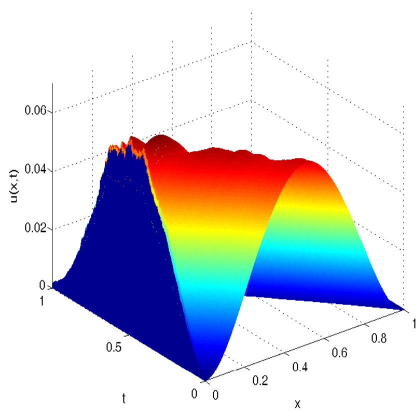

Figure 8: Mean solutions of stochastic advection diffusion equation using stochastic difference scheme.

5 Conclusion

This paper has provided a stochastic finite difference scheme for the numerical solution of stochastic advection diffusion equations. Consistency of the stochastic finite difference scheme is established. Stability conditions and convergence of the proposed stochastic difference scheme are given. Some numerical results are included to show the efficiency of the scheme.

References

[1] E.J. Allen, S. J. Novose and Z. C. Zhang, *Finite element and difference approximation of some linear stochastic partial differential equations*, Stochastic Rep. **64** (1998) 117–142.

[2] A. Ashyralyev, *On modified Crank-Nicolson differential scheme for stochastic parabolic equation*, Numer. Funct. Anal. Optim. **29** (2008) 268–282.

[3] M. Bishehniasar and A.R. Soheili, *Approximation of stochastic advection diffusion equation using compact finite difference technique*, Iran. J. Sci. Technol. Trans. A Sci. **37A3** (2013) 327–333.

[4] A.M. Davie and J.G. Gaines, *Convergence of numerical schemes for the solution of parabolic stochastic partial differential equations*, Math. Comp. **70** (2000) 121–134.

[5] I. Gyongy, *Lattice approximations for stochastic qusi–linear parabolic partial differential equations driven by space–time white noise II*, Potential Anal. **11** (1999) 1–37.

[6] I. Gyongy and C. Rovira, *On L^p–solution of semililinear stochastic partial differential equation,* Stoch. Proc. Appl. **90** (2000) 83–108.

[7] I. Gyongy, *Existence and uniqueness results for semilinear stochastic partial differential*, Stoch. Proc. Appl. **73** (1998) 271–299.

[8] M. Kamrani and S.M. Hosseini, *The role of the cofficients of a general SPDE on the stability and convergence of the finite difference method,* J. Comput. Appl. Math. **234** (2010) 1426–1434.

[9] M. Kamrani and S.M. Hosseini, *Spectral collocation method for stochastic Burgers equation driven by additive noise*, Math. Comput. Simulation **82** (2012) 1630–1644.

[10] P.E. Kloeden and E. Platen, *Numerical solution of stochastic differential equations*, Springer, Berlin, 1995.

[11] J.D. Murray, *Mathematical biology I, II.*, Springer, 2003.

[12] G.D. Prato and L. Tubaro, *Stochastic partial differential equations and application*, Longman scientific and technical, Harlow, 1992.

[13] C. Roth, *Difference methods for stochastic partial differential equations*, Z. Zngew. Math. Mech. **82** (2002) 821–830.

[14] A.R. Soheili, M.B. Niasar and M. Arezoomandan, *Approximation of stochastic parabolic differential equations with two different finite difference schemes*, Bull. Iranian Math. Soc. **37** (2011) 61–83.

[15] J.W. Thomas, *Numerical partial differential equations: finite difference methods*, Springer, 1998.

[16] A.M. Wazwaz, *Partial differential equations and solitary waves theory-higher education press*, Springer, 2003.

[17] J.B. Walsh, *Finite element methods for parabolic stochastic PDEs*, Potential Anal. **23** (2005) 1–43.

[18] H. Yoo, *semi-discretization of stochastic partial differential equation on R by a finite difference method*, Math. Comp. **69** (1999) 653–666.