Journal of Mathematical Modeling
Vol. 2, No. 2, 2015, pp. 187-216

A single server perishable inventory system with N additional options for service

Jeganathan Kathirvel*

Ramanujan Institute for Advanced Study in Mathematics,
University of Madras, Chepauk, Chennai-600 005, India
Email: jegan.nathan85@yahoo.com

Abstract. This article presents a perishable (s, S) inventory system under continuous review at a service facility in which a waiting hall for customers is of finite size M. The arrival instants of customers to the service station constitutes a Poisson process. The life time of each items is assumed to be exponential. All arriving customers demand the first "essential" service, whereas only some of them demand the second "optional" service, and the second service is multi-optional. The joint probability distribution of the number of customers in the waiting hall and the inventory level is obtained for the steady state case. Some important system performance measures in the steady state are derived, and the long-run total expected cost rate is also calculated. We have derived the Laplace-Stieljes transforms of waiting time distribution of customers in the waiting hall. The results are illustrated numerically.

Keywords: (s, S) policy, Continuous review, Perishable commodity, Optional service, Markov process.
AMS Subject Classification: 90B22, 60K25.

1 Introduction

Several researchers have studied the inventory systems in which demanded items are instantaneously distributed from stock (if available) to the cus-

*Corresponding author.
Received: 14 July 2014 / Revised: 2 September 2014 / Accepted: 4 September 2014.

tomers. During stock out period, the demands of a customer are either not satisfied (lost sales case) or satisfied only after getting the receipt of the ordered items (backlog case). In the backlog case, either all demands (full backlog case) or only a limited number of demands (partial backlog case) are satisfied during stock out period. To know the review of these works see Çakanyildirim et al. [7], Durán et al. [8], Elango and Arivarignan [9], Goyal and Giri [10], Kalpakam and Arivarignan [13, 14], Liu and Yang [16], Nahmias [17], Raafat [18] and Yadavalli et al. [20] and the references therein.

However, in the case of inventories maintained at service facilities, after some service is performed on the demanded items they are distributed to the customers. In such situations, the items are issued not on demanding rather it is done after a random time of service. It causes the formation of queues in front of service centres. As a result there is a need for study of both the inventory level and the queue length in the long run. Berman and Kim [2] analyzed a queueing - inventory system with Poisson arrivals, exponential service times and zero lead times. The authors proved that the optimal policy is never to order when the system is empty. Berman and Sapna [5] studied queueing - inventory systems with Poisson arrivals, arbitrary distribution service times and zero lead times. The optimal value of the maximum allowable inventory which minimizes the long - run expected cost rate has been obtained.

Berman and Sapna [6] discussed a finite capacity system with Poisson arrivals, exponential distributed lead times and service times. The existence of a stationary optimal service policy has been proved. Berman and Kim [3] addressed an infinite capacity queueing - inventory system with Poisson arrivals, exponential distributed lead times and service times. The authors identified a replenishment policy which maximized the system profit. Berman and Kim [4] studied internet based supply chains with Poisson arrivals, exponential service times, the Erlang lead times and found that the optimal ordering policy has a monotonic threshold structure. Krishnamoorthy et al. in [15] introduced an additional control policy (N-policy) into (s, S) inventory system with positive service time.

In all the above models, the authors assumed that after completion of the service (namely, regular service or main service or essential service), immediately the customers leave the system. But in many real life situation, all the arriving customers first require an essential service and only some may require additional optional service immediately after completion of the first essential service by the same server. The concept of the additional optional service with queue has been studied by several researchers in the past. As a related work we refer [11, 12].

In this article we have assumed Poisson demands for the commodity that are perishable and are issued to the customer after a random time of service performed on it. A (s, S) ordering policy with positive random lead time is adopted. The joint probability distribution for the inventory level and the number of customers is obtained in the steady-state case. Various measures of system performance are computed in the steady state case.

This paper is presented as follows. In the next section, the mathematical model and the notations used in this paper are described. Analysis of the model and the steady state solutions of the model are obtained in Section 3. In Section 4, we have derived the Laplace-Stieltjes transform of waiting time distribution of customers in the waiting hall. Some key system performance measures are derived in Section 5. In Section 6, we calculate the total expected cost rate. In Section 7, we present sensitivity analysis numerically. The last section is meant for conclusion.

2 Model description

Consider a continuous review perishable inventory system at a service facility with the maximum capacity of S units and N additional options for service. The waiting hall space is limited to accommodate a maximum number M of customers including the one at the service point. The waiting customers receive their service one by one and they demand single item. The arrival of customers is assumed to form a Poisson process with parameter $\lambda(> 0)$. Furthermore, customers who arrive and find either server busy or inventory level is zero must wait in the waiting hall until the server is available with positive inventory level.

The reorder level for the commodity is fixed as s and an order is placed when the inventory level reaches the reorder level s. The ordering quantity for the commodity is $Q(= S-s > s+1)$ item. The requirement $S-s > s+1$ ensures that after a replenishment the inventory level will be always above the reorder level. Otherwise it may not be possible to place reorder which leads to perpetual shortage. The lead time is assumed to be distributed as negative exponential with parameter $\beta(> 0)$. The life time of the commodity is assumed to be distributed as negative exponential with parameter $\gamma(> 0)$. We have assumed that an item of inventory that makes it into the service process cannot perish while in service.

There is a single server that provides the first essential service(regular service) as well as one of the N optional services (Type 1, Type 2, . . . , Type N) to each arriving customer. The items are issued to the demanding customers only after some random time due to some service on it. In this article the latter type of service referred to as first essential service(regular service). The first essential service of a customer is assumed to be exponentially distributed with parameter μ_{α_0}. As soon as the first essential service of a customer is completed, then with probability r_j the customer may ask for Type j service (i.e immediately customer requests additional service on their item), in which case his Type j service will immediately commence, or with probability r_0 he may opt to leave the system, in which case if both the inventory level and waiting hall size are positive, the customer will be taken for first essential service immediately by the server. Otherwise (i.e., either inventory level is zero or customer level is zero or both), server becomes idle. The service time of the jth optional service is assumed to be exponential with parameter μ_{α_j}, where $j = 1, 2, \dots, N$ and $\sum_{j=0}^{N} r_j = 1$.

Any arriving customer who finds the waiting hall full is considered to be lost. Various stochastic processes involved in the system are independent of each other.

2.1 Notations:

The following notations are used in the paper.

I : Identity matrix,

I_k : An identity matrix of order k,

π : a column vector of appropriate dimension containing all ones,

$\mathbf{0}$: Zero matrix,

$[A]_{ij}$: entry at $(i, j)^{th}$ position of a matrix A,

δ_{ij} : $\begin{cases} 1, & \text{if} \quad j = i, \\ 0, & \text{otherwise}, \end{cases}$

$\bar{\delta_{ij}}$: $1 - \delta_{ij}$,

$k \in V_i^j$: $k = i, i+1, \dots j$,

$H(x)$: $\begin{cases} 1, & \text{if} \ x \geq 0, \\ 0, & \text{otherwise}. \end{cases}$

3 Analysis

Let $L(t)$, $Y(t)$ and $X(t)$ respectively, denote the inventory level, the server status and the number of customers in the waiting hall at time t.

Further, server status $Y(t)$ is defined as follows:

$$Y(t)\ :\ \begin{cases} \alpha_i, & \text{if the server is idle at time } t,\\ \alpha_e, & \text{if the server is busy with essential service at time } t,\\ \alpha_1, & \text{if the server is busy with Type 1 service at time } t,\\ \alpha_2, & \text{if the server is busy with Type 2 service at time } t,\\ \alpha_3, & \text{if the server is busy with Type 3 service at time } t,\\ \vdots & \qquad\qquad\vdots\\ \alpha_{N-1}, & \text{if the server is busy with Type } N-1 \text{ service at time } t,\\ \alpha_N, & \text{if the server is busy with Type } N \text{ service at time } t. \end{cases}$$

From the assumptions made on the input and output processes, it can be shown that the stochastic process $I(t)=\{(L(t),Y(t),X(t)),t\geq 0\}$ is a continuous time Markov chain with state space given by $E=E_1\cup E_2\cup E_3\cup E_4$, where

$$\begin{aligned} E_1 &: \{(0,\alpha_i,i_3)\mid i_3=0,1,2,\ldots,M,\}\\ E_2 &: \{(i_1,\alpha_i,0)\mid i_1=1,2,\ldots,S,\}\\ E_3 &: \{(i_1,\alpha_e,i_3)\mid i_1=1,2,\ldots,S,\ i_3=1,2,\ldots,M,\}\\ E_4 &: \{(i_1,i_2,i_3)\mid i_1=0,1,2,\ldots,S,\ i_2=\alpha_1,\alpha_2,\ldots,\alpha_N,\ i_3=1,2,\ldots,M\}. \end{aligned}$$

Define the following ordered sets:
$\ll i_1,i_2\gg=$

$$\begin{cases} <i_1,\alpha_i,0>,<i_1,\alpha_i,1>,\ldots,<i_1,\alpha_i,M>, & i_1=0;\\ <i_1,\alpha_i,0>, & i_1=1,2,\ldots S;\\ <i_1,\alpha_e,1>,<i_1,\alpha_e,2>,\ldots,<i_1,\alpha_e,M>, & i_1=1,2,\ldots S;\\ <i_1,i_2,1>,<i_1,i_2,2>,\ldots,<i_1,i_2,M>, & i_1=0,1,2,\ldots S;\\ & i_2=\alpha_1,\alpha_2,\ldots,\alpha_N, \end{cases} \tag{1}$$

$$\lll i_1\ggg=\begin{cases} \ll i_1,\alpha_i,\gg, & i_1=0;\\ \ll i_1,\alpha_i,\gg, & i_1=1,2,\ldots S;\\ \ll i_1,\alpha_e\gg, & i_1=1,2,\ldots S;\\ \ll i_1,\alpha_1\gg,\ll i_1,\alpha_2\gg,\ldots,\ll i_1,\alpha_N\gg, & i_1=0,1,\ldots,S; \end{cases} \tag{2}$$

Then the state space can be ordered as $(\lll 0\ggg,\lll 1\ggg,\ldots,\lll S\ggg)$.

By ordering the state space $(\lll 0\ggg,\lll 1\ggg,\ldots,\lll S\ggg)$, the infinitesimal generator Θ can be conveniently written in a block partitioned matrix with entries

$$\Theta=\begin{array}{c} \\ \lll 0\ggg\\ \lll 1\ggg\\ \lll 2\ggg\\ \vdots\\ \lll S-1\ggg\\ \lll S\ggg \end{array} \begin{array}{c} \begin{array}{cccccc} \lll 0\ggg & \lll 1\ggg & \lll 2\ggg & \cdots & \lll S-1\ggg & \lll S\ggg \end{array}\\ \begin{pmatrix} A_{0,0} & A_{0,1} & A_{0,2} & \cdots & A_{0,S-1} & A_{0,S}\\ A_{1,0} & A_{1,1} & A_{1,2} & \cdots & A_{1,S-1} & A_{1,S}\\ A_{2,0} & A_{2,1} & A_{2,2} & \cdots & A_{2,S-1} & A_{2,S}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ A_{S-1,0} & A_{S-1,1} & A_{S-1,2} & \cdots & A_{S-1,S-1} & A_{S-1,S}\\ A_{S,0} & A_{S,1} & A_{S,2} & \cdots & A_{S,S-1} & A_{S,S} \end{pmatrix} \end{array}$$

More explicitly, due to the assumptions made on the demand and replenishment processes, we note that

$$A_{i_1,j_1} = \mathbf{0}, \quad \text{for} \quad j_1 \neq i_1, i_1 - 1, i_1 + Q.$$

We first consider the case A_{i_1,i_1+Q}. This will occur only when the inventory level is replenished. First we consider the inventory level is zero and the server is idle, that is $A_{0,Q}$. For this

Case (A) When there is no customer in the waiting hall and server is idle:

- At the time of replenishment the state of the system changes from $(0, \alpha_i, 0)$ to $(Q, \alpha_i, 0)$, with intensity of transition β. The sub matrix of the transition rates from $\ll 0, \alpha_i \gg$ to $\ll Q, \alpha_i \gg$, is given by

$$[F_1]_{i_3 j_3} = \begin{cases} \beta, & j_3 = 0, \quad i_3 = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Case (B) When there is a customer in the waiting hall and server is idle:

- At the time of replenishment takes the system state from $(0, \alpha_i, i_3)$ to (Q, α_e, i_3), $i_3 = 1, 2, \ldots, M$. The sub matrix of the transition rates from $\ll 0, \alpha_i \gg$ to $\ll Q, \alpha_e \gg$, is given by H

$$[H]_{i_3 j_3} = \begin{cases} \beta, & j_3 = i_3, \quad i_3 = 1, 2, \ldots, M, \\ 0, & \text{otherwise.} \end{cases}$$

Second, we consider the inventory level is zero and server is busy with Type $j (j = 1, 2, \ldots, N)$ service.

Case (C) At the time of a replenishment takes the system state from $(0, \alpha_j, i_3)$ to (Q, α_j, i_3) $j = 1, 2, \ldots, N,$. The sub matrix of the transition rates from $\ll 0, \alpha_j \gg$ to $\ll Q, \alpha_j \gg,$, $j = 1, 2, \ldots, N$, is given by

$$[J]_{i_3 j_3} = \begin{cases} \beta, & j_3 = i_3, \quad i_3 \in V_1^M, \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$[A_{0,Q}]_{i_2 j_2} = \begin{cases} F_1, & j_2 = i_2, \quad i_2 = \alpha_i, \\ H, & j_2 = \alpha_e, \quad i_2 = \alpha_i, \\ J, & j_2 = i_2, \quad i_2 = \alpha_1, \alpha_2, \ldots, \alpha_N, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

We denote $A_{0,Q}$ as C_1.

We now consider the case when the inventory level lies between one to s. We note that for this case, only the inventory level changes from i_1 to $i_1 + Q$. The other system states does not change. Simply, we have $[A_{i_1,i_1+Q}]_{i_2 j_2} = \beta I_{(N+1)M+1}$.

More explicitly,

$$[A_{i_1,i_1+Q}]_{i_2j_2} = \begin{cases} G, & j_2 = i_2, \quad i_2 = \alpha_i, \\ J, & j_2 = i_2, \quad i_2 = \alpha_e, \alpha_1, \alpha_2, \ldots, \alpha_N, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

$$[G]_{i_3j_3} = \begin{cases} \beta, & j_3 = i_3, \quad i_3 = 0, \\ 0, & \text{otherwise}. \end{cases}$$

A_{i_1,i_1+Q} is denoted by C.

Next, we consider the case $A_{i_1,i_1-1}, i_1 = 1, 2, \ldots, S$. This will occur only either when the essential service of a customer is completed or when any one of $i_1(i_1 = 1, 2, \ldots, S)$ items fails.

Now, we assume that the inventory level is one, that is $A_{1,0}$. For this, we have the following cases occur:

Case (D): When the server is idle and no customers in the waiting hall.

- Due to perishability takes the inventory level $(1, \alpha_i, 0)$ to $(0, \alpha_i, 0)$ with intensity of transition γ. The submatrix of the transition rates from $\ll 1, \alpha_i \gg$ to $\ll 0, \alpha_i \gg$, is given by

$$[D_0]_{i_3j_3} = \begin{cases} \alpha, & j_3 = i_3, \quad i_3 = 0, \\ 0, & \text{otherwise}. \end{cases}$$

Case (E): When the server is providing essential service to a customer and at least one customer in the waiting hall.

- The essential service of a customer is completed, both buffer size and inventory level decrease by one and then server becomes idle.

- With probability $r_j(j = 1, 2, \ldots, N)$, the serviced customer (essential service) may ask for Type j service, in which case his Type j service will immediately commence and the intensity of this transition $r_j\mu_0$ or with probability r_0 he may opt to leave the system, in which case if both the inventory level and waiting hall size are positive, the customer will be taken for first essential service immediately by the server. Otherwise (i.e., either inventory level is zero or no customer in the waiting hall), the server becomes idle.

Thus, the sub matrix of this transition rate from $\ll 1, \alpha_e \gg$ to $\ll 0, \alpha_i \gg$, or from $\ll 1, \alpha_e \gg$ to $\ll 0, \alpha_j \gg$, $j = 1, 2, \ldots, N$ is given by

For $i_2 = \alpha_1, \alpha_2, \ldots, \alpha_N$,

$$[D]_{i_3j_3} = \begin{cases} r_0\mu_{\alpha_0}, & j_3 = i_3 - 1, \quad i_3 \in V_1^M, \\ 0, & \text{otherwise}, \end{cases}$$

$$[H_{i_2}]_{i_3j_3} = \begin{cases} r_j\mu_{\alpha_0}, & j_3 = i_3, i_3 \in V_1^M \\ 0, & \text{otherwise}. \end{cases}$$

Case (F): When the server is busy with Type $j(j = 1, 2, \ldots, N)$ service and at least one customer in the waiting hall.

- Due to perishability takes the inventory level $(1, i_2, i_3)$ to $(0, i_2, i_3)$, $i_2 = \alpha_1, \alpha_2, \ldots, \alpha_N$, $i_3 = 1, 2, \ldots, M$, with intensity of transition γ. The sub matrix of the transition rates from $\ll 1, i_2 \gg$ to $\ll 0, i_2 \gg$, $i_2 = \alpha_1, \alpha_2, \ldots, \alpha_N$, is given by

$$[G_1]_{i_3 j_3} = \begin{cases} \gamma, & j_3 = i_3, \ \ i_3 \in V_1^M, \\ 0, & \text{otherwise.} \end{cases}$$

Hence $A_{1,0}$ is given by

$$[A_{1,0}]_{i_2 j_2} = \begin{cases} D_0, & j_2 = i_2, \ \ i_2 = \alpha_i, \\ D, & j_2 = \alpha_i, \ \ i_2 = \alpha_e, \\ H_{j_2}, & j_2 = \alpha_1, \alpha_2, \ldots, \alpha_N, \ \ i_2 = \alpha_e, \\ G_1, & j_2 = i_2, \ \ i_2 = \alpha_1, \alpha_2, \ldots, \alpha_N, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

$A_{1,0}$ is denoted by B_1.

Now, we have assume that the inventory level is more than one, that is $A_{i_1, i_1 - 1}, i_1 = 2, 3, \ldots, S$. For this, we have following cases occur.

Case (G): When the server is idle and no customer in the waiting hall:

- A transition from $(i_1, \alpha_i, 0)$ to $(i_1 - 1, \alpha_i, 0)$ will take place when any one of i_1 items perish at a rate of γ; thus intensity of transition is $i_1\gamma$, $i_1 = 2, \ldots, S$. The sub matrix of this transition rates from $\ll i_1, \alpha_i \gg$ to $\ll i_1 - 1, \alpha_i \gg$, $i_1 = 2, \ldots, S$, is given by

$$[D_{i_1}]_{i_3 j_3} = \begin{cases} i_1\gamma, & j_3 = i_3, \ \ i_3 = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Case (H): When the server is providing essential service to a customer and at least one customer in the waiting hall.

Using the similar argument (Case (E) and Case (G)), we have the sub matrices are given by F, F_{i_1}, H_{i_2}, $i_1 = 2, 3, \ldots, S$,$i_2 = \alpha_1, \alpha_2, \ldots, \alpha_N$.

$$[F_{i_1}]_{i_3 j_3} = \begin{cases} (i_1 - 1)\gamma, & j_3 = i_3, \ \ i_3 \in V_1^M, \\ r_0\mu_{\alpha_0}, & j_3 = i_3 - 1, \ \ i_3 \in V_2^M, \\ 0, & \text{otherwise,} \end{cases}$$

$$[H_{i_2}]_{i_3 j_3} = \begin{cases} r_j\mu_{\alpha_0}, & j_3 = i_3, \ \ i_3 \in V_1^N \\ 0, & \text{otherwise,} \end{cases}$$

$$[F]_{i_3 j_3} = \begin{cases} r_0\mu_{\alpha_0}, & j_3 = 0, \ \ i_3 = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Case (I): When the server is busy with type j, $j = 1, 2, \ldots, N$ service and at least one customer in the waiting hall.

Using the similar argument (Case (G)), we get the submatrix is given by G_{i_1}, $i_1 = 2, 3, \ldots, S$

$$[G_{i_1}]_{i_3 j_3} = \begin{cases} i_1\gamma, & j_3 = i_3, \quad i_3 \in V_1^M, \\ 0, & \text{otherwise.} \end{cases}$$

Hence $A_{i_1, i_1 - 1}$, is given by: For $i_1 = 2, 3, \ldots, S$,

$$[A_{i_1, i_1-1}]_{i_2 j_2} = \begin{cases} D_{i_1}, & j_2 = i_2, \quad i_2 = \alpha_i, \\ F, & j_2 = \alpha_i, \quad i_2 = \alpha_e, \\ F_{i_1}, & j_2 = i_2, \quad i_2 = \alpha_e, \\ H_{j_2}, & j_2 = \alpha_1, \alpha_2, \ldots, \alpha_N, \quad i_2 = \alpha_e, \\ G_{i_1}, & j_2 = i_2, \quad i_2 = \alpha_1, \alpha_2, \ldots, \alpha_N, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

We will denote A_{i_1, i_1-1}, $i_1 = 2, 3, \ldots, S$, as B_{i_1}.

Finally, we consider the case A_{i_1, i_1}, $i_1 = 0, 1, 2, 3, \ldots, S$. Here due to each one of the following mutually exclusive cases, a transition results:

- an arrival of a customer may occur;
- an optional service (Type 1, Type 2, ..., Type N) may be occurred.

If the inventory level is zero we have the following two states changes may arise:

Case (J): When the server is idle:

- An arrival of a customer increases, number of customers in the buffer increases by one and the state of the arrival process moves from $(0, \alpha_i, i_3)$ to $(0, \alpha_i, i_3 + 1)$, $i_3 = 0, 1, 2, \ldots, M - 1$, with intensity of this transition rates from $\ll 0, \alpha_i \gg$ to $\ll 0, \alpha_i \gg$, is denoted by L.

Case (K): When the server is busy with Type j service:

- At the time of service(optional service) completion of a customer, the state of the system moves from $(0, i_2, i_3)$ to $(0, \alpha_i, i_3 - 1)$, $i_2 = \alpha_1, \alpha_2, \ldots, \alpha_N$, $i_3 = 1, 2, \ldots, M$, with intensity of transition μ_{i_2}. The transition rates for any other transitions not considered above, when the inventory level is zero, are zero. The intensity of passage in the state $(0, i_2, i_3)$ is given by

$$-\sum_{(0, i_2, i_3) \neq (0, j_2, j_3)} a\left((0, i_2, i_3); (0, j_2, j_3)\right).$$

From the previous cases (case(I) and case (J)), we have construct the following matrices:

For $i_2 = \alpha_1, \alpha_2, \ldots, \alpha_N$

$$[L]_{i_3 j_3} = \begin{cases} \lambda, & j_3 = i_3 + 1, \quad i_3 \in V_0^{M-1}, \\ -(\bar{\delta}_{i_3 M}\lambda + \beta), & j_3 = i_3 + 1, \quad i_3 \in V_0^{M}, \\ 0, & \text{otherwise}, \end{cases}$$

$$[K_{i_2}]_{i_3 j_3} = \begin{cases} \mu_{i_2}, & j_3 = i_3 - 1, \quad i_3 \in V_1^{M}, \\ 0, & \text{otherwise}, \end{cases}$$

$$[L_{i_2}]_{i_3 j_3} = \begin{cases} -(\mu_{i_2} + \beta), & j_3 = i_3, \quad i_3 \in V_1^{M}, \\ 0, & \text{otherwise}. \end{cases}$$

Combining these matrices suitable form, we get

$$[A_{0,0}]_{i_2 j_2} = \begin{cases} L, & j_2 = i_2, \quad i_2 = \alpha_i, \\ K_{i_2}, & j_2 = \alpha_i, \quad i_2 = \alpha_1, \alpha_2, \ldots, \alpha_N, \\ E_{i_2}, & j_2 = i_2, \quad i_2 = \alpha_1, \alpha_2, \ldots, \alpha_N, \\ \mathbf{0}, & \text{otherwise}. \end{cases}$$

Hence the matrix $A_{0,0}$ is denoted by A_0. Similar to the above arguments it follows that:
For $i_1 = 1, 2, \ldots, s, s+1, \ldots, S$

$$[A_{i_1,i_1}]_{i_2 j_2} = \begin{cases} R, & j_2 = i_2, \quad i_2 = \alpha_i, \\ W, & j_2 = \alpha_e, \quad i_2 = \alpha_i, \\ R_{i_1}, & j_2 = i_2, \quad i_2 = \alpha_e, \\ L_{i_2}, & j_2 = \alpha_i, \quad i_2 = \alpha_1, \alpha_2, \ldots, \alpha_N, \\ V_{i_2}, & j_2 = \alpha_e, \quad i_2 = \alpha_1, \alpha_2, \ldots, \alpha_N, \\ U_{i_1}, & j_2 = i_2, \quad i_2 = \alpha_1, \alpha_2, \ldots, \alpha_N, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

$$[R]_{i_3 j_3} = \begin{cases} -(\lambda + i_1\alpha + H(s - i_1)\beta), & j_3 = i_3, \quad i_3 = 0, \\ 0, & \text{otherwise}, \end{cases}$$

$$[W]_{i_3 j_3} = \begin{cases} \lambda, & j_3 = 1, \quad i_3 = 0, \\ 0, & \text{otherwise}. \end{cases}$$

For $i_1 = 1, 2, 3, \ldots, S, \quad i_2 = \alpha_1, \alpha_2, \ldots, \alpha_N,$

$$[L_{i_2}]_{i_3j_3} = \begin{cases} \mu_{i_2}, & j_3 = 0, \quad i_3 = 1 \\ 0, & \text{otherwise.} \end{cases}$$

$$[V_{i_2}]_{i_3j_3} = \begin{cases} \mu_{i_2}, & j_3 = i_3 - 1, \quad i_3 \in V_2^M, \\ 0, & \text{otherwise.} \end{cases}$$

$$[U_{i_1}]_{i_3j_3} = \begin{cases} -(\bar{\delta}_{i_3M}\lambda + \mu_{i_2} + i_1\gamma + & \\ H(s-i_1)\beta), & j_3 = i_3, i_3 \in V_1^M \\ 0, & \text{otherwise.} \end{cases}$$

$$[R_{i_1}]_{i_3j_3} = \begin{cases} \lambda, & j_3 = i_3 + 1, \quad i_3 \in V_1^{M-1} \\ -(\bar{\delta}_{i_3M}\lambda + \sum_{j=1}^{N} r_j\mu_0 + & \\ (i_1-1)\gamma + H(s-i_1)\beta), & j_3 = i_3, \quad i_3 \in V_1^M \\ 0, & \text{otherwise.} \end{cases}$$

We denote $A_{i_1,i_1}, i_1 = 1, 2, \ldots, S$ as A_{i_1}. Hence the matrix Θ can be written in the following form

$$\Theta_{i_1j_1} = \begin{cases} A_{i_1} & j_1 = i_1, \ i_1 = 0, 1, 2, \ldots, S \\ B_{i_1} & j_1 = i_1 - 1, \ i_1 = 1, 2, \ldots, S-1, S \\ C & j_1 = i_1 + Q, \ i_1 = 1, 2, \ldots, s, \\ C_1 & j_1 = i_1 + Q, \ i_1 = 0, \\ \mathbf{0} & \text{Otherwise.} \end{cases}$$

More explicitly,

$$\Theta = \begin{array}{c} S \\ S-1 \\ \vdots \\ s+1 \\ s \\ s-1 \\ \vdots \\ 1 \\ 0 \end{array} \begin{pmatrix} A_S & B_S & & & & & & & & & \\ & A_{S-1} & B_{S-1} & & & & & & & & \\ & & \cdots & & & & & & & & \\ & & & \cdots & A_{s+1} & B_{s+1} & & & & & \\ C & & & & & A_s & B_s & & & & \\ & C & & & & & A_{s-1} & & & & \\ & & \cdots & & & & \cdots & & & & \\ & & C & & & & & \cdots & A_1 & B_1 \\ & & & C_1 & & & & & & A_0 \end{pmatrix}.$$

It may be noted that A_{i_1}, B_{i_1}, $i_1 = 1, 2, \ldots, S$, A_0, C_1 and C are square matrices of order $(N+1)M+1$. The sub matrices J, G_{i_1}, F_{i_1}, R_{i_1}, U_{i_1}, H_{i_2}, E_{i_2}, V_{i_2}, $i_1 = 1, 2, \ldots, S$, $i_2 = \alpha_1, \alpha_2, \ldots, \alpha_N$, are square matrices of order M. G, R, D_{i_1}, $i_1 = 2, 3, \ldots, S$ are square matrices of order 1. F, L_{i_2}, $i_2 = \alpha_1, \alpha_2, \ldots, \alpha_N$, are matrices of order $M \times 1$. D, K_{i_2}, $i_2 = \alpha_1, \alpha_2, \ldots, \alpha_N$, are matrices of order $M \times (M+1)$. L, W, D_0, F and H are matrices of order $(M+1) \times (M+1)$, $1 \times M$, $1 \times (M+1)$, $(M+1) \times 1$ and $(M+1) \times M$, respectively.

3.1 Steady State Analysis

It can be seen from the structure of Θ that the homogeneous Markov process $\{(L(t), Y(t), X(t)) : t \geq 0\}$ on the finite space E is irreducible, aperiodic and persistent non-null. Hence the limiting distribution

$$\phi^{(i_1,i_2,i_3)} = \lim_{t\to\infty} Pr[L(t) = i_1, Y(t) = i_2, X(t) = i_3 | L(0), Y(0), X(0)],$$

exists. Let $\mathbf{\Phi}$ denote the steady state probability vector of the generator Θ. The vector, $\mathbf{\Phi}$, partitioned as $\mathbf{\Phi} = (\mathbf{\Phi^{(0)}}, \mathbf{\Phi^{(1)}}, \ldots, \mathbf{\Phi^{(S)}})$, where

$$\begin{aligned}
\mathbf{\Phi^{(0)}} &= (\mathbf{\Phi^{(0,\alpha_i)}}, \mathbf{\Phi^{(0,\alpha_1)}}, \mathbf{\Phi^{(0,\alpha_2)}}, \ldots, \mathbf{\Phi^{(0,\alpha_N)}}),\\
\mathbf{\Phi^{(i_1)}} &= (\mathbf{\Phi^{(i_1,\alpha_i)}}, \mathbf{\Phi^{(i_1,\alpha_e)}}, \mathbf{\Phi^{(i_1,\alpha_1)}}, \mathbf{\Phi^{(i_1,\alpha_2)}}, \ldots, \mathbf{\Phi^{(i_1,\alpha_N)}}), i_1 = 1, 2, 3, \ldots, S;\\
\mathbf{\Phi^{(0,\alpha_i)}} &= (\phi^{(0,\alpha_i,0)}, \phi^{(0,\alpha_i,1)}, \ldots, \phi^{(0,\alpha_i,M)}),\\
\mathbf{\Phi^{(i_1,\alpha_i)}} &= (\phi^{(i_1,\alpha_i,0)}), \qquad i_1 = 1, 2, 3, \ldots, S;\\
\mathbf{\Phi^{(i_1,\alpha_e)}} &= (\phi^{(i_1,\alpha_e,1)}, \phi^{(i_1,\alpha_e,2)}, \ldots, \phi^{(i_1,\alpha_e,M)}),\ i_1 = 1, 2, 3, \ldots, S;\\
\mathbf{\Phi^{(i_1,i_2)}} &= (\phi^{(i_1,i_2,1)}, \phi^{(i_1,i_2,2)}, \ldots, \phi^{(i_1,i_2,M)}),\ i_1 = 0, 1, \ldots, S;\ i_2 = \alpha_1, \ldots, \alpha_N,
\end{aligned}$$

The computation of steady state probability vector $\mathbf{\Phi} = (\mathbf{\Phi^{(0)}}, \mathbf{\Phi^{(1)}}, \ldots, \mathbf{\Phi^{(S)}})$, can be done by solving the following set of equations,

$$\begin{aligned}
\mathbf{\Phi^{i_1}} B_{i_1} + \mathbf{\Phi^{i_1-1}} A_{i_1-1} &= \mathbf{0}, & i_1 &= 1, 2, \ldots, Q,\\
\mathbf{\Phi^{i_1}} B_{i_1} + \mathbf{\Phi^{i_1-1}} A_{i_1-1} + \mathbf{\Phi^{(i_1-1-Q)}} C_1 &= \mathbf{0}, & i_1 &= Q+1,\\
\mathbf{\Phi^{i_1}} B_{i_1} + \mathbf{\Phi^{i_1-1}} A_{i_1-1} + \mathbf{\Phi^{(i_1-1-Q)}} C &= \mathbf{0}, & i_1 &= Q+2, Q+3, \ldots, S,\\
\mathbf{\Phi^{S}} A_S + \mathbf{\Phi^{s}} C &= \mathbf{0},
\end{aligned}$$

subject to conditions $\mathbf{\Phi}\Theta = \mathbf{0}$ and $\sum\sum\sum_{(i_1,i_2,i_3)} \phi^{(i_1,i_2,i_3)} = 1$.

This is done by the following algorithm.

Step 1. Solve the following system of equations to find the value of $\mathbf{\Phi^{Q}}$

$$\mathbf{\Phi^{Q}} \left[\left\{ (-1)^Q \sum_{j=0}^{s-1} \left[\left(\mathop{\Omega}_{k=Q}^{s+1-j} B_k A_{k-1}^{-1} \right) C A_{S-j}^{-1} \left(\mathop{\Omega}_{l=S-j}^{Q+2} B_l A_{l-1}^{-1} \right) \right] \right\} B_{Q+1} + A_Q + \left\{ (-1)^Q \mathop{\Omega}_{j=Q}^{1} B_j A_{j-1}^{-1} \right\} C \right] = \mathbf{0},$$

and

$$\mathbf{\Phi^{Q}} \left[\sum_{i_1=0}^{Q-1} \left((-1)^{Q-i_1} \mathop{\Omega}_{j=Q}^{i_1+1} B_j A_{j-1}^{-1} \right) + I + \sum_{i_1=Q+1}^{S} \left((-1)^{2Q-i_1+1} \sum_{j=0}^{S-i_1} \left[\left(\mathop{\Omega}_{k=Q}^{s+1-j} B_k A_{k-1}^{-1} \right) C A_{S-j}^{-1} \left(\mathop{\Omega}_{l=S-j}^{i_1+1} B_l A_{l-1}^{-1} \right) \right] \right) \right] \boldsymbol{\pi} = 1.$$

Step 2. Compute the values of

$$\begin{aligned}
\mathbf{\Omega}_{i_1} &= (-1)^{Q-i_1}\mathbf{\Phi}^{\mathbf{Q}} \mathop{\Omega}_{j=Q}^{i_1+1} B_j A_{j-1}^{-1}, && i_1 = Q-1, Q-2, \dots, 0 \\
&= (-1)^{2Q-i_1+1}\mathbf{\Phi}^{\mathbf{Q}} \sum_{j=0}^{S-i_1} \left[\left(\mathop{\Omega}_{k=Q}^{s+1-j} B_k A_{k-1}^{-1} \right) C A_{S-j}^{-1} \left(\mathop{\Omega}_{l=S-j}^{i_1+1} B_l A_{l-1}^{-1} \right) \right], \\
& && i_1 = S, S-1, \dots, Q+1 \\
&= I, && i_1 = Q
\end{aligned}$$

Step 3. Using Step 1 ($\mathbf{\Phi}^{(\mathbf{Q})}$) and Step 2 ($\Omega_{i_1}, i_1 = 0, 1, \dots, S$) calculate the value of $\mathbf{\Phi}^{(\mathbf{i_1})}, i_1 = 0, 1, \dots, S$. That is,

$$\mathbf{\Phi}^{(\mathbf{i_1})} = \mathbf{\Phi}^{(\mathbf{Q})}\Omega_{i_1}, \quad i_1 = 0, 1, \dots, S.$$

4 Waiting time distribution

Generally, system performance measures in inventories are related to the availability of stock but are not customer oriented. However, inventory maintained at service facilities, queues may form and hence the waiting time of the customer cannot be neglected because it gives important information about the system performance from the customers point of view. Hence, in this section we derive the Laplace - Stieltjes transform of waiting time distribution for customers.

4.1 Waiting time of the customers

In this subsection, our aim is to derive the waiting time for the customer. We deal with the arriving (tagged) customer waiting time, defined as the time between the arrival epoch of a customer till the instant at which the customer request is satisfied. We will represent this continuous random variable as W_1. The objective is to describe the probability that a customer has to wait, the distribution of the waiting time and n^{th} order moments. Note that W_1 is zero when the arriving customer finds positive stock and the server is free. Consequently, the probability that the customer does not have to wait is given by

$$P\{W_1 = 0\} = \sum_{i_1=1}^{S} \phi^{(i_1, \alpha_i, 0)}$$

In order to get the distribution of W_1, we will define some auxiliary variables. Let us consider the Markov process at an arbitrary time t and suppose that it is at state $(i_1, i_2, i_3), i_3 > 0$. We tag any of those waiting customer and ${}^{(1)}W_{(i_1,i_2,i_3)}$ denotes the time until the selected customer gets the desired item. Let $W_1^*(y) = E[e^{-yW_1}]$ and ${}^{(1)}W_{(i_1,i_2,i_3)}^*(y) = E[e^{-y^{(1)}W_{(i_1,i_2,i_3)}}]$ be the corresponding Laplace-Stieltjes transforms for unconditional and conditional waiting time. Obviously we have that

$$W_1^*(y) = \sum_{i_1=1}^{S} \phi^{(i_1,\alpha_i,0)} + \sum_{i_1=0}^{S}\sum_{i_2=\alpha_1}^{\alpha_N}\sum_{i_3=1}^{M-1} \phi^{(i_1,i_2,i_3)(1)} W^*_{(i_1,i_2,i_3+1)}(y)$$
$$+ \sum_{i_1=1}^{S}\sum_{i_3=0}^{M-1} \phi^{(i_1,\alpha_e,i_3)(1)} W^*_{(i_1,\alpha_e,i_3+1)}(y) + \sum_{i_3=0}^{M-1} \phi^{(0,\alpha_i,i_3)(1)} W^*_{(0,\alpha_i,i_3+1)}(y) \tag{3}$$

To study ${}^{(1)}W^*_{(i_1,i_2,i_3)}$, we introduce an auxiliary Markov chain on the state space $E^* = E_1 \cup E_2 \cup E_3 \cup E_4 \cup \{*\}$, where $\{*\}$ represents an absorbing state. The absorption occurs when the customer gets his requested item.

Being on the state (i_1, i_2, i_3), we apply the first step argument in the auxiliary chain (i.e., we condition on the epoch of the next event and next state of this chain) in order to determine the Laplace-Stieltjes transform ${}^{(1)}W^*_{(i_1,i_2,i_3)}(y)$. The functions ${}^{(1)}W^*_{(i_1,i_2,i_3)}(y)$, $(i_1,i_2,i_3) \in E$ are the smallest non-negative solution to the system

For $i_1 = 0, \quad i_2 = \alpha_i, \quad 1 \le i_3 \le M$

$$w_2 {}^{(1)}W^*_{(i_1,i_2,i_3)}(y) - \lambda\bar{\delta}_{i_3M} {}^{(1)}W^*_{(i_1,i_2,i_3+1)}(y) - \beta\delta_{i_30} {}^{(1)}W^*_{(i_1+Q,i_2,i_3)}(y)$$
$$-\beta\bar{\delta}_{i_30} {}^{(1)}W^*_{(i_1+Q,i_2,i_3)}(y) = 0 \tag{4}$$

where

$$w_2 = y + \lambda\bar{\delta}_{i_3M} + \beta\delta_{i_30} + \beta\bar{\delta}_{i_30}$$

For $1 \le i_1 \le S, \quad i_2 = \alpha_e, \quad 1 \le i_3 \le M,$

$$w_3 {}^{(1)}W^*_{(i_1,i_2,i_3)}(y) - \lambda\bar{\delta}_{i_3M} {}^{(1)}W^*_{(i_1,i_2,i_3+1)}(y) - \beta H(s-i_1) {}^{(1)}W^*_{(i_1+Q,\alpha_e,i_3)}(y)$$
$$-(i_1-1)\gamma\bar{\delta}_{i_11} {}^{(1)}W^*_{(i_1-1,i_2,i_3)}(y) - r_0\mu_0\bar{\delta}_{i_11}\bar{\delta}_{i_31} {}^{(1)}W^*_{(i_1-1,i_2,i_3-1)}(y)$$
$$-r_0\mu_0\delta_{i_11} {}^{(1)}W^*_{(i_1-1,\alpha_i,i_3-1)}(y) - r_0\mu_0\delta_{i_31} {}^{(1)}W^*_{(i_1-1,\alpha_i,i_3-1)}(y)$$
$$-r_1\mu_0 {}^{(1)}W^*_{(i_1,\alpha_1,i_3)}(y) - r_2\mu_0 {}^{(1)}W^*_{(i_1,\alpha_2,i_3)}(y)$$
$$-r_3\mu_0 {}^{(1)}W^*_{(i_1,\alpha_3,i_3)}(y) \ldots - r_N\mu_0 {}^{(1)}W^*_{(i_1,\alpha_N,i_3)}(y) = 0 \tag{5}$$

where

$$w_3 = y + \lambda\bar{\delta}_{i_3M} + \beta H(s-i_1) + (i_1-1)\gamma\bar{\delta}_{i_11} + r_0\mu_0\bar{\delta}_{i_11}\bar{\delta}_{i_31}$$
$$+r_0\mu_0\delta_{i_11} + r_0\mu_0\delta_{i_31} + r_1\mu_0 + r_2\mu_0 + \ldots + r_N\mu_0$$

For $0 \le i_1 \le S, \quad \alpha_1 \le i_2 \le \alpha_N, \quad 1 \le i_3 \le M,$

$$w_4 {}^{(1)}W^*_{(i_1,i_2,i_3)}(y) - \lambda\bar{\delta}_{i_3M} {}^{(1)}W^*_{(i_1,i_2,i_3+1)}(y)$$
$$-i_1\gamma\bar{\delta}_{i_10} {}^{(1)}W^*_{(i_1-1,i_2,i_3)}(y) - \beta H(s-i_1) {}^{(1)}W^*_{(i_1+Q,i_2,i_3)}(y)$$
$$-\mu_{i_2}\bar{\delta}_{i_10} {}^{(1)}W^*_{(i_1,\alpha_i,i_3-1)}(y) - \mu_{i_2}\delta_{i_11}\delta_{i_31} {}^{(1)}W^*_{(i_1,\alpha_i,i_3-1)}(y)$$
$$-\mu_{i_2}\delta_{i_10}\bar{\delta}_{i_31} {}^{(1)}W^*_{(i_1,\alpha_e,i_3-1)}(y) = 0 \tag{6}$$

where

$$w_4 \quad = \quad y + \lambda\bar{\delta}_{i_3M} + i_1\gamma\bar{\delta}_{i_10} + \beta H(s-i_1) + \mu_{i_2}\delta_{i_10} + \mu_{i_2}\delta_{i_11}\delta_{i_31} + \mu_{i_2}\delta_{i_10}\bar{\delta}_{i_31}$$

Using the expression (3) we get $W_1^*(y)$ for a given y. This facilitates the applications of the Euler and Post-Widder algorithms in Abate and Whitt [1] for the numerical inversion of $W_1^*(y)$.

We can exploit the system of equations (4) - (6) to get a recursive algorithm for computing moments for the conditional and unconditional waiting times.

By differentiating $(n+1)$ times the system of equations (4) - (6), and evaluating at $y=0$, we arrive at

For $i_1 = 0, \quad i_2 = \alpha_i, \quad 1 \le i_3 \le M$

$$\begin{aligned} w_6 E\left[{}^{(1)}W^{(n+1)}_{(i_1,i_2,i_3)}\right] - \lambda\bar{\delta}_{i_3M} E\left[{}^{(1)}W^{(n+1)}_{(i_1,i_2,i_3+1)}\right] - \beta\delta_{i_30} E\left[{}^{(1)}W^{(n+1)}_{(i_1+Q,i_2,i_3)}\right] \\ -\beta\bar{\delta}_{i_30} E\left[{}^{(1)}W^{(n+1)}_{(i_1+Q,i_2,i_3)}\right] = (n+1)E\left[{}^{(1)}W^{(n)}_{(i_1,i_2,i_3)}\right] \end{aligned} \tag{7}$$

where

$$w_6 \quad = \quad \lambda\bar{\delta}_{i_3M} + \beta\delta_{i_30} + \beta\bar{\delta}_{i_30}$$

For $1 \le i_1 \le S, \quad i_2 = \alpha_e, \quad 1 \le i_3 \le M$,

$$\begin{aligned} w_7 E\left[{}^{(1)}W^{(n+1)}_{(i_1,i_2,i_3)}\right] - \lambda\bar{\delta}_{i_3M} E\left[{}^{(1)}W^{(n+1)}_{(i_1,i_2,i_3+1)}\right] \\ -\beta H(s-i_1) E\left[{}^{(1)}W^{(n+1)}_{(i_1+Q,\alpha_e,i_3)}\right] - (i_1-1)\gamma\bar{\delta}_{i_11} E\left[{}^{(1)}W^{(n+1)}_{(i_1-1,i_2,i_3)}\right] \\ -r_0\mu_0\bar{\delta}_{i_11}\bar{\delta}_{i_31} E\left[{}^{(1)}W^{(n+1)}_{(i_1-1,i_2,i_3-1)}\right] - r_0\mu_0\delta_{i_11} E\left[{}^{(1)}W^{(n+1)}_{(i_1-1,\alpha_i,i_3-1)}\right] \\ -r_0\mu_0\delta_{i_31} E\left[{}^{(1)}W^{(n+1)}_{(i_1-1,\alpha_i,i_3-1)}\right] - r_1\mu_0 E\left[{}^{(1)}W^{(n+1)}_{(i_1,\alpha_1,i_3)}\right] \\ -r_2\mu_0 E\left[{}^{(1)}W^{(n+1)}_{(i_1,\alpha_2,i_3)}\right] - r_3\mu_0 E\left[{}^{(1)}W^{(n+1)}_{(i_1,\alpha_3,i_3)}\right] \\ \ldots - r_N\mu_0 E\left[{}^{(1)}W^{(n+1)}_{(i_1,\alpha_N,i_3)}\right] = (n+1)E\left[{}^{(1)}W^{(n)}_{(i_1,i_2,i_3)}\right] \end{aligned} \tag{8}$$

where

$$\begin{aligned} w_7 \quad = \quad & \lambda\bar{\delta}_{i_3M} + \beta H(s-i_1) + (i_1-1)\gamma\bar{\delta}_{i_11} + r_0\mu_0\bar{\delta}_{i_11}\bar{\delta}_{i_31} \\ & + r_0\mu_0\delta_{i_11} + r_0\mu_0\delta_{i_31} + r_1\mu_0 + r_2\mu_0 + \ldots + r_N\mu_0 \end{aligned}$$

For $0 \le i_1 \le S, \quad \alpha_1 \le i_2 \le \alpha_N, \quad 1 \le i_3 \le M$,

$$\begin{aligned} w_8 E\left[{}^{(1)}W^{(n+1)}_{(i_1,i_2,i_3)}\right] - \lambda\bar{\delta}_{i_3M} E\left[{}^{(1)}W^{(n+1)}_{(i_1,i_2,i_3+1)}\right] - i_1\gamma\bar{\delta}_{i_10} E\left[{}^{(1)}W^{(n+1)}_{(i_1-1,i_2,i_3)}\right] \\ -\beta H(s-i_1) E\left[{}^{(1)}W^{(n+1)}_{(i_1+Q,i_2,i_3)}\right] - \mu_{i_2}\bar{\delta}_{i_10} E\left[{}^{(1)}W^{(n+1)}_{(i_1,\alpha_i,i_3-1)}\right] \\ -\mu_{i_2}\delta_{i_11}\delta_{i_31} E\left[{}^{(1)}W^{(n+1)}_{(i_1,\alpha_i,i_3-1)}\right] - \mu_{i_2}\delta_{i_10}\bar{\delta}_{i_31} E\left[{}^{(1)}W^{(n+1)}_{(i_1,\alpha_e,i_3-1)}\right] \\ = (n+1)E\left[{}^{(1)}W^{(n)}_{(i_1,i_2,i_3)}\right] \end{aligned} \tag{9}$$

where

$$w_8 \quad = \quad \lambda\bar{\delta}_{i_3M} + i_1\gamma\bar{\delta}_{i_10} + \beta H(s-i_1) + \mu_{i_2}\delta_{i_10} + \mu_{i_2}\delta_{i_11}\delta_{i_31} + \mu_{i_2}\delta_{i_10}\bar{\delta}_{i_31}$$

Equations (7) - (9) are used to determine the unknowns $E\left[{}^{(1)}W^{(n+1)}_{(i_1,i_2,i_3)}\right]$, $(i_1, i_2, i_3) \in E$ in terms of the moments of one order less. Noticing that $E\left[{}^{(1)}W^{(n)}_{(i_1,i_2,i_3,)}\right] = 1$, for $n = 0$. We can obtain the moments up to a desired order in a recursive way. For determine the moments of W_1 we differentiate $W_1^*(y)$ and evaluate at $y = 0$, we have

$$\begin{aligned} E[W_1^{(n)}] &= \delta_{0n} + (1-\delta_{0n})\sum_{1_3=0}^{M-1}\phi^{(0,\alpha_i,i_3)}E\left[{}^{(1)}W^{(n)}_{(0,\alpha_i,i_3+1)}\right] \\ &+ (1-\delta_{0n})\sum_{i_1=1}^{S}\sum_{i_3=1}^{M-1}\phi^{(i_1,\alpha_e,i_3)}E\left[{}^{(1)}W^{(n)}_{(i_1,\alpha_e,i_3+1)}\right] \\ &+ (1-\delta_{0n})\sum_{i_1=0}^{S}\sum_{i_2=\alpha_1}^{\alpha_N}\sum_{i_3=1}^{M-1}\phi^{(i_1,i_2,i_3)}E\left[{}^{(1)}W^{(n)}_{(i_1,i_2,i_3+1)}\right](1-\delta_{0n}) \end{aligned} \tag{10}$$

which provides the n^{th} moments of the unconditional waiting time in terms of conditional moments of the same order.

5 System performance measures

In this section, we derive some measures of system performance in the steady state. Using this, we calculate the total expected cost rate.

5.1 Expected inventory level

Let η_I denote the excepted inventory level in the steady state. Since $\Phi^{(i_1)}$ is the steady state probability vector that there are i_1 items in the inventory with each component represents a particular combination of the number of customers in the waiting hall and the status of the server, $\Phi^{(i_1)}\mathbf{e}$ gives the probability of i_1 item in the inventory in the steady state. Hence η_I is given by

$$\eta_I = \sum_{i_1=1}^{S} i_1\Phi^{(i_1)}\mathbf{e}$$

5.2 Expected reorder rate

Let η_R denote the expected reorder rate in the steady state. A reorder is placed when the inventory level drops from $s+1$ to s. This may occur in the following two cases:

- the server completes the essential service for a customer
- any one of the $(s+1)$ items fails when the server is idle,
- any one of the s items fails when the server is busy with an essential service,
- any one of the $(s+1)$ items fails when the server is busy with Type j service,

Hence we get

$$\eta_R = (s+1)\gamma\phi^{(s+1,\alpha_i,0)} + \sum_{i_2=\alpha_1}^{\alpha_N}\sum_{i_3=1}^{M}(s+1)\gamma\phi^{(s+1,i_2,i_3)} + \sum_{i_3=1}^{M}(r_0\mu_{\alpha_0}+s\gamma)\phi^{(s+1,\alpha_e,i_3)}$$

5.3 Expected perishable rate

Since $\Phi^{(i_1)}$ is the steady state probability vector for inventory level, the expected perishable rate η_P is given by

$$\eta_P = \sum_{i_1=1}^{S} i_1\gamma\phi^{(i_1,\alpha_i,0)} + \sum_{i_1=1}^{S}\sum_{i_3=1}^{M}(i_1-1)\gamma\phi^{(i_1,\alpha_e,i_3)} + \sum_{i_1=1}^{S}\sum_{i_2=\alpha_1}^{\alpha_N}\sum_{i_3=1}^{M} i_1\gamma\phi^{(i_1,i_2,i_3)}$$

5.4 Expected number of customers in the waiting hall

Let Γ denote the expected number of customers in the steady state. Since $\phi^{(i_1,i_2,i_3)}$ is a vector of probabilities with the inventory level i_1, the server status is i_2 and the number of customer in the waiting hall is i_3, the expected number of customers Γ in the steady state is given by

$$\Gamma = \sum_{i_1=1}^{S}\sum_{i_3=1}^{M} i_3\phi^{(i_1,\alpha_e,i_3)} + \sum_{i_1=0}^{S}\sum_{i_2=\alpha_1}^{\alpha_N}\sum_{i_3=1}^{M} i_3\phi^{(i_1,i_2,i_3)}$$

5.5 Expected waiting time

Let η_W denote the expected waiting time of the customers in the waiting hall. Then by Little's formula

$$\eta_W = \frac{\Gamma}{\eta_{AR}},$$

where Γ is the expected number of customers in the waiting hall and the effective arrival rate (Ross [19]), η_{AR} is given by

$$\eta_{AR} = \sum_{i_1=1}^{S}\lambda\phi^{(i_1,\alpha_i,0)} + \sum_{i_1=1}^{S}\sum_{i_3=1}^{M-1}\lambda\phi^{(i_1,\alpha_e,i_3)} + \sum_{i_1=0}^{S}\sum_{i_2=\alpha_1}^{\alpha_N}\sum_{i_3=1}^{M-1}\lambda\phi^{(i_1,i_2,i_3)}$$

5.6 Average customers lost to the system

Let η_{BP} denote the average customers lost to the system in the steady state. Any arriving customer finds the waiting hall is full and leaves the system without getting service. These customers are considered to be lost. Thus we obtain

$$\eta_{BP} = \lambda\phi^{(0,\alpha_i,M)} + \sum_{i_1=1}^{S} \lambda\phi^{(i_1,\alpha_e,M)} + \sum_{i_1=0}^{S}\sum_{i_2=\alpha_1}^{\alpha_N} \lambda\phi^{(i_1,i_2,M)}$$

5.7 Probability that server is busy with essential service

Let η_{SB} denote the probability that server is busy with essential service is given by

$$\eta_{SB} = \sum_{i_1=1}^{S}\sum_{i_3=1}^{M} \phi^{(i_1,\alpha_e,i_3)}$$

5.8 Probability that server is idle

Let η_{SI} denote the probability that server is idle is given by

$$\eta_{SI} = \sum_{i_1=0}^{S} \phi^{(i_1,\alpha_i,0)}$$

5.9 Probability that server is busy with optional service

Let η_{SO} denote the probability that server is idle is given by

$$\eta_{SO} = \sum_{i_1=0}^{S}\sum_{i_2=\alpha_1}^{\alpha_N}\sum_{i_3=1}^{M} \phi^{(i_1,i_2,i_3)}$$

6 Cost analysis

The expected total cost per unit time (expected total cost rate) in the steady state for this model is defined to be

$$TC(S,s,N,M) = c_h\eta_I + c_s\eta_R + c_p\eta_P + c_w\eta_W + c_l\eta_{BP},$$

c_h : The inventory carrying cost per unit item per unit time,
c_s : Setup cost per order,
c_p : Perishable cost per unit item per unit time,
c_w : Waiting time cost of a customer per unit time,
c_l : Cost of a customer lost per unit time,

Substituting the values of η's, we get

TC(S, s, N, M)=

$$c_s\left[(s+1)\gamma\phi^{(s+1,\alpha_i,0)}\right]+c_h\left[\sum_{i_1=1}^{S} i_1\phi^{(i_1)}\mathbf{e}\right]+c_w\left[\frac{\Gamma}{\eta_{AR}}\right]+$$

$$c_s\left[\sum_{i_3=1}^{M}\left((r_0\mu_0+s\gamma)\phi^{(s+1,\alpha_e,i_3)}+\sum_{i_2=\alpha_1}^{\alpha_N}\sum_{i_3=1}^{M}(s+1)\gamma\phi^{(s+1,i_2,i_3)}\right)\right]+$$

$$+c_p\sum_{i_1=1}^{S}\left[i_1\gamma\phi^{(i_1,\alpha_e,0)}+\sum_{i_1=1}^{M}(i_1-1)\gamma\phi^{(i_1,\alpha_e,i_3)}+\sum_{i_2=\alpha_1}^{\alpha_N}\sum_{i_3=1}^{M}i_1\gamma\phi^{(i_1,i_2,i_3)}\right]+$$

$$+c_l\left[\lambda\phi^{(0,\alpha_i,M)}+\sum_{i_1=1}^{S}\lambda\phi^{(i_1,\alpha_e,M)}+\sum_{i_1=0}^{S}\phi^{(i_1,0,0,i_4)}\lambda\phi^{(i_1,i_2,M)}\right]$$

7 Numerical illustrations

To study the behaviour of the model developed in this work, several examples were performed and a set of representative results is shown here. Although not showing the convexity of $TC(s,S)$ analytically, our experience with considerable numerical examples indicates the function $TC(s,S)$, to be convex. In some cases, it turned out to be an increasing function of s, and simple numerical search procedures are used to obtain the optimal values of TC, s and S (say TC^*, s^* and S^*). A typical three dimensional plot of the expected cost function is given in Figure 1. We have assumed constant values for other parameters and costs. Namely, $N=2$, $M=5$, $\lambda=8.2$, $\beta=0.02$, $\mu_{\alpha_0}=0.01$, $\mu_{\alpha_1}=0.3$, $\mu_{\alpha_2}=0.1$, $r_0=0.5$, $r_1=0.25$, $r_2=0.25$, $c_h=0.01$, $c_s=50$, $c_p=0.5$, $c_w=0.3$, $c_l=0.2$, and $c_{wl}=9$. The optimal cost value $TC^*=16.764817$ is obtained at $(5,26)$.

The effect of varying the system parameters and costs on the optimal values have been studied and the results agreed with as expected. First, we explore the behavior of the cost function by considering it as function of any two variables by fixing the others at a constant level. Table 1, gives the total expected cost rate for various combinations of S and s. We have assumed constant values for other parameters and costs. Namely, $\lambda_1=8.2$, $\beta=0.02$, $\gamma=0.01$, $\mu_{\alpha_0}=0.01$, $\mu_{\alpha_1}=0.3$, $\mu_{\alpha_2}=0.1$, $r_0=0.5$, $r_1=0.25$, $r_2=0.25$, $c_h=0.01$, $c_s=50$, $c_p=0.5$, $c_w=0.3$, $c_l=0.2$, $N=2$, $M=5$.

Table 2, gives the total expected cost rate for various combinations of s and M, by assuming fixed values for other parameters and costs. Namely, $\lambda_1=10$, $\beta=2$, $\gamma=0.13$, $\mu_{\alpha_0}=4$, $\mu_{\alpha_1}=0.3$, $\mu_{\alpha_2}=1$, $r_0=0.5$, $r_1=0.25$, $r_2=0.25$, $c_h=0.01$, $c_s=50$, $c_p=0.5$, $c_w=0.3$, $c_l=0.2$, $N=2$, $S=40$. Table 3, gives the total expected cost rate for various combinations of S and M, by assuming fixed values for other parameters and costs. Namely, $\lambda_1=9$, $\beta=2$, $\gamma=1$, $\mu_{\alpha_0}=4$, $\mu_{\alpha_1}=0.3$, $\mu_{\alpha_2}=1$, $r_0=0.5$, $r_1=0.25$, $r_2=0.25$, $c_h=0.01$, $c_s=50$, $c_p=0.5$, $c_w=0.3$, $c_l=0.2$, $N=2$, $s=3$. The value that is shown bold is the least among the values in that row and the value that is shown underlined is the least in that

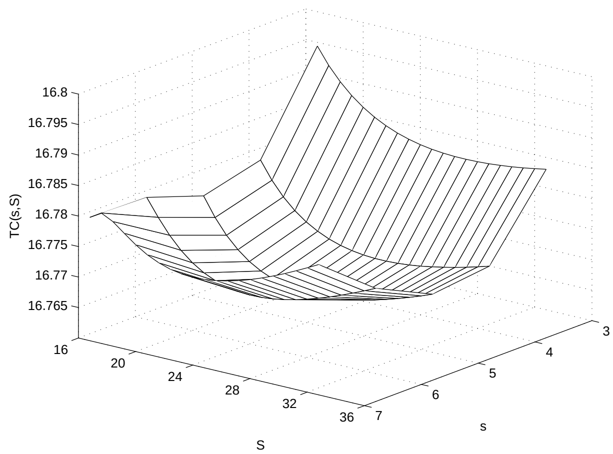

Figure 1: A three dimensional plot of the cost function TC(s, S).

column. It may be observed that, these values in each table exhibit a (possibly) local minimum of the function of the two variables.

Example 1. In this example, we study the impact of the setup cost c_s, holding cost c_h, perishable cost c_p, shortage cost c_l and the waiting cost c_w on the optimal values s^*, S^* and TC^*. Towards this end, we first fix the parameter values as $\lambda = 9,\ \beta = 0.2,\ \gamma = 0.01,\ \mu_{\alpha_0} = 0.01,\ \mu_{\alpha_1} = 0.3,\ \mu_{\alpha_2} = 1,\ r_0 = 0.5,\ r_1 = 0.25,\ r_2 = 0.25,\ N = 2, M = 5$. We observe the following from Table $4 - 10$:

1. The optimal cost increases, when c_s, c_h, c_p, c_l, and c_w increase. The optimal cost is more sensitive to c_w than to c_s, c_h, c_p and c_l.
2. As c_h increases, as is to be expected, the optimal values s^* and S^* decrease monotonically. This is to be expected since the holding cost increases, we resort to maintain low stock in the inventory.
3. When c_l increases, as is to be expected, the optimal values s^* and S^* increase monotonically. This is to be expected since for the high shortage cost to reduce the number of customer to be lost, we have to increase the waiting area size, maintain high inventory, and place order in the higher level.
4. If the setup cost c_s increases, it is a common decision that we have to maintain more stock to avoid frequent ordering. This fact is also observed in our model.
5. We note that, when the waiting cost c_w increases, the optimal values of s^* and S^* increase monotonically and when the perishable cost c_p increases, s^* and S^* decrease monotonically.

In the following all numerical examples, we select $c_h = 0.01$, $c_s = 50$, $c_p = 0.5$, $c_m = 0.3$, and $c_l = 0.2$.

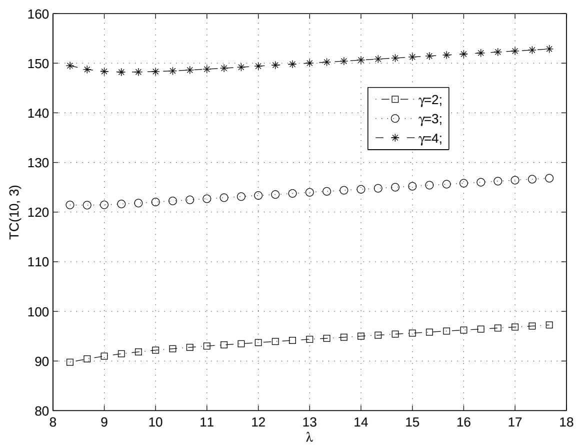

Figure 2: λ vs γ on TC.

Example 2. In this example, we look at the impact of the demand rate λ, the perishable rate γ, the lead time rate β, essential service rate μ_{α_0} and optional service rates μ_{α_1} and μ_{α_2} on the total expected cost rate. Towards this end, we first fix the parameter values as $S = 10$, $s = 3$, $N = 2$, $M = 5$, $r_0 = 0.5$, $r_1 = 0.25$ and $r_2 = 0.25$. From Figure 2 and 3, we observe the following:

1. The optimal expected cost rate increases when λ increases.
2. The optimal expected cost rate increases when γ increases.
3. The total expected cost rate initially decreases then it stabilizes or slightly increases when β increases.
4. The optimal expected cost rate decreases when μ_{α_0}, μ_{α_1} and μ_{α_2} increase.

Example 3. Here, we study the impact of arrival rate λ, the perishable rate γ, the lead time rate β, number of customers in the waiting area M and essential service rate μ_{α_0} on the expected number of customers in waiting area Γ. Towards this end, we first fix the parameter values as $S = 10$, $s = 3$, $N = 2$, $\mu_{\alpha_1} = 0.3$, $\mu_{\alpha_2} = 1$, $r_0 = 0.5$, $r_1 = 0.25$ and $r_2 = 0.25$. We observe the following from Figure 4 to 6.

1. The expected number of customers in the waiting area is an increasing function of perishable rate (see Figure 4) and this behaviour is maintained for various values of M, namely $M = 5$, 10 and 15. However, the expected number of customers in the waiting area is higher if M is larger.
2. The expected number of customers in the waiting area increases when λ and M increase.
3. The expected number of customers in the waiting area increases when the essential service rate increases.

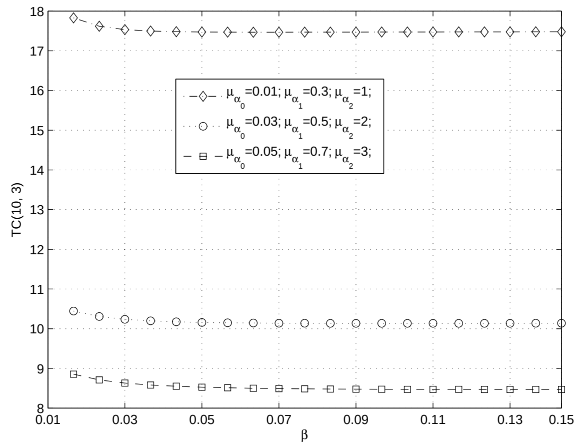

Figure 3: β vs $\mu_{\alpha_0},\ \mu_{\alpha_1}\ \mu_{\alpha_2}$ on TC.

4. The expected number of customers in the waiting area increases when re-order rate increases.

Next, the numerical results are obtained by considering different service cases as follows:

Case 1: $r_0 = 0.33$, $r_1 = 0.33$, $r_2 = 0.33$;
Case 2: $r_0 = 0.5$, $r_1 = 0.25$, $r_2 = 0.25$;
Case 3: $r_0 = 0.6$, $r_1 = 0.3$, $r_2 = 0.1$;
Case 4: $r_0 = 0.6$, $r_1 = 0.4$, $r_2 = 0$;

Figure 7, depicts the effect of the different service cases and the perishable rate on the total expected cost rate. From Figure 7, the total expected cost rate of the four different service cases, showed the following result:

$$TC_{\text{class 4}} > TC_{\text{class 1}} > TC_{\text{class 3}} > TC_{\text{class 2}} \tag{11}$$

Figure 8, depicts the effect of the different service cases and the perishable rate on the expected number of customers in the waiting area. The effect of the different service cases and customer arrival rate on the expected number of customers in the waiting area is shown in Figure 9. The effect of the different service cases and customer arrival rate on the expected shortage rate is shown in Figure 10. From Figures $8 - 10$, the comparison of the expected number of customers in the waiting area and expected shortage rate of the four different service cases, showed the following results:

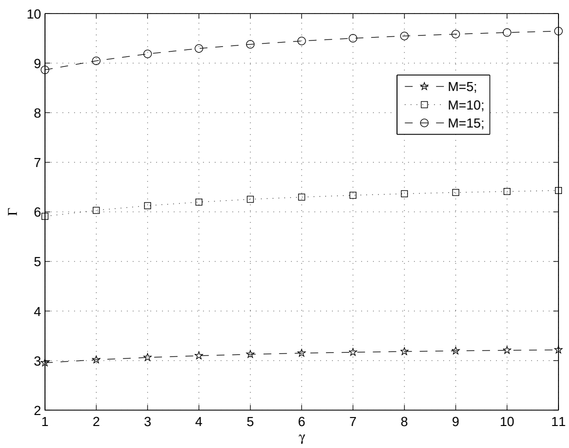

Figure 4: γ vs M on Γ.

Table 1: Total expected cost rate as a function of S and s

s / S	3	4	5	6	7
24	16.782647	16.767858	**16.764989**	16.768130	16.774529
25	16.782211	16.767598	**16.764827**	16.768010	16.774414
26	16.781902	16.767473	**16.764817**	16.768063	16.774493
27	16.781702	16.767465	**16.764936**	16.768263	16.774737
28	16.781596	16.767557	**16.765166**	16.768587	16.775123
29	16.781574	16.767736	**16.7654906**	16.769017	16.775628
30	16.781624	16.767990	**16.765896**	16.769539	16.776237

From Figure 8:

1. $\Gamma_{\text{class 1}} < \Gamma_{\text{class 2}} < \Gamma_{\text{class 3}} < \Gamma_{\text{class 4}}$.

From Figure 9:

1. The expected number of customers in the waiting area increases when λ increases.
2. $\Gamma_{\text{class 1}} < \Gamma_{\text{class 2}} < \Gamma_{\text{class 3}} < \Gamma_{\text{class 4}}$.

From Figure 10:

1. The expected shortage rate increases when λ increases.
2. $\eta_{BP\text{class 1}} < \eta_{BP\text{class 2}} < \eta_{BP\text{class 3}} < \eta_{BP\text{class 4}}$.

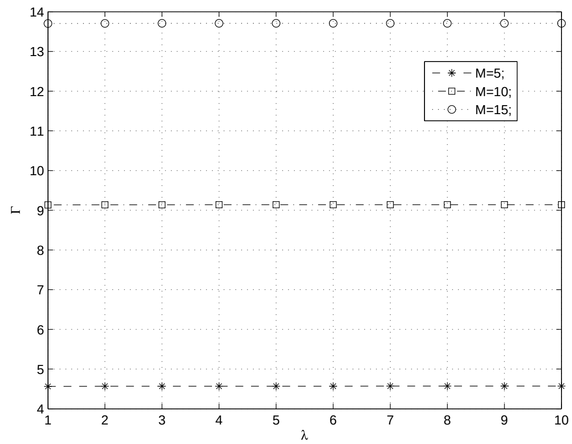

Figure 5: λ vs M on Γ.

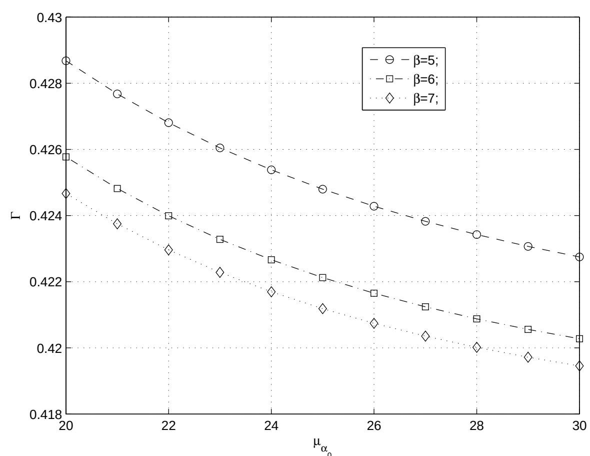

Figure 6: $\mu_{\alpha 0}$ vs β on Γ.

Table 2: Total expected cost rate as a function of s and M

M / s	2	3	4	5	6
9	7.366051	7.243717	**7.214509**	7.218362	7.234890
10	7.284831	7.167980	**7.141078**	<u>7.145958</u>	<u>7.162987</u>
11	<u>7.271173</u>	<u>7.163555</u>	<u>**7.140422**</u>	7.147073	7.164968
12	7.350037	7.255734	**7.238078**	7.247422	7.266592
13	7.547559	7.471338	**7.461386**	7.474618	7.495482

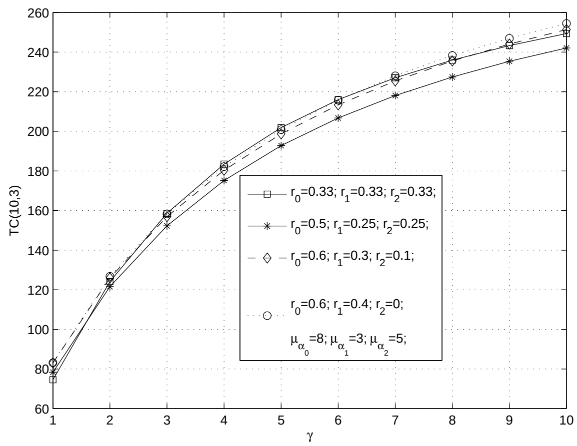

Figure 7: γ vs $Class$ on TC.

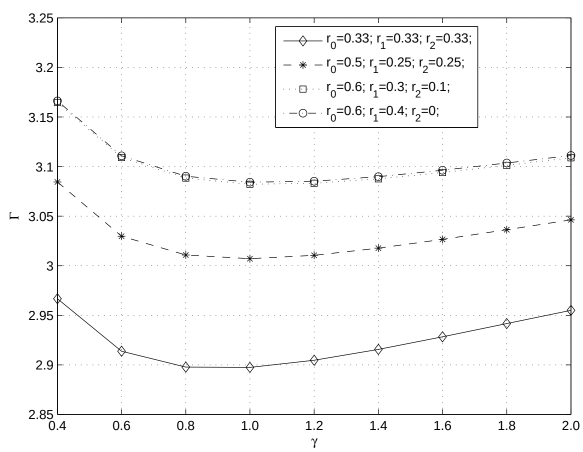

Figure 8: γ vs $Class$ on Γ.

Table 3: Total expected cost rate as a function of S and M

M / S	2	3	4	5	6
64	25.316333	25.185385	**25.150825**	25.153254	25.170301
65	25.314740	25.183799	**25.149013**	25.151185	25.168055
66	25.314124	25.183212	**25.148214**	25.150137	25.166833
67	25.314443	25.183581	**25.148385**	25.150066	25.166591
68	25.315656	25.184865	**25.149484**	25.150931	25.167287

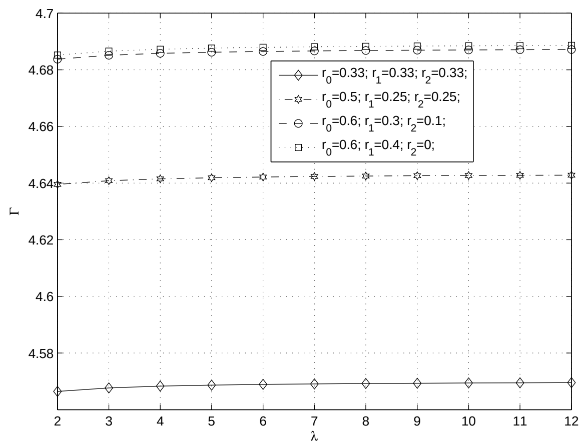

Figure 9: λ vs $Class$ on Γ.

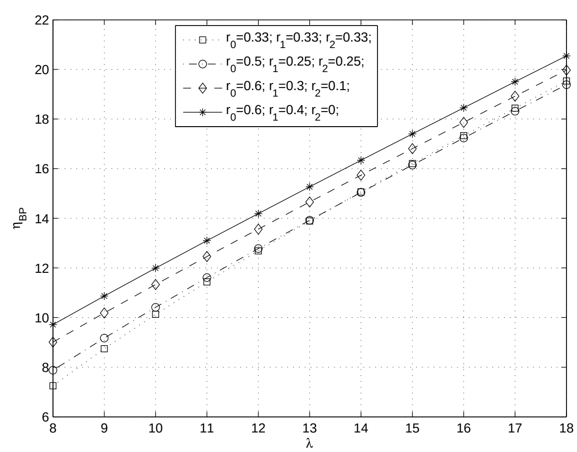

Figure 10: λ vs $Class$ on η_{BP}.

Table 4: Effect of c_s and c_h on the optimal cost rate
$c_p = 0.5, c_w = 0.3, c_l = 0.2$

c_s / c_h	50		55		60		65		70		75	
0.01	26	5	27	5	29	4	30	4	31	4	32	4
	16.764817		16.787804		16.810080		16.831205		16.852220		16.873131	
0.02	18	4	19	4	20	4	20	4	22	3	23	3
	16.839854		16.862746		16.885489		16.908013		16.929071		16.949224	
0.03	14	4	15	3	15	3	16	3	17	3	17	3
	16.961560		16.925443		16.947021		16.968053		16.989133		17.010184	
0.04	13	3	13	3	13	3	14	3	14	3	15	3
	16.954414		16.976689		16.998964		17.021007		17.042907		17.064801	
0.05	11	3	11	3	12	3	12	3	12	3	13	3
	17.000545		17.023779		17.046854		17.069568		17.092283		17.114872	

Table 5: Effect of c_w and c_h on the optimal cost rate
$c_s = 50, c_p = 0.5, c_l = 0.2$

c_w / c_h	0.3		0.4		0.5		0.6		0.7		0.8	
0.01	26	5	27	6	28	7	29	8	30	9	31	10
	16.764817		21.684164		26.596317		31.503624		36.407470		41.308744	
0.02	18	4	19	5	20	6	21	7	21	8	21	10
	16.839854		21.769901		26.690624		31.605205		36.497583		41.401813	
0.03	14	4	15	5	15	6	15	7	15	7	15	7
	16.901560		21.839411		26.750124		31.668161		36.586199		41.504234	
0.04	13	3	13	5	13	6	13	6	13	6	13	6
	16.954414		21.887715		26.813833		31.739751		36.665668		41.591586	
0.05	10	3	13	4	13	5	13	5	13	6	13	6
	17.023374		21.949291		26.875209		31.801127		36.727044		41.652962	

Table 6: Effect of c_w and c_s on the optimal cost rate
$c_h = 0.01, c_p = 0.5, c_l = 0.2$

c_w / c_s	0.3		0.4		0.5		0.6		0.7		0.8	
50	24	5	26	6	27	7	28	8	29	9	30	10
	16.143853		20.804908		25.458921		30.108207		34.754125		39.397538	
55	26	4	27	6	28	7	29	8	30	9	31	10
	16.166851		20.828924		25.483850		30.133955		34.780620		39.424726	
60	27	4	28	6	29	7	30	8	31	9	32	10
	16.188097		20.852750		25.508573		30.159480		34.806877		39.451658	
65	28	4	30	5	31	6	32	7	33	8	34	9
	16.209222		20.875131		25.532372		30.184081		34.831997		39.477167	
70	29	4	31	5	32	6	33	7	34	8	35	9
	16.230235		20.897368		25.555635		30.208229		34.856929		39.502809	

Table 7: Effect of c_s and c_l on the optimal cost rate

$c_h = 0.01, c_p = 0.5, c_w = 0.3$

c_l \ c_s	50		55		60		65		70		75	
0.2	26	5	27	5	29	4	30	4	31	4	34	4
	16.764817		16.787804		16.810080		16.831205		16.852220		16.873131	
0.4	27	6	28	5	29	4	30	4	31	4	34	4
	18.402853		18.425839		18.448132		18.469256		18.490270		18.511181	
0.6	28	6	29	5	30	5	30	5	31	5	34	5
	20.040888		20.063875		20.086184		20.107308		20.128321		20.149232	
0.8	28	6	29	5	30	5	30	5	31	5	34	5
	21.678924		21.701910		21.724236		21.745359		21.766372		21.787282	
1.0	29	7	30	6	30	6	31	6	32	6	35	6
	23.316960		23.339945		23.362287		23.383410		23.404423		23.425333	

Table 8: Effect of c_w and c_l on the optimal cost rate

$c_h = 0.01, c_p = 0.5, c_s = 50$

c_l \ c_w	0.3		0.4		0.5		0.6		0.7		0.8	
0.2	26	5	27	6	28	7	29	8	30	9	31	10
	16.764817		21.684164		26.596317		31.503624		36.407476		41.308744	
0.4	26	6	28	6	29	7	30	8	30	9	31	10
	18.402853		23.322187		28.234330		33.141630		38.045476		42.946739	
0.6	27	6	29	6	30	7	31	8	31	9	32	10
	20.040888		24.960209		29.872343		34.779635		39.683476		44.584734	
0.8	28	7	29	7	30	7	31	8	31	9	32	10
	21.678924		26.598232		31.570356		36.417641		41.321476		46.222730	
1.0	29	7	30	8	30	8	31	9	32	10	33	11
	23.320106		28.336255		33.148370		30.055647		42.959476		47.860725	

Table 9: Effect of c_s and c_p on the optimal cost rate

$c_h = 0.01, c_l = 0.2, c_w = 0.3$

c_p \ c_s	50		52		54		56		58		60	
0.5	26	5	26	5	27	5	27	5	27	5	29	4
	16.764817		16.774026		16.783231		16.792378		16.801525		16.810080	
0.7	23	5	24	5	25	4	25	4	26	4	26	4
	16.779882		16.789263		16.798592		16.807253		16.815881		16.824489	
0.9	22	5	23	4	23	4	23	4	24	4	24	4
	16.794211		16.803126		16.811908		16.820690		16.829449		16.838168	
1.1	21	4	21	4	21	4	22	4	22	4	22	4
	16.806772		16.815699		16.824625		16.833533		16.842384		16.851235	
1.3	19	4	20	4	20	4	20	4	20	4	21	4
	16.818716		16.827798		16.836808		16.845818		16.854828		16.863758	

Table 10: Effect of c_w and c_p on the optimal cost rate
$c_h = 0.01, c_l = 0.2, c_s = 50$

c_w / c_p	0.3		0.4		0.5		0.6		0.7		0.8	
0.5	26	5	27	6	28	7	29	8	30	9	31	10
	16.764817		21.684164		26.596317		31.503624		36.407476		41.308744	
0.7	23	5	25	6	26	7	27	8	28	9	28	10
	16.779882		21.701228		26.615162		31.524097		36.429437		41.331971	
0.9	22	5	23	6	24	7	25	8	25	9	26	10
	16.794211		21.717422		26.633050		31.543493		36.450081		41.352427	
1.1	21	4	21	6	22	7	23	8	24	8	25	9
	16.806772		21.732887		26.650090		31.561840		36.469069		41.356240	
1.3	21	4	21	6	22	7	23	7	23	8	24	9
	16.818716		21.746913		26.666300		31.579169		36.469815		41.374045	

8 Conclusions

The stochastic model discussed here is useful in studying a perishable inventory system with N additional options for service and (s, S) ordering policy. The joint probability distribution of the number of customers in the waiting hall and the inventory level is derived in the steady state. Various system performance measures and the long-run total expected cost rate are derived. By assuming a suitable cost structure on the inventory system, we have presented extensive numerical illustrations to show the effect of change of values for constants on the total expected cost rate. The authors are working in the direction of MAP (Markovian arrival process) arrival for the customers and service times follow PH-distributions.

Acknowledgements

The author wishes to thank the anonymous referee who has provided valuable suggestions for improving the presentation of this paper.

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